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Distance from a curve in the Heisenberg group (preliminary version)

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Abstract

Some properties of the function "Carnot distance from a curve" are considered, both in the case of a horizontal and of a nowhere horizontal curve. Application to: (i) the cut-locus of a surface; (ii) approximating a surface by means of a surface without characteristic points.

Contents

1. Introduction

In two joint papers [5] [6] with Fausto Ferrari the problem of the regularity "signed Carnot distance from a surface in the Heisenberg group" was studied. (See [8] for an extension of the result to step-two Carnot groups and [10] for an interesting application to PDE's in Carnot groups). To prove the regularity result we introduced the "metric normal" to a surface: a geodesic which plays, in this sub-Riemannian setting, the rôle played by the tangent vector normal to a surface in Riemannian geometry. In Riemannian geometry there is one-to-one correspondence between geodesics starting at a point and unit tangent vectors based at that same point. In the sub-Riemannian world this is not true in any obvious way. Usually, one considers priviledged tangent vactors (*horizontal vectors*); but each of them is tangent to infinitely many geodesics. On the other hand, geodesics have horizontal tangents: the rôle of non-horizontal vectors, if there is one, is not as transparent as that they play in the Riemannian world.

In this seminar I consider the distance from a curve instead of the distance from a surface. Let $\Gamma : [\alpha, \beta] \to \mathbb{H}$ be a smooth (simple, open) curve in the Heisenberg rgoup $\mathbb{H}$ and let $d_\Gamma : \mathbb{H} \to [0, +\infty)$ be the function measuring the (Carnot) distance from Γ:

$$d_\Gamma(Q) = \inf\{d(Q, P) : \ P \in \Gamma([\alpha, \beta])\}.$$

We consider various problems.

(i) Is the function d_Γ regular near Γ?

(ii) Given P in Γ, what can we say about the set $\mathcal{N}_P\Gamma = \{Q \in \mathbb{H} : \ d(Q, P) = d_\Gamma(Q)\}$?

We will see that the answer to question (i) is negative if Γ is a horizontal curve and it is positive if Γ is nowhere horizontal. In Riemannian geometry (consider the case of a three-dimensional manifold for better comparison of results), the set $\mathcal{N}_P\Gamma$ is the union of arcs of the geodesics which are normal to Γ at P. The answer to (ii) identifies the corresponding sub-Riemannian object in the union of a corresponding one-parameter family of geodesics of $\mathbb{H}$, starting at P. We can not say that such geodesics are normal to Γ in the usual sub-Riemannian sense; unless the curve Γ is horizontal (but this is the bad case for the distance function d_Γ).

We give two applications of the results just reviewed. The first one concerns the "negative case" of the horizontal curves: the cut-locus of a subset of $\mathbb{H}$ can not be a horizontal curve. More precisely, let E be a closed subset of $\mathbb{H}$ and let K be its cut-locus. There is no open subset U of $\mathbb{H}$ such that $K \cap U$ is a horizontal curve. Actually, this result does not directly follow from the analysis of the function "distance from a curve"; but its proof is a variation on that of the Theorem concerning the "negative case".

The second result is based on the "positive case" of nowhere horizontal curves. Let Ω be a bounded open subset of $\mathbb{H}$ with rectifiable boundary S. Let $\mathcal{H}^s$ denote s-Hausdorff measure w.r.t. the Carnot metric. For each $\epsilon > 0$ there is an open subset Ω' whose boundary S' is a smooth surface *without characteristic points* and such that $\mathcal{H}^4(\Omega \Delta \Omega') < \epsilon$ and $|\mathcal{H}^3(S) - \mathcal{H}^3(S')| < \epsilon$. Here $\Omega \Delta \Omega'$ stands for the symmetric set difference of the two sets.

In the case when Γ is a straight line in $\mathbb{H}$ (i.e. a coset of a one parameter subrgroup of $\mathbb{H}$), the distance from Γ was studied in [4], in collaboration with A. Baldi. It was shown there that the metric tube around Γ could be identified with a metric disc in the Gruschin plane. It was clear from the results of [4] that the properties of the metric tube were different in the case where Γ is horizontal and in that where it is not. In [4], however, this point was not further developed because the main theme of the article was the projection of the Heisenberg group onto the Gruschin plane *per se.*

