Stability of isometric maps in the Heisenberg group *

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Abstract

In this paper we prove some approximation results for biLipschitz maps in the Heisenberg group. Namely, we show that a biLipschitz map with biLipschitz constant close to one can be pointwise approximated, quantitatively on any fixed ball, by an isometry. This leads to an approximation in BMO norm for the map's Pansu derivative. We also prove that a global quasigeodesic can be approximated by a geodesic on any fixed segment.

1 Introduction

In 1961 Fritz John proved the following stability estimates. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a biLipschitz map such that $f(0) = 0$ and the Lipschitz constant of f and f^{-1} is less than $1 + \varepsilon$. Then for any ball $B = B(0, R)$, there is $T \in O(n)$ such that

$$|f(x) - Tx| \leq C_n \varepsilon R, \quad \forall\, x \in B \quad \text{and} \tag{1.1}$$

$$\frac{1}{|B(0,R)|} \int_B |f'(x) - T| dx \leq C'_n \varepsilon. \tag{1.2}$$

Here C_n and C'_n are dimensional constant. f' is the differential of f. Estimates (1.1) and (1.2) and their improvements are object of considerable interest in geometric function theory and nonlinear elasticity; see references below.

In this article we study approximation results extending (1.1) and (1.2) from Euclidean space to the Heisenberg group $\mathbb{H} = \{(z; t) \in \mathbb{C} \times \mathbb{R}\}$ equipped with its Lie group structure and its control distance d. See Section 2 for all the background.

The first issue is to establish what the correct extensions are. It is well known that an isometry $T : \mathbb{H} \to \mathbb{H}$ which fixes the origin has the form $(z; t) \mapsto (Az; (\det A)t)$

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where $A \in O(2)$. Moreover, a notion of differentiability for maps in the Heisenberg group has been introduced by Pansu [P] and the Pansu differential can be identified with a 2×2 matrix. Therefore it is reasonable to guess that the extensions of (1.1) and (1.2) have the form

$$d\big(f(z;t),(Az,(\det A)t\big) \le C(\varepsilon)R, \quad \forall\ (z;t) \in B(0,R) \quad \text{and} \tag{1.3}$$

$$\frac{1}{|B(0,R)|}\int_{B(0,R)} |Jf(z;t)-A|dzdt \le C'(\varepsilon). \tag{1.4}$$

Here f is a $(1+\varepsilon)$ biLipschitz map from $\mathbb{H}$ onto itself fixing the origin and once $R>0$ is chosen, there is $A \in O(2)$ such that both estimates hold. Jf is the Jacobian of f in the sense of Pansu, see Section 2. d is the control distance and $B(0,R) = \{(z;t) \in \mathbb{H} : d((0;0),(z;t)) < R\}$. $|B(0,R)|$ denotes the standard Lebesgue measure of the ball. Lebesgue measure is the Haar measure of $\mathbb{H}$.

The main goal of our paper is proving both (1.3) and (1.4), with a *quantitative* estimate on the constant $C(\varepsilon)$ and $C'(\varepsilon)$. A *qualitative* version of the first inequality (1.3) can rather easily be obtained by Arzelà Theorem, but it does not give any estimate of the rate of convergence to 0 of $C(\varepsilon)$. Our search for quantitative estimates for $C(\varepsilon)$ and $C'(\varepsilon)$, as $\varepsilon \to 0$ involves the understanding of a number of fine properties of the Carnot-Carathéodory distance in $\mathbb{H}$ which may have some independent interest in subriemannian geometry.

John's proof of the pointwise (1.1), see [J, Lemma IV and Theorem 3], is rather elementary, but it heavily relies on the Euclidean structure on $\mathbb{R}^n$, in particular on the isotropic nature of its geometry. Due to the non isotropic structure of the Heisenberg group, the proof of (1.3) cannot be obtained so easily. In order to get estimate (1.3), we examine the behaviour under biLipschitz maps of different subsets of $\mathbb{H}$ and in so doing we consider $\mathbb{H}$ as a metric space, making very little use of its differential structure. The study of the geometry of subsets of the Heisenberg group and more generally of Carnot groups is very rich and intricate and it has been object of many recent papers. See for instance [G], [FSSC], [BHT], [BRSC], [Ba], [AF], just to quote a few.

We shall make a substantial use of the explicit form of the geodesics for the metric d; see Section 2. Although their equations are known, they are not always easy to manage, and this introduces several new difficulties with respect to the Euclidean situation. In order to avoid further complications, we work here in the first Heisenberg group. Geodesics in the Heisenberg group have been recently used by several authors, in order to discuss a number of different properties of $\mathbb{H}$ with its control distance. See for instance Gaveau [Gav], Korányi [Kor], Monti and Serra Cassano [MSC], Ambrosio and Rigot [AR], Arcozzi and Ferrari [AF].

The first result we prove concerns the behaviour of Heisenberg quasigeodesics. This is a rather studied topic, expecially in the hyperbolic setting (see, e.g. [GH], [Bo]).

It is well known that for any $\theta \in [0, 2\pi]$, the path $\gamma(s) = (se^{i\theta}; 0)$, $s \in \mathbb{R}$, is a global geodesic for the control metric in $\mathbb{H}$, i.e. $d(\gamma(s), \gamma(s')) = |s - s'|$ for $s, s' \in \mathbb{R}$. Moreover, all global geodesics have this form. A $(1+\varepsilon)-$quasigeodesic is, by definition, any path $\gamma : \mathbb{R} \to \mathbb{H}$, such that $(1+\varepsilon)^{-1}|s-s'| \leq d(\gamma(s), \gamma(s')) \leq (1+\varepsilon)|s-s'|$, for any $s, s' \in \mathbb{R}$.

It is known that any quasigeodesic γ is a horizontal path. Denote by $\dot{\gamma}_H = (a, b)$ the invariant components of $\dot{\gamma}$ in the standard horizontal orthonormal frame $\{X, Y\}$: $\dot{\gamma} = aX(\gamma) + bY(\gamma)$ almost everywhere. Then

Theorem A (approximation of quasigodesics). *There are $\varepsilon_0 > 0$ and C_0 absolute constants such that, given a $(1 + \varepsilon)-$quasigeodesic $\gamma : \mathbb{R} \to \mathbb{H}$ with $\varepsilon \leq \varepsilon_0$, then its horizontal speed $\dot{\gamma}_H$ satisfies*

$$1 - C\varepsilon^{1/2} \leq \Big| \frac{1}{|I|} \int_I \dot{\gamma}_H(s) ds \Big| \leq 1 + C\varepsilon, \tag{1.5}$$

Contrary to the Euclidean case, (1.5) is not trivially equivalent to the definition of quasigeodesic. A peculiarly subriemannian consequence of (1.5) is that, in the hypothesis of Theorem A, any quasigeodesic γ passing through a point P_0 in $\mathbb{H}$ at time s_0 is forever forced to avoid a certain metric cone; extrinsically speaking, a paraboloid, having vertex at P_0. See Corollary 3.3. If $\varepsilon = 0$ in (1.5), γ is a global geodesic.

The constant $\varepsilon^{1/2}$ does not probably exhibit the right order of growth with respect to ε. In proving Theorem A and all the results stated below, we use the known comparison between the control distance d and the Euclidean one stated in (2.1), which usually is not sharp. This forces us to take several time a square root of ε. The problem of getting sharp asymptotics as $\varepsilon \to 0$ seems to be rather complicated and it probably requires new non trivial ideas.

In $\mathbb{H}$ there are two different kind of Euclidean planes: laterals of two-dimensional subgroups of $\mathbb{H}$ and planes with a characteristic point. Our second step is studying how a biLipschitz map transforms a plane with a characteristic point. Up to a translation, it suffices to consider the plane $t = 0$. We prove the following.

Theorem B (biLipschitz image of a horizontal plane). *There is $\varepsilon_0 > 0$ and $C > 0$ such that, if f is $(1+\varepsilon)-$biLipschitz with $\varepsilon \leq \varepsilon_0$ and $f(0) = 0$, for any $R > 0$ there is $A \in O(2)$ such that*

$$d(f(z; 0), (Az; 0)) \leq C\varepsilon^{1/16}R, \quad \textit{for any } z \in \mathbb{R}^2,\ |z| \leq R. \tag{1.6}$$

Then, we examine how a biLipschitz map transforms the t axis $\{(0; t) \in \mathbb{C} \times \mathbb{R}\}$, the center of $\mathbb{H}$. Recall that, from the point of view of the metric d, the $t-$axis is unrectifiable and its Hausdorff dimension is 2, see (2.8). The behaviour of the t-axis under quasiconformal mappings has been object of some interest. See especially Heinonen and Semmes [HS], Question 25. Here we show that the image of the $t-$axis under a biLipschitz map lays in a metric cone around the $t-$axis itself. Note that biLipschitz is a smaller class than quasiconformal.

Theorem C (biLipschitz image of the $t-$axis). *Let f be a $(1+\varepsilon)-$biLipschitz map such that $f(0)=0$ and $\varepsilon<\varepsilon_0$. Then, after possibly applying the isometry $(x,y,t)\mapsto(x,-y,-t)$, we have, for some absolute $C>0$,*

$$d\big(f(0;t),(0;t)\big)\leq C\varepsilon^{1/32}d\big((0;0),(0;t)\big),\quad \forall\, t\in\mathbb{R}.$$

Finally, combining all the result obtained, we obtain the extension of John's pointwise approximation theorem.

Theorem D (pointwise approximation). *There exist $\varepsilon_0>0$ and $C>0$ such that, if f is a $(1+\varepsilon)$-biLipschitz map of $\mathbb{H}$, $R>0$ and P_0 is a fixed point in $\mathbb{H}$, then there exists an isometry T of $\mathbb{H}$ such that*

$$d(f(P),T(P))\leq C\varepsilon^{1/2^{11}}R, \tag{1.7}$$

whenever $d(P,P_0)\leq R$.

As in the Euclidean case, Theorem D and Rademacher's Theorem, which was proved in the Heisenberg group by Pansu [P], imply that the Jacobian of f in the sense of Pansu (see Section 2) can be approximated by means of an isometry.