2. Notation and preliminaries

In $\mathbb{H} = \mathbb{R}^3 \ni (x, y, t)$ consider the vector fields $X = \partial_x - \frac{y}{2}\partial_t$, $Y = \partial_y + \frac{x}{2}\partial_t$ and $T = [X, Y] = \partial_t$ and the product for which $\mathfrak{h} = \text{span}\{X, Y, T\}$ is the Lie algebra of the corresponding group $\mathbb{H}$,

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 1/2(x_1 y_2 - y_1 x_2)).$$

The graded structure of $\mathfrak{h}$ has a horizontal stratum $H = V_1 = \{X, Y\}$ and a commutator $V = V_2 = [V_1, V_2] = \text{span}\{T\}$. The CC-length of an absolutely continuous horizontal curve $\Gamma : [a, b] \to \mathbb{H}$, $\dot{\Gamma} \in H$, is

$$\text{length}_{CC}(\Gamma) = \int_a^b |\dot{\Gamma}(\tau)|_H d\tau,$$

where $|\cdot|_H$ is the horizontal norm of a vector in H. The *Carnot* distance between two points $P,\ Q$ in $\mathbb{H}$ is

$$d(P,Q) = \inf\left\{\text{length}_{CC}(\Gamma) :\ \Gamma(a) = P,\ \Gamma(b) = Q\right\}.$$

A nice interpretation of length and distance in $\mathbb{H}$ can be given in terms of areas and perimeters in the Euclidean plane. Consider a smooth curve $\gamma : [\alpha,\beta] \to \mathbb{R}^2$ in the (x,y)-plane (assume WLOG that $\gamma(\alpha) = (0,0)$) and let $\Gamma : [\alpha,\beta] \to \mathbb{H}$ be its *horizontal lift:*

$$\Gamma(\tau) = (\gamma(\tau), t(\tau)),$$

where

$$t(\tau) = \int_\alpha^\tau \frac{xdy - ydx}{2}$$

is (by Stokes' Theorem) the (signed) Euclidean area enclosed between the arc of γ corresponding to the interval $[\alpha,\tau]$ and the straight line returning from $\gamma(\tau)$ and $(0,0)$. Then, Γ is a horizontal curve in $\mathbb{H}$ and $\text{length}_{CC}(\Gamma) = \text{length}_{Euclidean}(\gamma)$. It is easily checked that the group law in $\mathbb{H}$ is the one for which this interpretation of horizontal curves and length can be moved around the plane by translations, dropping the special assumption that $\gamma(0) = (0,0)$.

The Carnot distance in realized by the length of geodesics. The interpretation in terms of areas rightly suggests that geodesics are horizontal lifts of circles, which are solutions to isoperimetric problems.

2.0.1. *Geodesics.* The equations of the unit-speed geodesics starting at O and having initial speed $-Y$ are $\eta_\kappa = (x,y,t)$.

$$\begin{cases} x(\tau) = \frac{1}{\kappa}(1-\cos(\kappa\tau)), \\ y(\tau) = -\frac{1}{\kappa}\sin(\kappa\tau)), \\ t(\tau) = \frac{\kappa\tau - \sin(\kappa\tau)}{2\kappa^2}. \end{cases} \tag{1}$$

Each η_κ is length-minimizing in an interval of length $1/2\pi\kappa$. The number κ is the curvature of η_κ. Here we let $\kappa > 0$, but the case $\kappa \in \mathbb{R}$ will also be considered. Observe that

$\dot\eta_\kappa(0)=(0,-1,0)=Y$. More generally,

$$\begin{cases}\dot x(\tau)=\sin(\kappa\tau),\\ \dot y(\tau)=-\cos(\kappa\tau),\\ \dot t(\tau)=\frac{1-\cos(\kappa\tau)}{2\kappa}.\end{cases}$$

We look for efficient ways to parametrize the set of geodesics. The rotation $R_\theta:(z,t)\mapsto(e^{i\theta}z,t)$ is an isometric isomorphism of $\mathbb{H}$ and we let $\eta_{\theta,\kappa}=R_\theta\eta_\kappa$, $\theta\in\mathbb{T}$, $\kappa\ge0$. Our parameter space for geodesics will be $\Theta=\mathbb{T}\times\mathbb{R}^+$, with $\mathbb{R}^+=[0,+\infty)$.

Given $\epsilon>0$, the geodesic sphere $\partial B(O,\epsilon)$ is the set of endpoints of geodesics whose parameters $(\theta,\kappa)\in\Theta$ satify $0\le\kappa\le2\pi/\epsilon$.