Theorem E (approximation of derivatives). *There are constants $\varepsilon_0>0$ and $C>0$ such that, if f is $(1+\varepsilon)$-biLipschitz with $0\leq\varepsilon<\varepsilon_0$, $f(0)=0$ and Jf is the Jacobian matrix of f in the sense of Pansu, if $R>0$, then there is $A\in O(2)$ such that*

$$\frac{1}{|B(0,R)|}\int_{B(0,R)}\|Jf(Q)-A\|dQ\leq C\varepsilon^{1/2^{12}}. \tag{1.8}$$

Equation (1.8) says that Jf belongs to $BMO(\mathbb{H})$. By the John-Nirenberg inequality which holds in this setting [Bu], exponential integrability can be easily obtained, see Corollary 6.2.

John's result in Euclidean space is stronger in at least two respects. First, he only assumed f to be (locally) biLipschitz in a bounded, open subset of $\mathbb{R}^n$. In order to avoid further complication in the proofs, we chose to work with globally biLipschitz maps. More important, John deduced the validity of (1.7) and (1.8) with a factor ε on the right-hand side, in place of our nonsharp power of ε. The example at the beginning of Section 7 shows that in $\mathbb{H}$ the power can not be better than $\varepsilon^{1/2}$.

John-type estimates (1.1) and (1.2) in $\mathbb{R}^n$ have been improved in recent literature. Concerning estimate (1.1), we mention the papers [ATV], [Ma], [Ka], [GM]. See also the monograph [R]. For estimate (1.2) see e.g. the papers [Ko], [FJM] and [CFM]. It seems that a similar stability theory for maps in a subriemannian settings is still essentially lacking (with the exception of the qualitative results in [D]). Here we give a first contribution to research in this direction.

Before closing this introduction, we mention that biLipschitz maps are a subclass of quasiconformal. A characterization of biLipschitz maps among quasiconformal has been given by Balogh, Holopainen and Tyson [BHT] by means of some modulus estimates. Geometric function theory in homogeneous groups has been developed by several authors, see Korányi and Reimann [KR1, KR2], Pansu [P], Heinonen and Koskela [HK], Capogna [C1, C2], Capogna and Tang [CT], Capogna and Cowling [CC], Balogh [Ba], just to quote a few.

The article is structured as follows. In §2 with summarize some preliminaries on $\mathbb{H}$ and we prove several lemmata which will be used in subsequent sections. In §3 we prove Theorem A. In §4 we prove Theorem B. In §5 we prove Theorem C and D. In §6 we prove Theorem E and in §7 we discuss some examples. In Appendix A we provide an elementary proof of the known classification theorem of $\mathbb{H}$'s isometries, which corresponds to the case $\varepsilon = 0$ in Theorem D. The proof of this very special case guided us towards the proof of Theorem D and it might help the reader to follow the general structure of the article.

Notation. We write $(x, y, t) \simeq (x + iy; t) = (z; t) \in \mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ to denote points in the Heisenberg group $\mathbb{H}$. Sometimes we use a synthetic notation $P, Q, \dots$ to denote points in $\mathbb{H}$. Clearly $O = (0, 0, 0)$. A map $f : \mathbb{H} \to \mathbb{H}$ will be sometimes split in its coordinate projections as follows: $f(z;t) = (\zeta(z;t); \tau(\zeta;t)) = (\xi(\zeta;t), \eta(\zeta;t), \tau(z;t))$.

Since we will take several times the square root of $\varepsilon > 0$, we fix for brevity the notation $\varepsilon_k = \varepsilon^{1/2^k}$, so that $\varepsilon_{k+1} = \sqrt{\varepsilon_k}$. We denote by C positive absolute constants. The symbol b will denote any real or complex function bounded by an absolute constant, $|b| \le C$. Both C and b may change even in the same formula.

Finally, denote by $|v|$ the Euclidean norm of a vector $v \in \mathbb{R}^n$. Write $d_0(P) = d(0, P)$. Denote spheres by $S(P, r) = \{Q : d(P, Q) = r\}$. $S^+(0, r) = S(0, r) \cap \{t \ge 0\}$. Spheres and balls centered at origin are also denoted by $S_r := S(0, r)$, and $B_r := B(0, r)$.

2 Preliminary facts

The control distance in the Heisenberg group. Let $\mathbb{H} = \mathbb{R}^3$ be the first Heisenberg group with the product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')),$$

for any $(x, y, t), (x', y', t') \in \mathbb{R}^3$. Denote by L_P the left translation $L_P Q := P \cdot Q$, $P, Q \in \mathbb{H}$. Consider on $\mathbb{H}$ the left invariant vector fields $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. The bundle $\mathcal{H}$ spanned by X and Y is called the horizontal bundle. A path $\gamma : [\alpha, \beta] \to \mathbb{H}$

is said to be be a *horizontal curve* if γ is absolutely continuous and there are a, b measurable functions such that $\dot{\gamma}(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$, for a.e. $t \in [\alpha, \beta]$. The *length* of γ is $\text{length}(\gamma) := \int_\alpha^\beta \sqrt{a^2(t) + b^2(t)}dt$. Given $P, Q \in \mathbb{H}$, the *control distance* $d(P, Q)$ is defined as the infimum (actually minimum) among the lengths of horizontal paths connecting P and Q. Later on we will discuss the family of the corresponding geodesics.

The ball of center P and radius $R > 0$ in $\mathbb{H}$ is denoted by $B(P, R) = \{Q \in \mathbb{H} : d(P, Q) < R\}$. The Lebesgue measure $dxdydt$ on $\mathbb{H}$ is, at the same, the bi-invariant Haar measure on $\mathbb{H}$ and, modulo a multiplicative constant, the Hausdorff measure H_d^4 associated with d. Note the exponent 4, which comes from the homogeneous dimension of $\mathbb{H}$. $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{H}$.

The control distance (see [NSW]) locally satisfies the estimates

$$k_1|(z; t) - (z'; t')| \le d((z; t), (z'; t')) \le k_2|(z; t) - (z'; t')|^{1/2}, \tag{2.1}$$

$(z; t), (z'; t') \in K$ where $K \subset \mathbb{H}$ is compact and k_1, k_2 depend on K. More precisely,

$$d((z; t), (z'; t')) \approx |z - z'| + |t - t' - 2\text{Im}\, z\overline{z'}|^{1/2}, \tag{2.2}$$

with global equivalence constants.

A map f from $\mathbb{H}$ to itself is $(1 + \varepsilon)$-*biLipschitz*, $\varepsilon > 0$, if

$$\frac{1}{1+\varepsilon} d(P, Q) \le d(f(P), f(Q)) \le (1 + \varepsilon)d(P, Q), \ P, Q \in \mathbb{H}. \tag{2.3}$$

An isometry is a 1-biLipschitz map from $\mathbb{H}$ to itself.

Isometries and dilations. The left translations $L_P : Q \mapsto P \cdot Q$ are isometries of the Heisenberg group and they preserve the length of a curve. Let $\theta \in \mathbb{R}$. The rotation by an angle of θ around the t-axis, is the map $\mathcal{R}_\theta : (z; t) \mapsto (e^{i\theta} z; t)$. It is known, see [KR1], [C1], [T] and [Ki], that the only isometries of $\mathbb{H}$ are the compositions of rotations, left translations and of the map $J : (z; t) \mapsto (\overline{z}; -t)$. A simple proof of this fact, relying directely on properties of geodesics, is given in the Appendix.

The dilation with parameter $\lambda > 0$ of $\mathbb{H}$ is the map $\delta_\lambda : (z; t) \mapsto (\lambda z; \lambda^2 t)$. The length of a curve is homogeneous of degree 1 with respect to δ_λ, i.e. $\text{length}(\delta_\lambda(\gamma)) = \lambda\, \text{length}(\gamma)$, hence the same is true for the distance function, $d(\delta_\lambda P, \delta_\lambda Q) = \lambda d(P, Q)$.

Pansu calculus. These notions will be used in Section 6. Let $f : \mathbb{H} \to \mathbb{H}$ be a Lipschitz map. The Pansu differential $Df(P)$ of f at $P \in \mathbb{H}$ is the map from $\mathbb{H}$ to $\mathbb{H}$ defined by

$$Df(P)(Q) = \lim_{\sigma \to 0} \delta_{\sigma^{-1}} \big\{ f(P)^{-1} \cdot f(P \cdot \delta_\sigma Q) \big\},$$

where the limit must be uniform in Q belonging to compact sets of $\mathbb{H} \simeq \mathbb{R}^3$. Pansu proved that the differential of a Lipschitz map exists almost everywhere and it is a morphism of the group $(\mathbb{H}, \cdot)$ into itself. It is rather easy to check that that any morphism of $\mathbb{H}$ must have the form $(u, v, w) \mapsto (\alpha u + \beta v, \gamma u + \delta v, (\alpha\delta - \beta\gamma)w)$, for suitable constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Therefore it can be identified with the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and written as $(u, v, w) \mapsto \left(A\binom{u}{v}; \det(A)w\right)$. Given a point P where the differential of f exists and is a group morphism, we denote by $Jf(P)$ its associated 2×2 matrix, so that

$$Df(P)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Jf(P)\binom{u}{v} \\ \det(Jf)(P)w \end{pmatrix}. \tag{2.4}$$

Note that a smooth function need not be differentiable in Pansu sense: the function $f(x, y, t) = (0, 0, y)$ is not differentiable at $(0, 0, 0)$. Moreover the mere existence of the Pansu differential at a point does not ensure that the latter is a morphism of $\mathbb{H}$, as the function $f(x, y, t) = (x, y, 2t)$ shows at the origin.

But, if we know that f is differentiable in Pansu sense at P and Df is a morphism, writing $f(P) = (\xi(P), \eta(P), \tau(P))$, its Pansu Jacobian matrix has the form

$$Jf(P) = \begin{pmatrix} X\xi(P) & Y\xi(P) \\ X\eta(P) & Y\eta(P) \end{pmatrix}. \tag{2.5}$$

Geodesics and balls. We say that a curve $\gamma : I \to \mathbb{H}$, I open interval of $\mathbb{R}$, absolutely continuous in the Euclidean sense is a *geodesic* if for any $t \in I$ there is $J \subset I$ containing t such that for all $\alpha < \beta$, $\alpha, \beta \in J$, $d(\gamma(\alpha), \gamma(\beta)) = \text{length}\left(\gamma|_{[\alpha,\beta]}\right)$. Let $P \in \mathbb{H}$. If $P = (z; t)$ with $z \neq 0$, then there is a unique curve γ joining O and P, such that $\text{length}(\gamma) = d(O, P)$. If $P = (0; t)$, there are infinitely many curves with this property.

The explicit form of geodesics in the Heisenberg group has been calculated by several authors: see e.g. [Gav], [Kor], [Str], [Bel], [Mon]. For each $\phi \in \mathbb{R}$ and $\alpha \in [0, 2\pi]$, we have the unit-speed geodesic from the origin

$$\gamma_{\phi,\alpha}(s) = \begin{cases} x(s) = \sin(\alpha)\frac{1-\cos(\phi s)}{\phi} + \cos(\alpha)\frac{\sin(\phi s)}{\phi}, \\ y(s) = \sin(\alpha)\frac{\sin(\phi s)}{\phi} - \cos(\alpha)\frac{1-\cos(\phi s)}{\phi}, \\ t(s) = 2\frac{\phi s - \sin(\phi s)}{\phi^2}. \end{cases} \tag{2.6}$$

In the limiting case $\phi = 0$, geodesics are straight lines. The geodesic $\gamma_{\phi,\alpha}$ is length-minimizing for s varying over any interval I with $|I| \leq \frac{2\pi}{|\phi|}$. We say that $2\pi/|\phi|$ is the *total lifetime* of the geodesic γ. The geodesics between arbitrary pairs of points can be obtained by left translation. The parameter $|\phi|$ has an intrinsic geometric meaning, because $\frac{2\pi}{|\phi|}$ is the length over which γ is length-minimizing. Hence, $|\phi|$ is invariant under isometries and covariant under dilations. The geodesics for which $\phi \geq 0$ are the ones pointing upward (this means that as s grows, $t(s)$ grows).

¿From the equation of the geodesics, we obtain the equation of the geodesic sphere centered at the origin. We denote it by S_r or $S(0,r)$. It contains all $(z;t)$ in $\mathbb{H}$ s.t.

$$\begin{cases} |z| = |z|(r,\phi) = 2\frac{\sin(\phi r/2)}{\phi} \\ t = t(r,\phi) = 2\frac{\phi r - \sin(\phi r)}{\phi^2} \end{cases} \text{or} \begin{cases} |z|^2 = |z|^2(r,\phi) = \frac{2}{\phi^2}(1-\cos(\phi r)) \\ t = t(r,\phi) = 2\frac{\phi r - \sin(\phi r)}{\phi^2} \end{cases} \tag{2.7}$$

for some $\phi \in [-2\pi/r, 2\pi/r]$. See Figure 1.

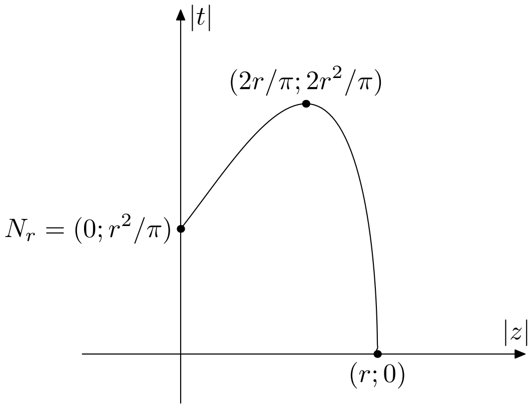

Figure 1: The sphere of radius r.

Observe that $\phi = 0 \Rightarrow (|z|;t) = (r;0)$, while $\phi = \frac{2\pi}{r} \Rightarrow (|z|;t) = (0;\frac{r^2}{\pi})$, so that

$$d((0;t),(0;s)) = \sqrt{\pi(t-s)}. \tag{2.8}$$

The maximum and minimum values for t are reached when $\phi r = \pm\pi$ (note that $t(r,\phi) = r^2 t(1,\phi r)$ and take a derivative of $t(1,\xi)$ with respect to ξ).

The upper half of the unit sphere $S^+(0,1) := S(0,1) \cap \{t > 0\}$ will be also written as a graph of the form $t = u(|z|)$. Although the function u is not explicit it can be easily seen, looking at (2.7), that

$$u(0) = \frac{1}{\pi}, \quad u\Big(\frac{2}{\pi}\Big) = \frac{2}{\pi} \quad \text{and} \quad u'\Big(\frac{2}{\pi}\Big) = 0. \tag{2.9}$$

A more careful look shows that $u'(0) = \frac{2}{\pi}$. The local behaviour of u near 0 is

$$u(|z|) = \frac{1}{\pi} + \frac{2}{\pi}|z|(1 + O(|z|)), \qquad \text{as } |z| \to 0. \tag{2.10}$$

Moreover an easy dilation argument shows that the equation of the upper half sphere of radius $r > 0$ is

$$\frac{t}{r^2} = u\Big(\frac{|z|}{r}\Big), \quad |z| < r. \tag{2.11}$$

We may also write the unit unit sphere as a graph of the $|z|$ variable, locally near $(|z|;t) = (1;0)$. The set $S(0,1) \bigcap \Big(\mathbb{C} \times (-1/\pi, 1/\pi)\Big)$ can be written as $\{(z;t) : |z| = v(t), |t| < 1/\pi\}$, where the function v satisfies, for some $C > 0$,

$$v(|t|) = 1 - Ct^2 + o(t^2), \text{ as } t \to 0. \tag{2.12}$$

A portion of the Heisenberg ball is convex in the Euclidean sense. The following lemma is implicit in [AF], but its proof is elementary, so we will give it here.

Lemma 2.1 *The convex envelope $B_{co}(O,r)$, in the Euclidean sense, of the ball $B(O,r)$ is the solid having as boundary the union of the portion of $S(O,r)$ corresponding to $|\phi r| \leq \pi$ in (2.7) and the two discs $\{(z;t) :\ t = \pm\frac{2}{\pi}r^2,\ |z| \leq \frac{2}{\pi}r\}$.*

Proof. Consider the equation of $S^+(O,r) = \partial B(O,r) \cap \{t \geq 0\}$ in (2.7). As functions of $\phi r \in [0, 2\pi]$, $|z|$ is decreasing while t increases on $[0,\pi]$ and decreases on $[\pi, 2\pi]$. Hence, t increases as $|z|$ varies in $[0, \frac{2}{\pi}r]$ and decreases as $|z|$ varies in $[\frac{2}{\pi}r, r]$. This shows that the disc $\{(z;t) : t = \frac{2}{\pi}r^2,\ |z| \leq \frac{2}{\pi}r\}$ is contained in the convex envelope's boundary.

Let now $P = (z;t)$ be a point on $S^+(O,r)$ such that $r\phi \in [0,\pi]$. The total lifetime of the geodesic γ between P and O is $2\pi/\phi$. Since $r\phi \leq \pi$, this means that the length of the path γ from O to P is less or equal than one half of γ's lifetime. Consider the arc of γ starting at P, containing O and having length π/ϕ, exactly one half of the lifetime of γ. Let A be its other endpoint. Apply now the left translation L mapping A to O, letting $LP = P'$ and $LO = O'$. Consider the ball $LB(A,R) = B(O,R)$, where $R = \pi/\phi$. $P' \in \partial B(O,R)$ is the point in (2.7), with R instead of r, corresponding to $\phi = \pi/R$: one of the point having maximum height. Hence, $B(O,R)$ stays below the set $\{t = \frac{2}{\pi}R^2\}$, its tangent plane at P'. Finally, we show that $B(O',r) = LB(O,r)$ is contained in $B(O,R)$. In fact, if $Q \in B(O',r)$, then

$$d(Q,O) \leq d(Q,O') + d(O',O) < d(P',O') + d(O',O) = d(P',O) = R,$$

where we have used the triangle inequality, the fact that $P' \in \partial B(O',r)$, the alignment of P', O' and O on the same length minimizing geodesic and the fact that $P' \in \partial B(O,R)$. The inequality is strict for any $Q \neq P'$ on the closed ball $\{Q : d(Q,O') \leq d(P',O')\}$. Then, $B(O',r)$ stays on one side of its tangent plane at P'. By translation invariance, the same must hold with P and O replacing P' and O'. $\square$

Cones. The *cone* with center at O and aperture $a \in \mathbb{R}$ is the set

$$\Gamma_a = \{(z;t) \in \mathbb{H} :\ t = a|z|^2\}.$$

We could also consider the degenerate cones $\Gamma_{\pm\infty} = \{(0;t) :\ t \in \mathbb{R}^\pm\} \cup \{O\}$. The cones centered at O in $\mathbb{H}$ are the orbits of the group generated by rotations and dilations centered at O, acting on $\mathbb{H}$, closed by adding the origin. $\Gamma_{P,a} = L_P\Gamma_a$ is the cone with center at P and aperture a.

We introduce now a coordinate for points in $\mathbb{H}$, which will be useful in Section 3.

Definition 2.2 *A point $P = (z;t)$ has coordinate $\lambda \geq 1$ if the geodesic γ starting at O and passing through P has a total lifetime $\lambda \cdot d(O,P)$.*

Let O' be the other endpoint of γ. Then, $d(O,O') = \lambda \cdot d(O,P)$. The definition of λ is dilation invariant, $\lambda(\delta_r P) = \lambda(P)$, $r > 0$. The relation between λ and the parameter ϕ of γ in (2.6) is

$$\lambda = \frac{2\pi}{|\phi| r}.$$

The points P for which $\lambda(P) = \lambda$ is constants lay on the union of two cones, $\Gamma_{\pm a(\lambda)}$. We mention that from (2.6) or (2.7) one deduces that $a(\lambda) \sim \frac{2\pi}{3\lambda}$ as $\lambda \to \infty$ and that $a(\lambda) \sim \frac{1}{\pi(\lambda-1)^2}$ as $\lambda \to 1$.