For the record, we write down the equation of $\eta_{\theta,\kappa}$ and of $\dot\eta_{\theta,\kappa}$.

$$\begin{cases}x(\tau)=\frac{\cos(\theta)}{\kappa}(1-\cos(\kappa\tau))+\frac{\sin(\theta)}{\kappa}\sin(\kappa\tau))=\frac{1}{\kappa}\left(\cos(\theta)-\cos(\theta+\kappa\tau)\right),\\ y(\tau)=\frac{\sin(\theta)}{\kappa}(1-\cos(\kappa\tau))-\frac{\cos(\theta)}{\kappa}\sin(\kappa\tau))=\frac{1}{\kappa}\left(\sin(\theta)-\sin(\theta+\kappa\tau)\right),\\ t(\tau)=\frac{\kappa\tau-\sin(\kappa\tau)}{2\kappa^2}.\end{cases}$$

$$\begin{cases}\dot x(\tau)=\cos(\theta)\sin(\kappa\tau)+\sin(\theta)\cos(\kappa\tau)=\sin(\theta+\kappa\tau),\\ \dot y(\tau)=\sin(\theta)\sin(\kappa\tau)-\cos(\theta)\cos(\kappa\tau)=-\cos(\theta+\kappa\tau),\\ \dot t(\tau)=\frac{1-\cos(\kappa\tau)}{2\kappa}.\end{cases}\tag{2}$$

In particular, $\dot\eta_{\theta,\kappa}(0)=(\sin(\theta),-\cos(\theta),0)$.

3. The metric tube around a non-horizontal curve.

Let $\Gamma:(\alpha,\beta)\to\mathbb{H}$ be a smooth curve in the Heisenberg group, which we assume for the moment to be nonhorizontal pointwise. We want to find the geodesics which are *normal* to Γ at some $P=\Gamma(\tau_0)$. Assume WLOG that $\Gamma(0)=O$. The geodesics η we look for are s.t. for some $\epsilon>0$ and all $0\le\tau\le\epsilon$ one has $d(\eta(\tau),\Gamma)=\tau$ (think of the Euclidean case for comparison). We assume throughtout that the projection $\dot\Gamma_H$ of $\dot\Gamma(0)$ on H satisfies $|\dot\Gamma_H|=1$. After a rotation we can assume that

$$\dot\Gamma(0)=X+mT=\partial_x+m\partial_t=(1,0,m).$$

Proposition 3.1. *Let Γ be as above. The normal (length minimizing) geodesics starting at O are the geodesics η^b ($-\infty \le b \le +\infty$) such that (if $b \neq \pm\infty$) having initial speed*

$$\dot{\eta}^b(0) = \left(\frac{m}{\sqrt{m^2+b^2}}, \frac{b}{\sqrt{m^2+b^2}}, 0 \right)$$

and curvature

$$\kappa^b = \frac{4}{\sqrt{m^2+b^2}}.$$

If $b = \pm\infty$, then $\dot{\eta}^{\pm\infty}(0) = (0, \pm 1, 0)$, $\kappa^{\pm\infty} = 0$ and $\eta^{\pm\infty}$ has equation

$$\eta^{\pm\infty}(\tau) = \begin{cases} x(\tau) = 0, \\ y(\tau) = \pm\tau, \\ t(\tau) = 0. \end{cases}$$

That is, there exists $\epsilon > 0$ such that for all $b \in [-\infty, +\infty]$ and $\tau \in [0, \epsilon]$:

$$d(\eta^b(\tau), \Gamma) = d(\eta^b(\tau), O) = \tau.$$

Moreover, if $P \neq O$ and $P = \Gamma(\sigma)$ for some σ, then $d(\eta^b(\tau), P) > 0$. That is, in the metric disc $\{\eta^b(\tau) : \ -\infty \le b \le +\infty, \ 0 \le \tau \le \epsilon\}$ we have the unique projection property.