Lemma 2.3 *The following two facts hold.*

(A) Given $P = (z;t) \in S_1$, $t > 0$, and $R > 1$, if $\lambda(P) < R$, then $\operatorname{dist}(P, S_R) > R - 1$. Moreover, if $\lambda(P) \le R$, then the distance $\operatorname{dist}(P, S_R)$ is realized by the North Pole $N_R := (0; R^2/\pi) \in S_R$, and by it only. Finally, if $\lambda(P) > R$, then $d(P, S_R) = R-1$ and the distance is realized by a point different from the north pole.

(B) There exist $R_0 > 0$ and $C_0 > 0$, large but absolute constants, such that, for all $R > R_0$, $(z;t) \in \mathbb{H}$, and $r \in [\frac{1}{2}, 2]$,

$$\begin{cases} \lambda(z;t) \ge R \\ (z;t) \in S_r^+ \end{cases} \Rightarrow \begin{cases} t \le C_0 R^{-1} \\ 0 \le r - |z| \le C_0 R^{-2}. \end{cases}$$

Proof of (A). Suppose that $\lambda(P) < R$ and let Q be a point realizing the distance $d(Q,P) = d(P, S_R)$. Then $1 + d(P,Q) = d(O,P) + d(P,Q) > d(O,Q) = R$, the strict inequality holding since there is no geodesic passing through O, P, Q. Thus $d(P,Q) = \operatorname{dist}(P, S_R) > R - 1$, as desired.

In order to prove the second statement, call $\mathcal{T}$ the closed, smooth surface obtained by taking the union of all the geodesics joining O and N_R. Observe that $\mathcal{T} - \{N_P\} \subset B(O,R)$ and that, since $\lambda(P) \le R$, P lies in the closure of the open set delimited by $\mathcal{T}$. Let Q be be any point in $S_R - \{N_R\}$. Take a geodesic γ between P and Q and let U be the the last point where γ meets $\mathcal{T}$. Since O, U and N_R lie on the same geodesic, while O, U and Q do not, we have that $d(O,U) + d(U, N_R) = R < d(O,U) + d(U,Q)$, hence $d(U, N_R) < d(U,Q)$. Thus,

$$d(P, N_R) \le d(P,U) + d(U,N_R) < d(P,U) + d(U,Q) = d(P,Q).$$

The second statement in (A) is proved.

The third statement follows easily from the definition of lifetime of a geodesic.

Proof of (B). We prove (B) for $r = 1$. The proof for $r \in [\frac{1}{2}, 2]$ is analogous. Take a geodesic of lifetime R, i.e. with $\phi = \frac{2\pi}{R}$. By (2.7) with $r = 1$, we have

$$\begin{aligned} t(1) &= \frac{R^2}{2\pi^2}\Big(\frac{2\pi}{R} - \sin\Big(\frac{2\pi}{R}\Big)\Big) = \frac{2}{3}\pi R^{-1} + o(R^{-1}), \quad \text{while} \\ |z(1)| &= \frac{R}{\pi}\sin\Big(\frac{\pi}{R}\Big) = 1 - \frac{\pi^2}{6R^2} + o(R^{-2}). \end{aligned}$$

as $R \to +\infty$. This immediately proves the statement (B). $\square$

The properties of the set $\mathcal{T}$ in the proof are related to the fact that all points in the vertical axis are conjugate to O. It is conjectured that $\mathcal{T}$ is the extremal for the isoperimetric inequality in $\mathbb{H}$.

3 Quasigeodesics in the Heisenberg group

A (global) $(1+\varepsilon)-$quasigeodesic in $\mathbb{H}$ is a curve $\gamma : \mathbb{R} \to \mathbb{H}$ such that

$$(1+\varepsilon)^{-1}|s-\sigma| \leq d(\gamma(s), \gamma(\sigma)) \leq (1+\varepsilon)|s-\sigma|, \quad \text{for all} \quad s, \sigma \in \mathbb{R}. \tag{3.1}$$

A quasigeodesic is in particular a Lipschitz map between $\mathbb{R}$ and $\mathbb{H}$ equipped with the control distance. By the differentiability theorem in [P], or by [HK, Proposition 11.4] the path γ is horizontal, i.e. $\dot{\gamma}(s) = a(s)X(\gamma(s)) + b(s)Y(\gamma(s))$ a.e. Denote $\dot{\gamma}_H = (a, b)$.

Theorem 3.1 *There exist $\varepsilon_0 > 0$, $C > 0$ such that for any $(1+\varepsilon)-$quasigeodesic γ, $\varepsilon < \varepsilon_0$, and for any interval I in $\mathbb{R}$*

$$1 - C\sqrt{\varepsilon} \leq \frac{1}{|I|}\left|\int_I \dot{\gamma}_H(s)ds\right| \leq 1+\varepsilon. \tag{3.2}$$

The statement of Theorem 3.1 can be explained as follows. Let $I = [s_1, s_2]$ and $\gamma(s) = (\zeta(s); \tau(s))$, where ζ is nothing but the Euclidean orthogonal projection of γ on the plane $t = 0$. Then (3.2) reads

$$1 - C\sqrt{\varepsilon} \leq \frac{|\zeta(s_2) - \zeta(s_1)|}{s_2 - s_1} \leq 1 + \varepsilon, \tag{3.3}$$

which implies that ζ is a $(1+C\sqrt{\varepsilon})-$biLipschitz map of $\mathbb{R}$ in the plane $t=0$ endowed with the Euclidean metric.

Theorem 3.1 clearly will follow from the following statement.

Proposition 3.2 *There exist $\varepsilon_0 > 0$, $C > 0$ such that for any $(1+\varepsilon)-$quasigeodesic γ, $\varepsilon < \varepsilon_0$, such that $\gamma(0) = 0$, if $\gamma(s) = (\zeta(s); \tau(s))$, then*

$$\begin{cases} 1 - C\sqrt{\varepsilon} \leq \dfrac{|\zeta(s)|}{|s|} \leq 1 + C\varepsilon, \\ |\tau(s)| \leq C\varepsilon^{1/4}s^2, \end{cases} \qquad s \in \mathbb{R}. \tag{3.4}$$

We formulated here a statement for a global geodesic. Actually Theorem 3.1 holds on $I = [0, L]$ for a quasigeodesic $\gamma : \left[-\frac{L}{\sqrt{\varepsilon}}, \frac{L}{\sqrt{\varepsilon}}\right]$ satisfying (3.1) for any s, σ in the mentioned interval.

¿From the previous results we get that given a $(1+\varepsilon)-$quasigeodesic $\gamma = (\zeta;\tau)$ with $\gamma(0) = 0$, we have for some function θ and some bounded function b,

$$(\zeta(1);\tau(1)) = (e^{i\theta}(1+b\sqrt{\varepsilon}); b\varepsilon^{1/4}). \tag{3.5}$$

Therefore, in view of (2.1), the estimate

$$d((\zeta(1);\tau(1)),(\zeta(1);0)) \leq C\varepsilon^{1/8} \tag{3.6}$$

holds. It is actually easy to realize that the distance of $\gamma(1)$ from the plane $t=0$ satisfies the estimate $\mathrm{dist}(\gamma(1),\{t=0\}) \leq Ct^{1/4}$, with an exponent which is better than (3.6).

By dilation invariance, this last remark implies that for $\varepsilon < \varepsilon_0$, a $(1+\varepsilon)-$quasigeodesic γ starting at O is forced to stay outside a cone.

Corollary 3.3 *There exist $\varepsilon_0 > 0$, $C > 0$ such that for any $(1+\varepsilon)-$quasigeodesic γ, $\varepsilon < \varepsilon_0$, γ never intersects the (dilation invariant) set $\{(z;t) : |t| > C\varepsilon^{1/4}|z|^2\}$.*

The proof of Proposition 3.2 is based on the fact that the distance between a point on the metric sphere $S(O,r)$ and the larger, concentric sphere $S(O,R)$ can be *larger* than $R-r$. On a qualitative level, this is a consequence of the fact that all spheres centered at O contain points P which are conjugate to O along a geodesic.

Proof of Proposition 3.2. Introduce the numbers

$$\sigma = 4\sqrt{\varepsilon}, \quad R = \frac{1}{\sqrt{\varepsilon}}. \tag{3.7}$$

Recall that $\gamma(1) \in S_{\eta_1}$ and $\gamma(R) \in S_{\eta_2 R}$, where $\eta_j \in [(1+\varepsilon)^{-1},(1+\varepsilon)]$. Denote by $N_{\eta_2 R} = \left(0;\frac{(\eta_2 R)^2}{\pi}\right)$ the north pole of $S_{\eta_2 R}$. Denote $\gamma(1) := (z;t)$, recall Definition 2.2 of the $\lambda-$coordinate and distinguish the following two cases.

Case A. $\lambda(z,t) \geq \dfrac{1}{2\sqrt{\varepsilon}}$.

Case B. $\lambda(z,t) \leq \dfrac{1}{2\sqrt{\varepsilon}}$.

In *Case A,* the required estimates (3.4) follow immediately from Lemma 2.3, part (B), which provides the estimates $|t| \leq C\sqrt{\varepsilon}$ and $\big||z|-\eta_1\big| \leq \varepsilon$ (even with better powers than the ones in (3.4)).

The discussion of *Case B* is articulated in 3 steps. The following three statements hold for $\varepsilon \leq \varepsilon_0$, where ε_0 and C are absolute constants.

Step B.1: $\quad (z;t) = \gamma(1) \in B\left(N_{\eta_2 R}, R\eta_2 - \eta_1 + \sigma\right) := B_*$.

Step B.2: z satisfies

$$|z| > 1 - C\sqrt{\varepsilon} \tag{3.8}$$

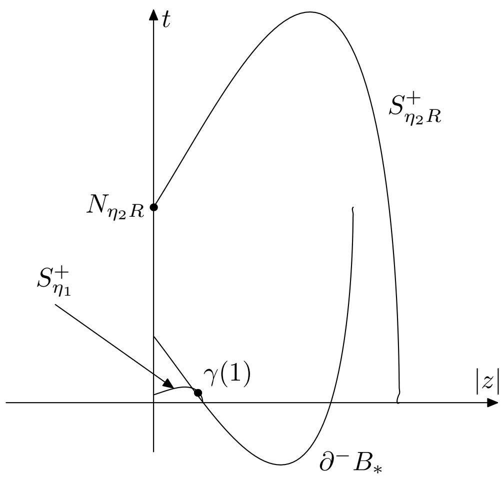

Figure 2: A graphic description of *Case B.*

and, as a consequence, the first line of (3.4) holds.

Step B.3: t satisfies the estimate $|t| \leq C\varepsilon^{1/4}$, so that the second line of (3.4) holds too.

Proof of Step B.1. Assume by contradiction that $\gamma(1) = (z;t) \notin B_*$. Then it would be $d\left(\gamma(1), N_{\eta_2 R}\right) > R\eta_2 - \eta_1 + \sigma$. Since we are assuming $\lambda(z;t) \leq \frac{1}{2\sqrt{\varepsilon}} = \frac{1}{2}R$, if ε is small enough we can assert by Lemma 2.3, part A, that the distance between $\gamma(1)$ and $S_{\eta_2 R}$ is realized by the point $N_{\eta_2 R}$. Then

$$d(\gamma(R), \gamma(1)) \geq \operatorname{dist}(\gamma(1), S_{\eta_2 R}) = d\left(\gamma(1), N_{\eta_2 R}\right) > R\eta_2 - \eta_1 + \sigma,$$

because we are assuming $\gamma(1) \notin B_*$. On the other hand, the quasigeodesic property gives $d(\gamma(R), \gamma(1)) \leq (R-1)(1+\varepsilon)$. Thus we get $R\eta_2 - \eta_1 + \sigma \leq (R-1)(1+\varepsilon)$. Since $\eta_j \in [(1+\varepsilon)^{-1}, 1+\varepsilon]$ and R and σ are prescribed in (3.7), we get

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\eta_2 - \eta_1 + 4\sqrt{\varepsilon} \leq \Big(\frac{1}{\sqrt{\varepsilon}} - 1\Big)(1+\varepsilon) \\
&\qquad \Rightarrow \frac{1}{(1+\varepsilon)\sqrt{\varepsilon}} - (1+\varepsilon) + 4\sqrt{\varepsilon} \leq \Big(\frac{1}{\sqrt{\varepsilon}} - 1\Big)(1+\varepsilon).
\end{aligned}$$

It is now an easy to see that the last inequality can not hold for small ε. This contradiction finishes the proof of *Step B.1.*

Proof of Step B.2. The idea here is to study the shape of the boundary of the ball $B_* = B\left(\left(0; \frac{(\eta_2 R)^2}{\pi}\right), R\eta_2 - \eta_1 + \sigma\right)$, for small ε, with R and σ given by (3.7). Recall that the center of B_* is $N_{\eta_2 R}$. Note that if we would choice $\sigma = 0$, then the boundary of B_* would touch $\partial B(0, \eta_1)$ exactly along a circle. Choosing $\sigma > 0$ we enlarge the ball, of the precised amount prescribed by (3.7). Next computation gives us an information on the intersection between the two balls. In Figure 2, we represented the small upper-half sphere sphere S_{η_1}, the half sphere $\partial^- B_*$, lower boundary of B_* and the largest upper half spere $S_{\eta_2 R}$. The point $(z;t)$ belongs to the very small region intersection of B_* and B_{η_1}.

The equation of the (lower hemi)sphere $S^-(0,\varrho)$ is $t = -\varrho^2 u\Big(\frac{z}{\varrho}\Big)$, $|z| \le \varrho$, where u satisfies (2.10). Taking $\varrho = \eta_2 R - \eta_1 + \sigma$ and translating up of the amount $\frac{(\eta_2 R)^2}{\pi}$, we get that any point belonging to B_* should satisfy

$$\begin{aligned} t &> \frac{(\eta_2 R)^2}{\pi} - (\eta_2 R - \eta_1 + \sigma)^2 u\left(\frac{|z|}{\eta_2 R - \eta_1 + \sigma}\right) \\ &= \frac{(\eta_2 R)^2}{\pi} - (\eta_2 R - \eta_1 + \sigma)^2 \left[\frac{1}{\pi} + \frac{2}{\pi}\frac{|z|}{\eta_2 R - \eta_1 + \sigma}\Big(1 + O\Big(\frac{1}{R}\Big)\Big)\right] \\ &= \frac{1}{\pi}\left[2(\eta_1 - \sigma)\eta_2 R - (\eta_1 - \sigma)^2\right] - \frac{2}{\pi}\big(\eta_2 R - \eta_1 + \sigma\big)\left(1 + O(1/R)\right)|z|. \end{aligned} \tag{3.9}$$

We have used here expansion (2.10), then we made only algebraic simplifications and we wrote $O(1/R)$ instead of $O(|z|/(\eta_2 R - \eta_1 + \sigma))$ (this is correct because we know that $|z| \le 2$ and we may choose $R = \varepsilon^{-1/2}$ large enough).

To prove (3.8), use the fact that $\eta_j \in [(1+\varepsilon)^{-1}, (1+\varepsilon)]$. Thus (3.9) implies

$$\begin{aligned} t > \frac{1}{\pi}&\left[2\big((1+\varepsilon)^{-1} - 4\sqrt{\varepsilon}\big)\frac{1}{(1+\varepsilon)\sqrt{\varepsilon}} - (1+\varepsilon-4\sqrt{\varepsilon})^2\right] \\ &- \frac{2}{\pi}\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}} - \frac{1}{1+\varepsilon} + 4\sqrt{\varepsilon}\Big)\left(1 + O(\sqrt{\varepsilon})\right)|z|. \end{aligned}$$

Some short computations show that, as $\varepsilon \to 0$,

$$2\big((1+\varepsilon)^{-1} - 4\sqrt{\varepsilon}\big)\frac{1}{(1+\varepsilon)\sqrt{\varepsilon}} = \frac{2}{\sqrt{\varepsilon}}(1 - 4\sqrt{\varepsilon} + O(\varepsilon)),$$

and $(1+\varepsilon-4\sqrt{\varepsilon})^2 = 1 + O(\sqrt{\varepsilon})$. Moreover, since $\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}} - \frac{1}{1+\varepsilon} + 4\sqrt{\varepsilon}\Big) = \frac{1}{\sqrt{\varepsilon}}(1 - \sqrt{\varepsilon} + O(\varepsilon))$, we have

$$\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}} - \frac{1}{1+\varepsilon} + 4\sqrt{\varepsilon}\Big)(1 + O(\sqrt{\varepsilon})) \le \frac{1}{\sqrt{\varepsilon}}(1 + C_1\sqrt{\varepsilon}),$$

where the constant C_1 is positive and ε is small enough.

Therefore (3.9) becomes

$$\begin{aligned} t &> \frac{2}{\pi}\frac{1}{\sqrt{\varepsilon}}\left[1 - 4\sqrt{\varepsilon} + O(\varepsilon) - \frac{\sqrt{\varepsilon}}{2}(1 + O(\sqrt{\varepsilon}))\right] - \frac{2}{\pi}\frac{1}{\sqrt{\varepsilon}}\left[1 + C_1\sqrt{\varepsilon} + O(\varepsilon)\right]\ |z| \\ &\ge \frac{2}{\pi}\frac{1}{\sqrt{\varepsilon}}\left[1 - C_2\sqrt{\varepsilon}\right] - \frac{2}{\pi}\frac{1}{\sqrt{\varepsilon}}\left[1 + C_1\sqrt{\varepsilon}\right]\ |z|. \end{aligned}$$

The latter implies $|z| > -C_3\sqrt{\varepsilon}t + 1 - C_4\sqrt{\varepsilon}$, which immediately provides (3.8) (note that $(z;t) \in B_{1+\varepsilon}$ ensures $|t| \le C$). Step B.2. is finished.

Proof of Step B.3. Now we know that $\gamma(1) = (z; t)$ satisfies (3.8). To better understand the situation, note that, as $\varepsilon \to 0$, (3.8) becomes $|z| > 1$. Together with the shape of the unit ball, which is convex near $|z| = 1$, this suggests that the point $(z; t)$ stays near the circle $|z| = 1$, $t = 0$, as ε approaches 0. To make this statement quantitative, recall also that our point $\gamma(1)$ belongs to the set S_{η_1} by the biLipschitz property, $\eta_1 \in [(1+\varepsilon)^{-1}, 1+\varepsilon]$. To prove *Step B.3* write, in a neighborhood of $|z| = \eta_1$ and $t = 0$, the sphere S_{η_1} in the form $|z| = \eta_1\, v(t/\eta_1^2)$, where $|z| = v(t)$ is the equation of S_1 near $|z| = 1$, which satisfies (2.12). Recall that $\eta_1 \in ((1+\varepsilon)^{-1}, (1+\varepsilon))$. Thus we have the estimate

$$|z| < (1+\varepsilon)\, v\Big(\frac{t}{(1+\varepsilon)^2}\Big) \le (1+\varepsilon)\left[1 - C\frac{t^2}{(1+\varepsilon)^4}\right] = 1 + \varepsilon - C\frac{t^2}{(1+\varepsilon)^3}.$$

This, together with (3.8), implies the estimate in the second line of (3.4). The proof of Proposition 3.2 is completed. □

Proof of Theorem 3.1. It suffices to prove it for $I = [0, 1]$. Write again $(z; t) = \gamma(1)$, where γ satisfy the ODE $\dot\gamma = aX(\gamma) + bY(\gamma)$, $\gamma(0) = (0, 0, 0)$ with $a(s)^2 + b(s)^2 \in [(1+\varepsilon)^{-1}, (1+\varepsilon)]$ a.e. Then $z = z(1) = \int_0^1 (a(s), b(s))ds$. Therefore the first estimate of (3.4) (with $s = 1$) implies

$$1 - C\sqrt{\varepsilon} \le \Big|\int_0^1 (a(s), b(s))ds\Big| \le 1 + C\varepsilon.$$

The proof is finished. □

4 BiLipschitz image of a horizontal plane

In this section we prove Theorem B. This requires the understanding of how the different quasigeodesics, $(se^{i\phi}; 0)$ and $(se^{i\theta}; 0)$, $s \in \mathbb{R}$, are transformed by f, as $\theta, \phi \in [0, 2\pi]$. The key point is in the following geometric result.