Sketch of the proof. Think of a closed Carnot-metric ball B which is already tangent to Γ at O. This means that (assuming the ball is small enough, depending smallness on several things which include the Euclidean -or Riemannian w.r.t. suitable Riemannian metrics- curvature of Γ) there is a plane in $\mathbb{H}$ which is tangent to both Γ and B. There is in fact a one-parameter family of such planes. They have equation

$$mx + by = t,$$

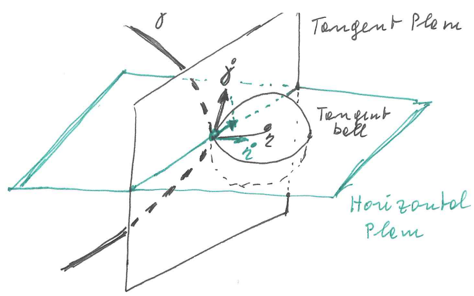
where b is a free real parameter. (We have indeed the exceptional plane $y = 0$, corresponding to $b = \infty$). Call $\Pi = \Pi_b$ the plane. Its characteristic point is $C = C_b = (-b/2, m/2, 0)$, which describes a straight line in the $t = 0$ plane. Observe that the straight line OC is the only horizontal line (direction) on Π at O.

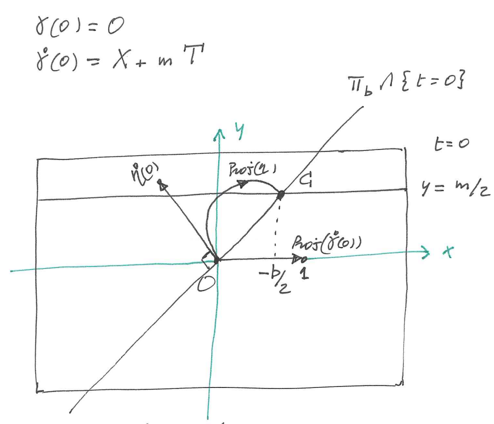
The geodesic η we look for is the metric normal to Π at O. The starting velocity $\dot{\eta}(0) = \nu$ is the horizontal normal to Π at O while $|d(O,P)|/2$ is the radius of curvature of η, that is $\kappa = 2/d(O,P)$. We will choose a positive direction for η in the following way.

Consider the circle having diameter OC on the plane $t = 0$, counterclockwise oriented. Then, $\dot{\eta}(0)$ is the normalized velocity of this circle at O. Namely,

$$\nu = \dot{\eta}(0) = \left(\frac{m}{\sqrt{m^2 + b^2}}, \frac{b}{\sqrt{m^2 + b^2}}, 0 \right).$$

With this choice, for (small) positive τ's we see $\eta(\tau)$ *above* $t = 0$ and *below* Π.





Rule and compass construction

We now translate the above into concrete equations. Namely, $\eta = \eta_{\theta,\kappa}$ with

$$\begin{cases} \kappa = 4/\sqrt{b^2+m^2}, \\ \cos(\theta) = -b/\sqrt{b^2+m^2}, \\ \sin(\theta) = m/\sqrt{b^2+m^2}. \end{cases} \tag{3}$$

That is,

$$\begin{cases} x(\tau) = -\frac{b}{4}(1-\cos(\kappa\tau)) + \frac{m}{4}\sin(\kappa\tau)), \\ y(\tau) = \frac{m}{4}(1-\cos(\kappa\tau)) + \frac{b}{4}\sin(\kappa\tau)), \\ t(\tau) = \frac{b^2+m^2}{32}\left[\frac{4\tau}{\sqrt{b^2+m^2}} - \sin\left(\frac{4\tau}{\sqrt{b^2+m^2}}\right)\right]. \end{cases}$$

Observe that $b=0$ gives the maximal curvature corresponding to a given choice for m. We have

$$\begin{aligned} \kappa_{\max} &= 4/|m|, \\ \theta_{\max} &= \operatorname{sign}(m)\frac{\pi}{2}, \\ C_{\max} &= (0, m/2, 0). \end{aligned}$$

Let us go back to the curve Γ. Suppose $\dot{\Gamma}(s) = \cos(\alpha)(s)X + \sin(\alpha(s))Y + m(s)T$. The geodesic we have computed before has to be rotated by an angle of $\alpha(s)$, then left translated by $\Gamma(s)$. That is, for each $b \in \mathbb{R}$ we have a geodesic $\eta^{s,b}(\tau) = \Gamma(s)\cdot\eta_{\alpha(s),m(s),b}(\tau)$, where $\eta_{\alpha(s),m(s),b}(\tau)$ is given by