Proposition 4.1 *Define $\varrho(\theta) = d\big((1; 0), (e^{i\theta}; 0)\big)$, $\theta \in [-\pi, \pi]$. The function ϱ is even, smooth on $]0, \pi[$ and for any $\lambda > 0$ there is $C_\lambda > 0$ such that:*

$$\varrho'(\theta) > C_\lambda \theta^{-1/2} \quad \textit{for all } \theta \in]0, \pi - \lambda[. \tag{4.1}$$

An immediate consequence of (4.1) is the estimate

$$|\varrho(\theta) - \varrho(\phi)| \ge C'_\lambda\big|\theta - |\phi|\big|, \quad \forall\, \theta \in [0, \pi - \lambda], \phi \in [-\pi + \lambda, \pi - \lambda]. \tag{4.2}$$

The fact that ϱ has a maximum at $\theta = \pi$ suggests that estimate (4.2) no longer holds for $\lambda = 0$.

We postpone the proof of the Proposition to the second part of the section.

Proof of Theorem B. By dilation invariance it suffices to prove it for $R = 1$.

Step 1. Proof of estimate (1.6) *for* $|z| = 1$. After a rotation we may assume by Proposition 3.2, that $f(1,0,0) = (1 + b\varepsilon_1; b\varepsilon_2)$. Observe that an information on the position of the point $f(-1;0)$ can be easily extracted. Indeed, write as usual $f(1;0) = (\zeta(1;0), \tau(1;0))$. Formula (3.3) applied on the interval $(-1,1)$ gives

$$|\zeta(-1;0) - (1 + b\varepsilon_1)| = |\zeta(-1;0) - \zeta(1;0)| = \left|\int_{-1}^{1} c\right| = 2 + b\varepsilon_1. \tag{4.3}$$

Here we denoted $c(s) = \frac{d}{ds}(\zeta(s;0))$. Moreover, (3.4) gives $|\zeta(-1;0)| = 1 + b\varepsilon_1$, which means $\zeta(-1;0) = (1 + b\varepsilon_1)e^{i\psi}$, for some ψ. Inserting into (4.3) we get $\psi = \pi + b\varepsilon_2$. Therefore

$$f(-1;0) = \big((1 + b\varepsilon_1)e^{i(\pi+b\varepsilon_2)}; b\varepsilon_2\big). \tag{4.4}$$

In order to prove the required estimate (1.6), we will prove that, after possibly applying the isometry $(x,y,t) \mapsto (x,-y,-t)$, we can write

$$f(e^{i\theta};0) = ((1 + b\varepsilon_1)e^{i(\theta+b\varepsilon_3)}; b\varepsilon_2), \quad \theta \in [-\pi,\pi]. \tag{4.5}$$

Note that, by (2.2), (4.5) implies $d\big(f(e^{i\theta};0),(e^{i\theta};0)\big) \le C\varepsilon_4$, which is (1.6) when $|z| = 1$ and $A = I$.

Note first that we already know that (4.5) holds for $\theta = 0$ and $\theta = \pi$. This follows from the assumption $f(1;0) = (1 + b\varepsilon_1; b\varepsilon_2)$ and from (4.4), which implies

$$\begin{aligned} &d(f(1;0),(1;0)) \le C\varepsilon_3 \qquad \text{and} \\ &d(f(-1;0),(-1;0)) = d\big(((1 + b\varepsilon_1)e^{i(\pi+b\varepsilon_2)}; b\varepsilon_2),(-1;0)\big) \le C\varepsilon_3. \end{aligned} \tag{4.6}$$

We first prove (4.5) for $\theta \in [0, \frac{3}{4}\pi]$. Estimate (4.2) with $\lambda = \pi/4$ will be used. By the results of the previous section we may write $f(e^{i\theta};0) = \big((1+b\varepsilon_1)e^{i\phi(\theta)}; b\varepsilon_2\big)$, where the function $\theta \mapsto \phi(\theta)$ is defined by the last equality and satisfies $\phi(0) = 0$. After possibly applying the isometry $(x,y,t) \mapsto (x,-y,-t)$ we may assume that $\sin(\phi(\pi/2)) > 0$, i.e. the second coordinate of $e^{i\phi(\pi/2)}$ is positive. The biLipschitz property gives

$$d\big(f(e^{i\theta};0), f(1;0)\big) = (1 + b\varepsilon)d\big((e^{i\theta};0),(1;0)\big) = \varrho(\theta)(1 + b\varepsilon),$$

by definition of ϱ. By the triangle inequality and the first line of (4.6), we also have

$$\begin{aligned} d\big(f(e^{i\theta};0), f(1;0)\big) &= d\big(f(e^{i\theta};0),(1;0)\big) + b\varepsilon_3 = d\big(((1 + b\varepsilon_1)e^{i\phi(\theta)}; b\varepsilon_2),(1;0)\big) + b\varepsilon_3 \\ &= d\big((e^{i\phi(\theta)}; b\varepsilon_2),(1;0)\big) + b\varepsilon_3 = \varrho(\phi(\theta)) + b\varepsilon_3. \end{aligned}$$

Therefore we have proved that $\varrho(\phi(\theta)) = \varrho(\theta) + b\varepsilon_3$. Thus, estimate (4.2) gives $\big|\theta - |\phi(\theta)|\big| \le C\varepsilon_3$. Since the function ϕ is continuous and $\phi(\pi/2) > 0$, we can drop the absolute value:

$$\big|\theta - \phi(\theta)\big| \le C\varepsilon_3, \quad \theta \in \Big[0, \frac{3}{4}\pi\Big]. \tag{4.7}$$

The same argument works for $\theta \in [-\frac{3}{4}\pi, 0]$ and estimate (4.7) also holds in the latter interval.

In order to prove (4.5) for the values of θ near π (say $\pi/2 \leq |\theta| \leq \pi$), an analogous argument can be used, changing the "central" point $(1;0)$ with its opposite $(-1;0)$, whose image's position is narrowed down by the second line of (4.6). Step 1 is concluded.

Step 2. Proof of (1.6) *for* $|z| \leq 1$. We now assume (4.5). Also, we may assume that $(z;0) = (r;0)$, $r \in [0,1]$. We know from (2.2) that $d(f(r;0),(r;0)) \approx |\zeta(r;0) - r| + |\tau(r;0) + 2r\mathrm{Im}\,\zeta(r;0)|^{1/2}$, thus we can estimate the two summands separately. We begin with $|\zeta(r;0)-r|$. Let P be the point on the segment between O and $\zeta(1;0)$ such that $|P-O| = r$. Since, by (4.5), the angle with vertex O and rays $O\zeta(1;0)$, $O(1;0)$ has amplitude $b\varepsilon_3$ and we have the relations $|(r;0)-O| = r$, we have $|P-(r;0)| = b\varepsilon_3$. Consider now the case when $r \geq 1/2$. First we estimate the angle α having vertex in O and rays $O\zeta(1;0)$, $O\zeta(r;0)$. We claim that $|\alpha| = b\varepsilon_2$. Indeed, by the Generalized Pythagorean (GPT) Theorem,

$$|\zeta(1;0) - \zeta(r;0)|^2 = |\zeta(1;0)|^2 + |\zeta(r;0)|^2 - 2|\zeta(1;0)|\,|\zeta(r;0)|\cos(\alpha).$$

But now, by Theorem 3.1, we have $|\zeta(1;0)-\zeta(r;0)| = (1-r)(1+b\varepsilon_1)$, $|\zeta(1;0)| = 1+b\varepsilon_1$ and $|\zeta(r;0)| = r(1+b\varepsilon_1)$. Inserting these estimates into the previous equation and taking $r \geq 1/2$ into account, we get $|1-\cos\alpha| \leq C\varepsilon_1$, which ensures $\alpha = b\varepsilon_2$. Again the GPT applied to the triangle $O\zeta(r;0)P$ gives $|P - \zeta(r;0)| = b\varepsilon_2$. Therefore, the triangle inequality in the plane gives $|(r;0) - \zeta(r;0)| = b\varepsilon_3$.

In the case $r \leq 1/2$ we proceed much the same way, considering the triangle $P\zeta(1;0)\zeta(r;0)$ and its angle β having vertex in $\zeta(1;0)$ instead, in order to have the estimate for $|P - \zeta(r;0)|$.

Finally, to estimate the second term, $|\tau(r;0) + 2r\mathrm{Im}\,\zeta(r;0)|^{1/2}$, observe first that $|\tau(r;0)| \leq C\varepsilon_2$, if $r \leq 1$. We also know now that $\zeta(r;0) = r(1+b\varepsilon_1)e^{ib\varepsilon_3}$, so that $|\mathrm{Im}\zeta(r,0)| \leq C\varepsilon_3$. Hence $|\tau(r;0) + 2r\,\mathrm{Im}\zeta(r;0)|^{1/2} \leq C\varepsilon_4$. This ends the proof of Theorem B. $\square$

Proof of Proposition 4.1. Recall first that the geodesic balls with center at the origin are radial in both $|z|$ and $|t|$, i.e. $d_0(z;t) = d_0(|z|;|t|)$. The group law gives

$$\varrho(\theta) = d\big((1;0),(e^{i\theta};0)\big) = d_0\big(2\sin(\theta/2); 4\sin(\theta/2)\cos(\theta/2)\big). \tag{4.8}$$

The equation of the upper half of the sphere S_r is given by (2.7) and has the explicit form $|z|^2 = \frac{2}{\alpha^2}(1-\cos\alpha)r^2$, $t = \frac{2}{\alpha^2}(\alpha - \sin\alpha)r^2$, where $0 < \alpha < 2\pi$. Here we use the coordinate $|z|^2$ instead of $|z|$, in order to make computations easier. It is convenient to introduce the function

$$G(\alpha, r) = \Big(\frac{2}{\alpha^2}(1-\cos\alpha)r^2, \frac{2}{\alpha^2}(\alpha - \sin\alpha)r^2\Big) := r^2 g(\alpha), \quad r>0, \quad 0<\alpha<2\pi \tag{4.9}$$

Moreover, the point $(z;t)$ appearing in the right hand side of (4.8) satifies $|z|^2 = (2\sin(\theta/2))^2$, $t = 4\sin(\theta/2)\cos(\theta/2)$, where $0 < \theta < \pi$. Define the path

$$H(\theta) = \Big(4\sin^2\Big(\frac{\theta}{2}\Big), 4\sin\Big(\frac{\theta}{2}\Big)\cos\Big(\frac{\theta}{2}\Big)\Big) = 2\big(1-\cos\theta, \sin\theta\big), \quad 0 \le \theta \le \pi. \tag{4.10}$$

Observe that $H(\theta)$ describes the upper half of the circle of radius 2 centered at $(2,0)$.