$$\begin{cases} x(\tau) = \frac{1}{\kappa(m(s),b)}\left(\cos(\alpha(s)+\theta(m(s),b)) - \cos(\alpha(s)+\theta(m(s),b)+\kappa(m(s),b)\tau)\right), \\ y(\tau) = \frac{1}{\kappa(m(s),b)}\left(\sin(\alpha(s)+\theta(m(s),b)) - \sin(\alpha(s)+\theta(m(s),b)+\kappa(m(s),b)\tau)\right), \\ t(\tau) = \frac{\kappa(m(s),s)\tau - \sin(\kappa(m(s),s)\tau)}{2\kappa^2(m(s),s)}, \end{cases} \tag{4}$$

and θ, κ are given by (3). We record the velocity:

$$\begin{cases} \dot{x}(\tau) = \sin(\alpha(s)+\theta(m(s),b)+\kappa(m(s),b)\tau), \\ \dot{y}(\tau) = -\cos(\alpha(s)+\theta(m(s),b)+\kappa(m(s),b)\tau), \\ \dot{t}(\tau) = \frac{1-\cos(\kappa(m(s),s)\tau)}{2\kappa(m(s),s)}. \end{cases} \tag{5}$$

The metric exponential map starting on the curve Γ is

$$\mathcal{E}\mathrm{xp} : \Gamma \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{H}, \quad \mathcal{E}\mathrm{xp}(\Gamma(s), b, \tau) = \Gamma(s) \cdot \eta_{\alpha(s),m(s),b}(\tau).$$

Here, $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$.

Theorem 3.1. *If $\Gamma : [a, b] \to \mathbb{H}$ is a C^2 regular curve which is nowhere horizontal and $\epsilon > 0$ is suitably small, then $\mathcal{E}xp$ defines a diffeomorphism of $\Gamma \times \mathbb{R}^* \times [-\epsilon, \epsilon]$ onto a closed region T in $\mathbb{H}$.*

Moreover, T contains in its interior the open part $\Gamma((a, b))$.

The key step in proving the theorem is an estimate which is provided in a subsection below.

Fix $\epsilon > 0$ small enough and consider the ϵ-tube $S^\epsilon_\Gamma = \mathrm{Tube}^\epsilon_\Gamma(\Gamma \times \mathbb{R}^*)$ around Γ, where

$$\mathrm{Tube}^\epsilon_\Gamma(s, b) = \mathcal{E}\mathrm{xp}(\Gamma(s), b, \epsilon).$$

By definition and construction, the metric normal to S^ϵ_Γ at $\mathrm{Tube}^\epsilon_\Gamma(s_0, b_0)$ is the geodesic $\Delta\tau \mapsto \mathcal{E}\mathrm{xp}(\Gamma(s_0), b_0, \epsilon + \Delta\tau)$. The imaginary curvature at $\mathrm{Tube}^\epsilon_\Gamma(s_0, b_0)$ is then $\kappa(s_0, b_0)$, the horizontal normal is

$$\nu_H(s_0, b_0, \epsilon) = \left.\frac{d}{d\tau}\right|_{\tau=\epsilon} \mathcal{E}\mathrm{xp}(\Gamma(s_0), b_0, \tau),$$

which can easily computed. In principle, this gives the mean curvature as well. The horizontal field v can be reconstructed from ν_H. Overall, we could compute the Hessian of $\delta_{\mathrm{Tube}^\epsilon_\Gamma}$, using the results of [5].

Corollary 3.1. *If Γ is smooth and everywhere non-horizontal, then for small $\epsilon > 0$ the surface $Tube^\epsilon_\Gamma$ is smooth and free of characteristic points.*

3.1. **The metric tube around non-horizontal curve: the differential.** We give here the estimate which is the key in proving Theorem 3.1.

Let $\Gamma : I \to \mathbb{H}$ be a non-horizontal curve, smooth and without self-intersections, which we parametrize so to have unit speed projection Γ on $\mathbb{R}^2$,

$$\dot\Gamma(s) = \cos(\alpha(s))X + \sin(\alpha(s))Y + m(s)T, \ \dot\Gamma(s) = e^{i\alpha} \tag{6}$$

in complex notation. The exponential map, projected on $\mathbb{R}^2$, is

$$w = z + e^{i(\alpha+\theta)}\frac{1-e^{i\kappa\tau}}{\kappa} \tag{7}$$

where $z = z(s) = \Gamma(s)$, $\theta = \theta(s,b)$ and $\kappa = \kappa(s,b)$ are as in (3) and the expression is deduced from (4). We are interested in the tube $\{\tau = \epsilon\}$ and we want to show that, for $\epsilon > 0$ small enough it is a smooth surface in $\mathbb{H}$. Since θ depends on s through $m = m(s)$, we will write θ_m and $\theta_s = \theta_m m'$ to denote the partial derivatives w.r.t. these coordinates and the same we will do with κ. From (3) we deduce some useful formulas:

$$\begin{aligned}
\Delta &:= \sqrt{b^2+m^2},\ \kappa = 4/\Delta \\
-\sin\theta\cdot\theta_m &= \partial_m(\cos\theta) = \partial_m\left(-\frac{b}{\Delta}\right) = \frac{bm}{\Delta^3} = \sin\theta\cdot\frac{b}{\Delta^2} \implies \theta_m = -\frac{b}{\Delta^2} \\
\cos\theta\cdot\theta_b &= \partial_b(\sin\theta) = \partial_b\left(\frac{m}{\Delta}\right) = -\frac{bm}{\Delta^3} = \cos\theta\cdot\frac{m}{\Delta^2} \implies \theta_b = \frac{m}{\Delta^2} \\
\kappa_b &= -\frac{b}{\Delta^3} \text{ and } \kappa_m = -\frac{m}{\Delta^3}.
\end{aligned} \tag{8}$$

Since $\dot{z} = e^{i\alpha}$, we have

$$\begin{aligned}
w_s &= e^{i\alpha} + e^{i(\alpha+\theta)}i(\alpha_s+\theta_s)\frac{1-e^{i\kappa\epsilon}}{\kappa} + e^{i(\alpha+\theta)}\frac{-e^{i\kappa\epsilon}i\kappa\epsilon - (1-e^{i\kappa\epsilon})}{\kappa^2}\kappa_s \\
&= e^{i\alpha} + e^{i(\alpha+\theta)}i(\alpha_s+\theta_s)\frac{1-e^{i\kappa\epsilon}}{\kappa} - e^{i(\alpha+\theta)}\frac{1-e^{i\kappa\epsilon}}{\kappa}\frac{1-i\kappa\epsilon}{\kappa}\kappa_s - e^{i(\alpha+\theta)}\frac{i\epsilon\kappa_s}{\kappa}.
\end{aligned} \tag{9}$$

Also

$$w_b = e^{i(\alpha+\theta)}i\theta_b\frac{1-e^{i\kappa\epsilon}}{\kappa} - e^{i(\alpha+\theta)}\frac{1-e^{i\kappa\epsilon}}{\kappa}\frac{1-i\kappa\epsilon}{\kappa}\kappa_b - e^{i(\alpha+\theta)}\frac{i\epsilon\kappa_b}{\kappa}. \tag{10}$$

Observe that $w_s = e^{i\alpha}w_s'$ (where w_s' is like w_s with $\alpha = 0$) and that $w_b = e^{i\alpha}w_b'$ (same convention). This means that

$$J\begin{pmatrix} w \\ s,b \end{pmatrix} = e^{i\alpha}J\begin{pmatrix} w' \\ s,b \end{pmatrix},$$

where we have (ab)used a mixed real-complex notation. We record:

$$\begin{aligned}
w_s' &= 1 + e^{i\theta}i(\alpha_s+\theta_s)\frac{1-e^{i\kappa\epsilon}}{\kappa} - e^{i\theta}\frac{1-e^{i\kappa\epsilon}}{\kappa}\frac{1-i\kappa\epsilon}{\kappa}\kappa_s - e^{i\theta}\frac{i\epsilon\kappa_s}{\kappa}, \\
w_b' &= e^{i\theta}i\theta_b\frac{1-e^{i\kappa\epsilon}}{\kappa} - e^{i\theta}\frac{1-e^{i\kappa\epsilon}}{\kappa}\frac{1-i\kappa\epsilon}{\kappa}\kappa_b - e^{i\theta}\frac{i\epsilon\kappa_b}{\kappa}.
\end{aligned}$$

In order to use the Inverse Mapping Theorem, we show that w_s' and w_b' are independent for small ϵ.

The Taylor expansion w.r.t. ϵ gives (after lengthy calculations):

$$\begin{aligned} \det J(w_s', w_b') &= \cos(\kappa\epsilon) \cdot \frac{-\epsilon b^2}{\Delta^3} + \sin(\kappa\epsilon)\frac{1}{4} + o(\epsilon) = \left(-\frac{b^2}{\kappa\Delta^3} + \frac{1}{4}\right)\epsilon + o(\epsilon) \\ &= \frac{m^2}{4\Delta^2}\epsilon + o(\epsilon). \end{aligned} \tag{11}$$

Since $m \neq 0$ and m is a continuous function of τ, for $\epsilon > 0$ small enough the Jacobian is non-vanishing. There seems to be a problem for $b \to \pm\infty$, which is seen to be apparent after changing to the variable $1/b \to 0$ and verifyng the Jacobian in the new variables is still of the order of ϵ.

Corollary 3.2. *Let $\Gamma : [a,b] \to \mathbb{H}$ be a smooth non-horizontal curve. Then, $[a,b] \times \mathbb{R} \mapsto \mathcal{E}xp(\Gamma(s), b, \epsilon)$ has non-singular jacobian provided $0 < \epsilon \leq \epsilon_0$ with ϵ_0 small enough.*

4. Approximating surfaces by surfaces without characteristic points

Let S be a smooth, compact, connected, boundaryless surface in $\mathbb{H}$, $S = \partial\Omega$, where Ω is a bounded open subset of $\mathbb{H}$. If, for instance, Ω is simply connected, then S necessarily has characteristic points. Can we approximate S by surfaces without characteristic points? We can do this, in general, only at the expenses of boundedness or byaltering the topology. Here we sacrifice boundedness.

Theorem 4.1. *Let Ω be a bounded open set with rectifieble boundary S. Fix $\epsilon > 0$. Then, there exists a (possibly unbounded) open set Ω_ϵ in $\mathbb{H}$ such that $\S_\epsilon := \partial\Omega_\epsilon$ is smooth and free of characteristic points,*

$$\mathcal{H}^4(\Omega_\epsilon \Delta \Omega) < \epsilon$$

and

$$\left|\mathcal{H}^3(S_\epsilon) - \mathcal{H}^3(S)\right| < \epsilon.$$

Moreover, Ω and Ω_ϵ are homeomorphic.

Sketch of the proof. First, we approximate Ω by a bounded open set Ω' such that: (i) $S' := \partial\Omega'$ is smooth and has just isolated (forcibly, finitely many) characteristic points; (ii) it satisfies the desired estimates of volume and perimeter with $\epsilon/2$ instead of ϵ. This can be done by approximating first S by polyhedra, then by smoothing their edges and vertices, having care not to introduce curves of characteristic points. We can arrange things so to have all characteristic points on flat parts ofS'.

Next, for each characteristic point $P$of S', we consider a simple, smooth, everywhere non-horizontal curve Γ_P (i) starting at P; (ii) lying in $\{P\} \cup \mathbb{H} \setminus \overline{\Omega}$; (iii) going to infinity (i.e. eventually leaving all compact subsets of $\mathbb{H}$); whose starting arc and final (infinite) arc are vertical (i.e. of the form $\tau \mapsto (x_0, y_0, \tau)$). We can ask such curves not to intersect each other.

For each such curve and characteristic point P, remove a small disc of S' around P and consider a thin surface of rotation like the ones described in the subsection below which is smoothly glued to the disc's boundary. We shall see that such surface is free of characteristic points. Then, we smoothly glue on the final circle of the surface of rotation the metric tube (with a small radius) around the non-horizontal curve until we reach the final (infinite) vertical arc of Γ_P. Since the curve Γ_P is everywhere non-horizontal, by the previous results the metric tube is free of characteristic points, provided its radius is small enough. At the end of the metric tube we glue another surface of rotation which becomes thinner and thinner, so to have globally small perimeter and volume. In this approximating procedure we have replaced a small disc of S' around a characteristic point by an infinite tube having small perimeter and volume, which is free of characteristic points.

If we do this for all characteristic points while keeping the total perimeter and volume under control, and taking care thatthe tubes do not intersect, we obtain a surface S with the desired properties.

4.1. **Surfaces of rotation.** We give here a less qualitative description of the surface of rotation alluded to above. Let T be a surface having equation $(x, y, t) = F(t, \theta)$, $F : [a, b] \times [-\pi, \pi] \to \mathbb{H}$,

$$F : (t, \theta) \mapsto (\varphi(t) \cos(\theta), \varphi(t) \sin(\theta), t).$$

Here $\varphi : [a, b] \to (0, \infty)$is smooth.