By definition of H and G, $\varrho(\theta)$ is the unique number with the property that

$$G(\alpha, \varrho(\theta)) = H(\theta) \tag{4.11}$$

for some $\alpha \in [0, 2\pi]$. In fact, (4.11) uniquely determines (α, ϱ) as a function of θ. To see this, observe that $G(\alpha_1, \varrho_1) = G(\alpha_2, \varrho_2)$ only when $(\alpha_1, \varrho_1) = (\alpha_2, \varrho_2)$, otherwise we would have either two intersecting metric spheres with the same center and different radii, or a point on a metric sphere whose distance from the center is realized by geodesics with different values of the parameter ϕ.

The proof is articulated as follows.

Step 1. There exist $C_0, C_1 > 0$ such that $C_1\theta^{1/2} \le \varrho(\theta) \le C_0\theta^{1/2}$ for any θ in $[0,\pi]$.
Step 2. ϱ is smooth on $]0,\pi[$ and $\varrho'(\theta)$ is strictly positive for any $\theta \in]0,\pi[$.
Step 3. There exist $\sigma_0 > 0$, $C_0 > 0$ such that $\varrho'(\theta) \ge C_0\theta^{-1/2}$ for any $\theta \le \sigma_0$.

Proof of Step 1. H parametrizes a circle with speed 2 and it is easy to verify that $\frac{4}{\pi}\theta \le |H(\theta)| \le 2\theta$, for any $\theta \in [0,\pi]$. On the other side we have the estimate

$$\varrho(\theta)^2 \inf_{[0,2\pi]} |g| \le |G(\alpha, \varrho(\theta))| \le \varrho(\theta)^2 \sup_{[0,2\pi]} |g|, \quad \forall \alpha \in [0, 2\pi],$$

where g is defined in (4.9). The required inequalities follow from the fact that $0 < \inf_{[0,2\pi]} |g| < \sup_{[0,2\pi]} |g| < +\infty$.

Proof of Step 2. Let $\theta_0 \in]0,\pi[$. Write $\varrho_0 = \varrho(\theta_0)$. Then we have for a suitable $\alpha_0 \in]0,\pi[$ the equation $G(\alpha_0, \varrho_0) = H(\theta_0)$. The idea is to study the equation (4.11) for θ near a value $\theta = \theta_0$. We already know that there is a unique solution $(\alpha(\theta), \varrho(\theta))$ for any θ in $[0,\pi]$. Moreover we will show that the function ϱ satisfies $\varrho'(\theta_0) > 0$.

In order to apply the inverse function theorem to the function G, which is smooth near (α_0, ϱ_0) we compute

$$\begin{aligned} &\partial_r G(\alpha_0, \varrho_0) = \frac{4\varrho_0}{\alpha_0^2}(1-\cos\alpha_0, \alpha_0 - \sin\alpha_0) \quad \text{and} \\ &\partial_\alpha G(\alpha_0, \varrho_0) = -\frac{4\varrho_0^2}{\alpha_0^3}(1-\cos\alpha_0, \alpha_0-\sin\alpha_0) + \frac{2\varrho_0^2}{\alpha_0^2}(\sin\alpha_0, 1-\cos\alpha_0). \end{aligned} \tag{4.12}$$

Then

$$\det[\partial_\alpha G(\alpha_0, \varrho_0) \quad \partial_r G(\alpha_0, \varrho_0)] = -\frac{32\varrho_0^3}{\alpha_0^4}\Big[\sin\Big(\frac{\alpha_0}{2}\Big) - \Big(\frac{\alpha_0}{2}\Big)\cos\Big(\frac{\alpha_0}{2}\Big)\Big]\sin\Big(\frac{\alpha_0}{2}\Big).$$

It is easy to see that the function in the square bracket is strictly positive for any $\alpha_0 \in]0, 2\pi[$. Thus, by the inverse function theorem, equation (4.11) can be solved for any θ near θ_0. Denote by $\alpha(\theta)$, $\varrho(\theta)$ the solutions. The functions $\theta \mapsto \alpha(\theta)$ and $\varrho(\theta)$ are smooth near θ_0.

In order to get the estimate $\varrho'(\theta_0) \neq 0$ differentiate equation (4.11). This gives $\partial_\alpha G(\alpha_0, \varrho_0)\ \alpha'(\theta_0) + \partial_r G(\alpha_0, \theta_0)\varrho'(\theta_0) = H'(\theta_0)$. By Cramer's rule

$$\varrho'(\theta_0) = \frac{\det[\partial_\alpha G(\alpha_0, \varrho_0) \quad H'(\theta_0)]}{\det[\partial_\alpha G(\alpha_0, \varrho_0) \quad \partial_r G(\alpha_0, \varrho_0)]}. \tag{4.13}$$

Observe that the second line of (4.12) can be semplified as follows:

$$\partial_\alpha G(\alpha_0, \varrho_0) = \frac{8\varrho_0^2}{\alpha_0^3}\Big[\Big(\frac{\alpha_0}{2}\Big)\cos\Big(\frac{\alpha_0}{2}\Big) - \sin\Big(\frac{\alpha_0}{2}\Big)\Big]\Big(\sin\Big(\frac{\alpha_0}{2}\Big), -\cos\Big(\frac{\alpha_0}{2}\Big)\Big).$$

Therefore

$$\det[\partial_\alpha G(\alpha_0, r_0) \quad H'(\theta_0)] = \frac{16\varrho_0^2}{\alpha_0^3}\Big[\Big(\frac{\alpha_0}{2}\Big)\cos\Big(\frac{\alpha_0}{2}\Big) - \sin\Big(\frac{\alpha_0}{2}\Big)\Big]\sin\Big(\frac{\alpha_0}{2} + \theta_0\Big),$$

so that

$$\varrho'(\theta_0) = \frac{1}{\varrho(\theta_0)}\ \frac{\alpha_0/2}{\sin(\alpha_0/2)}\sin\Big(\frac{\alpha_0}{2} + \theta_0\Big). \tag{4.14}$$

Now we are in a position to prove that $\varrho' \neq 0$. Assume by contradiction that $\varrho'(\theta_0) = 0$ for some $\theta_0 \in]0, \pi[$. Then it must be $\frac{\alpha_0}{2} + \theta_0 = \pi$. Equation $G(\alpha_0, \varrho_0) = H(\theta_0)$ and the explicit form of G and H immediately furnish

$$\frac{1 - \cos\alpha_0}{\alpha_0 - \sin\alpha_0} = \tan\Big(\frac{\theta_0}{2}\Big) = \tan\Big(\frac{\pi}{2} - \frac{\alpha_0}{4}\Big) = \frac{\cos(\alpha_0/4)}{\sin(\alpha_0/4)} = \frac{\sin(\alpha_0/2)}{1 - \cos(\alpha_0/2)}.$$

Observe that $\sin(\alpha_0/4) \neq 0$ as $\alpha_0 \in]0, 2\pi[$. Now let $\alpha_0/2 = s \in]0, \pi[$. Then

$$\frac{2\sin^2 s}{2s - 2\sin s\cos s} = \frac{\sin s}{1 - \cos s}.$$

But it is easy to see that the latter fails for any $s \in]0, \pi[$. Therefore $\varrho' \neq 0$ and Step 2 is accomplished.

Proof of Step 3. We are interested in studying equation (4.11) near $\theta = 0$. If (4.11) holds, then

$$L(\theta) := \frac{1 - \cos\theta}{\sin\theta} = \frac{1 - \cos\alpha}{\alpha - \sin\alpha} := R(\alpha).$$

Note that $L(\theta) = \tan(\theta/2)$ increases on $[0, \pi]$, from $L(0) = 0$ to $L(\pi-) = +\infty$. One readily verifies that R is a strictly monotone function decreasing from $R(0^+) = +\infty$ to $R(2\pi) = 0$. Hence α is a monotonically decreasing function of θ, $\alpha(0) = 2\pi$, $\alpha(\pi) = 0$. Keeping into account that $\theta = 2\arctan R(\alpha)$, a calculation shows that (i) $\frac{d\alpha}{d\theta} < 0$ for all $\theta \in]0,\pi]$; (ii) $\frac{d\alpha}{d\theta} = -\frac{\pi}{2\pi-\alpha}(1+o(1))$, as $\theta \to 0$ ($\Leftrightarrow$ $\alpha \to 2\pi$); (iii) $\frac{d\alpha}{d\theta} = -\frac{3}{2}(1+o(1))$, as $\theta \to \pi$ ($\Leftrightarrow \alpha \to 0$). As a consequence, there are C_1 and $C_2 > 0$ so that,

$$C_1\sqrt{\theta} \le \frac{2\pi-\alpha}{2} \le C_2\sqrt{\theta}. \tag{4.15}$$

Finally we go back to (4.14). For θ close to 0 (which means $\frac{\alpha}{2}$ close to π), we have the estimates $\sin\frac{\alpha}{2} \ge C_3\Big(\pi - \frac{\alpha}{2}\Big)$ and $\sin\Big(\frac{\alpha}{2}+\theta\Big) \le C_4\Big(\pi - \frac{\alpha}{2} - \theta\Big)$. Therefore,

$$\varrho'(\theta) \ge \frac{1}{\varrho(\theta)}\frac{C}{\pi - \frac{\alpha}{2}}\left(\pi - \frac{\alpha}{2} - \theta\right).$$

The latter, together with (4.15) and the estimate $\varrho(\theta) \simeq \theta^{1/2}$, as $\theta \to 0$, concludes the proof of Proposition 4.1. □