Lemma 4.1. *The* **imaginary curvature** *of T is*

$$\kappa = \frac{\sqrt{1+\frac{\varphi^2(\varphi')^2}{4}}}{\varphi'}$$

The mean curvaure of T is

$$h_T(t,\theta) = \frac{\frac{1}{\varphi}-\frac{1}{4}\varphi''\varphi^2}{\frac{1}{4}\varphi'^2\varphi^2+1}.$$

Hence, T is free of characteristic points.

The part of the lemma we need for the approximation theorem is the formula for the imaginary curvature. The formula for the mean curvature can have independent interest. With little more effort, one can find the horizontal curves ruling the surface T.

5. The metric tube around a horizontal curve.

Let $\Gamma : [a,b] \to \mathbb{H}$ be a smooth horizontal curve and let $P = \Gamma(\tau_0)$ with $a < \tau_0 < b$. WLOG assume that $P = O = \Gamma(0)$. Let B be a small closed metric ball touching Γ at O and let Q be its center. Clearly, $\dot\Gamma(0)$ must lie on the space $T_O\partial B$ tangent to ∂B at O, hence it lies in the only horizontal direction in $T_O\partial B$. The geodesic η joining Q and O is (if the ball is suitably small) the metric normal to $T_O\partial B$ at O (identified with a plane in $\mathbb{H}$). Hence, $\dot\eta(0)$, the velocity of η at $\eta(0) = O$, is the unit horizontal vector normal to $\dot\Gamma(0)$, by the *horizontal Gauss Lemma* in [5]. This means that all geodesics metrically normal to Γ at $\Gamma(0) = O$ have initial velocity $\dot\eta(0) = \pm\nu$, where ν is one of the two horizontal vectors normal to $\dot\Gamma(0)$. This state of affairs is radically different from the one we have verified in the case of the non-horizontal curves. In the present situation, which is peculiarly sub-Riemannian, all geodesics normal to Γ at $\Gamma(0)$ have the same initial velocity vector. Since the infimum of the length over which such geodesics are length minimizing is zero, some of them will never reach the metric tube of radius ϵ around Γ, no matter how small we choose Γ.

These euristics make believable, I think, the following theorem.

Theorem 5.1. *Let Γ be as above a horizontal curve in $\mathbb{H}$. Then, for all $\epsilon > 0$ the tube*

$$T_\epsilon = \{Q \in \mathbb{H} : \ d(Q,\Gamma) = \epsilon\}$$

is not a C^1 surface in the Riemannian sense.

In fact, there are points Q in T_ϵ such that there are two distinct geodesics η_1 and η_2 which minimize the distance between Q and Γ and such geodesics have different horizontal tangent vector at Q.

6. The cutlocus does not contain horizontal curves

The result we are going to present here does not follow, strictly speaking, from the one about the distance from a horizontal curve, but its proof is a slight variation of that of Theorem 5.1.

Let E be a closed subset of $\mathbb{H}$ and let K be the *cut-locus* of E. The set K is define as follows. Let η be a geodesic starting at some point $P = \eta(0)$ of E with the property that $d(\eta(\tau), K) = \tau$ for $0 \le \tau \le \epsilon$ and $\epsilon > 0$. Let ϵ_0 be the supremum of the $\epsilon > 0$ for which the property holds. If $\epsilon_0 < \infty$, the point $\eta(\epsilon_0)$ belongs to the cutlocus. If the supremum is 0 and P belongs to the boundary ∂E of E, we say as well that P belongs to the cut-locus. See [1] and [5] for some properties of the cut-locus in $\mathbb{H}$.

Theorem 6.1. *Let E be a closed subset of $\mathbb{H}$ and let K be its cutlocus. Let Ω be an open subset of $\mathbb{H} \setminus E$. Then, $K \cap \Omega$ is not a horizontal curve.*

To better put into context this result, we recall some well known facts.

- The cut-locus of the plane $t = 0$ is the vertical axis, which is a smooth non-horizontal curve.
- The cut-locus of a single point is a non-horizontal curve as well (a coset of the vertical axis, in fact).
- The cutlocus of any closed set E can not contain isolated points.

These facts and the result just stated seem to justify the following.

Conjecture 6.1. *Let E, K and Ω be as above. Then,*

$$\mathcal{H}^2(\Omega \cap K) > 0.$$

The example

$$E = \{(\pm 1, y, t) : \ y, t \in \mathbb{R}\},$$

for which

$$K = \{(0, y, t) : \ y, t \in \mathbb{R}\},$$

shows that it can happen that $\mathcal{H}^3(\Omega \cap K) > 0$.

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