5 Image of points outside the plane $t = 0$

Let f be a biLipschitz map and assume $f(0) = 0$. By Theorem B we know how the plane $t = 0$ transforms: for any $R > 0$ there is a suitable $A \in O(2)$ such that,

$$d((f(z;0),(Az;0)) \le C\varepsilon_4 R, \quad |z| \le R. \tag{5.1}$$

In the current section, in order to study where points outside the plane $\{t = 0\}$ are mapped, we will make a sistematic use of the following family of geodesics $s \mapsto \gamma(s) = (x(s), y(s), t(s))$, where

$$\begin{aligned} x(s) &= q\cos\alpha\Big(1-\cos\left(\frac{s}{q}\right)\Big) - q\sin\alpha\sin\left(\frac{s}{q}\right), \\ y(s) &= q\sin\alpha\Big(1-\cos\left(\frac{s}{q}\right)\Big) + q\cos\alpha\sin\left(\frac{s}{q}\right), \\ t(s) &= 2q^2\Big(\frac{s}{q} - \pi - \sin\left(\frac{s}{q}\right)\Big). \end{aligned} \tag{5.2}$$

The parameter q is positive, while $\alpha \in [0, 2\pi]$. Note that for $0 \le s/q \le 2\pi$, $d(\gamma(s), O) \approx q$. The path γ is a unit speed geodesics with lifetime $2\pi q$. It can be obtained from (2.6) with a translation and by changing ϕ with $1/q$. Moreover

$$\gamma(0) = (0,0,-2q^2\pi), \quad \gamma(\pi q) = (2q\cos\alpha, 2q\sin\alpha, 0), \text{ and } \gamma(2\pi q) = (0,0,2q^2\pi). \tag{5.3}$$

The distance $\mathrm{dist}\big((0;2\pi q^2),\{t=0\}\big)$ is realized by any point of the form $(2qe^{i\theta};0)$ and its value is

$$\mathrm{dist}\big((0;2\pi q^2)),\{t=0\}\big) = d\big((0;2\pi q^2),(2qe^{i\theta};0)\big) = \pi q, \quad \forall\theta \in [0,2\pi].$$

All the points of the circle $(2qe^{i\theta}; 0)$, $\theta \in [0, 2\pi]$ are "projections" of $(0; 2\pi q^2)$ on the plane $\{t = 0\}$. This is the reason why statement 1. in Proposition 5.1 below is false for $s = 0$.

By means of an accurate analysis of the mentioned geodesics, we will obtain the following quantitative result.

Proposition 5.1 *There exist universal constants $\sigma_0 > 0$ and $C_0 > 0$ such that, for any $\sigma < \sigma_0$ the following statement holds. For any $q \in]0, \infty[$ consider the unit speed geodesic γ of total lifetime $2\pi q$ such that $\gamma(0) = (0, 0, -2q^2\pi)$ and $\gamma(\pi q) = (2q, 0, 0) := Q$. Take any number s with*

$$\sigma^{1/8} \leq s/q \leq \pi - \sigma^{1/16} \tag{5.4}$$

and denote $P = \gamma(s)$. Then:

1. The closure of the ball $B(P, d(P, Q))$ touches the plane $t = 0$ only in Q.

2. Let $\Pi_1(z; t) = (z; 0)$ be the vertical projection on $\{t = 0\}$. The enlarged ball $B\big(P, (1 + \sigma)d(P, Q)\big)$ satisfies the following property.

$$\Omega_1 = \Pi_1\Big(B\big(P, (1 + \sigma)d(P, Q)\big) \cap \{t > 0\}\Big) \subset \{z \in \mathbb{C} : |z - 2q| \leq C_0 q\sigma^{1/4}\}. \tag{5.5}$$

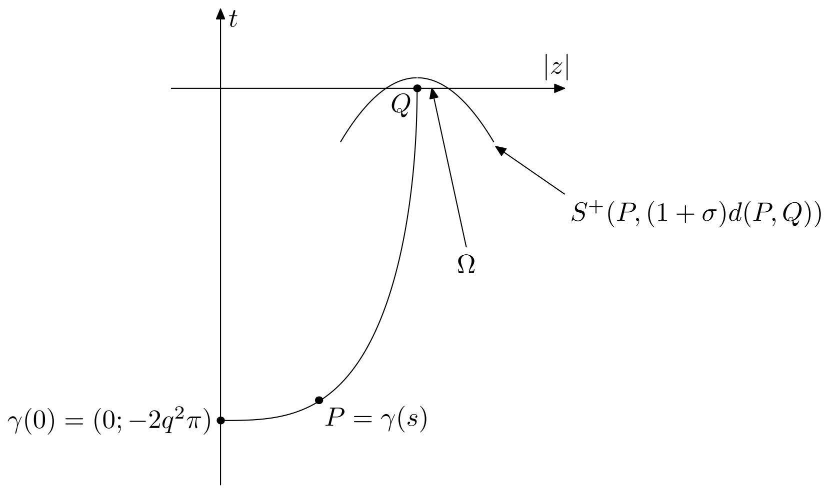

Figure 3: A bidimensional qualitative representation of inclusion (5.5).

Proof of Proposition 5.1. Observe first that statement 1. follows from the proof of Lemma 2.1. In order to prove statement 2., note that, by dilation invariance, we may choose $q = 1$. Letting $\alpha = 0$ and $q = 1$ in the geodesic (5.2) gives

$$P = \gamma(s) = \big(\big(1 - \cos s\big), \sin s, 2(s - \pi - \sin s)\big) .$$

It is $d(P, Q) = \pi - s$. Put $R := (1 + \sigma)d(P, Q) = (1 + \sigma)(\pi - s)$.

Before proving (5.5), we show that $\Omega_1 = \Omega$, where Ω is the seemingly smaller set

$$\Omega := B\big(P, (1+\sigma)d(P,Q)\big) \cap \{t=0\}.$$

The equality $\Omega_1 = \Omega$ holds if and only if the surface $S_{up} = S(P,R) \cap \{t > 0\}$ can be viewed as the graph of a function, $(z;t) \in S_{up}$ iff $t = t(z)$, and this is true if and only if the "equator" E of $S(P,R)$ lies in $\{t \le 0\}$. By equator, we mean the set $E = P \cdot (S(O,R) \cap \{t=0\})$, the set of the points in $S(P,R)$ which have parameter $\phi = 0$ (see Section 2). The equator of $S(O,R)$ has parametrization $(R\cos\alpha, R\sin\alpha, 0)$, $0 \le \alpha < 2\pi$. After a left translation by P, we see that the t-coordinate of a point in E has equation

$$\begin{aligned} t(s,\alpha) &= 2(s - \pi - \sin s) + 2R\sin s\cos\alpha - 2R\sin\alpha(1-\cos s) \\ &= 2(s-\pi-\sin s) + 4R\sin(s/2)\cos(s/2+\alpha) \\ &\le 2(s-\pi-\sin s) + 4R\sin(s/2) = k(s). \end{aligned}$$

We want then to show that $k(s) \le 0$ when $\sigma^{1/8} \le s \le \pi - \sigma^{1/16}$. After changing coordinate to $u = \pi - s$ and replacing R by its explicit expression, the inequality holds if

$$0 < u + \sin u - 2u(1+\sigma)\cos(u/2) = h(u), \text{ if } \sigma^{1/16} \le u \le \pi. \tag{5.6}$$

A Taylor expansion shows that, for $u = \sigma^{1/16}$, (5.6) becomes

$$2\sigma^{1/16} - \frac{1}{4}\sigma^{1/16+1/8}(1+o(1)) < 2\sigma^{1/16} - \frac{1}{6}\sigma^{3/16}(1+o(1)),$$

which is true if $\sigma < \sigma_0$ is small enough. For the other values of u, we take a derivative in (5.6),

$$h'(u) = 2\cos^2(u/2) - 2\cos(u/2) + u\sin(u/2) - b\sigma = g(u) - b\sigma.$$

Observe that $g(\sigma^{1/16}) = 1/4 \cdot \sigma^{1/8}(1+b\sigma)$ and that

$$g'(u) = 2\sin(u/2) - 2\sin(u/2)\cos(u/2) + \frac{u}{2}\cos(u/2) \ge 0 \text{ when } 0 \le u \le \pi.$$

Hence, $h(u) = g(u) - b\sigma \ge g(\sigma^{1/16}) - b\sigma > 0$, if $\sigma^{1/16} \le u \le \pi$. This shows that $k(s) \le 0$, hence that $\Omega = \Omega_1$.

We now return to the proof of (5.5) for $\Omega_1 = \Omega$. The generic point A of the upper half sphere $S^+(0,R)$ has coordinates

$$A(\alpha,\phi) = \left(2\frac{\sin(\phi R/2)}{\phi}\cos\alpha, 2\frac{\sin(\phi R/2)}{\phi}\sin\alpha, 2\frac{\phi R - \sin(\phi R)}{\phi^2}\right), \tag{5.7}$$

where $|\alpha| \le 2\pi$, and $0 \le \phi R \le 2\pi$. Denote by (x,y,t) the coordinates of the point $P \cdot A := (x,y,t) \in S^+(P,R)$. Then

$$\begin{aligned} x &= (1-\cos s) + 2\frac{\sin(\phi R/2)}{\phi}\cos\alpha, \qquad y = \sin s + 2\frac{\sin(\phi R/2)}{\phi}\sin\alpha \\ t &= 2(s-\pi) + \frac{2R}{\phi} - 2\Big\{\sin s + \frac{\sin(\phi R)}{\phi^2}\Big\} + \frac{8}{\phi}\sin\left(\frac{s}{2}\right)\sin\left(\frac{\phi R}{2}\right)\cos\left(\alpha + \frac{s}{2}\right). \end{aligned} \tag{5.8}$$

Note that letting $\alpha = -\frac{s}{2}$, $\phi = 1$ and $\sigma = 0$, i.e. $R = \pi - s$, we have $t = 0$ for all $s \in (0, \pi)$, as expected.

To prove the proposition, take a small σ, fix s satisfying (5.4) and consider the function $t = t(\alpha, \phi)$ defined in the last line of (5.8). We prove the following two statements.

Step 1. For small σ, the $t-$coordinate corresponding to $\alpha = -\frac{s}{2}$ and $\phi = 1$ is positive.

Step 2. For any point of the form

$$\mu(\psi) = (\alpha, \phi) = \left(-\frac{s}{2} + \sigma^{1/4}\cos\psi, 1 + \sigma^{1/4}\sin\psi\right), \tag{5.9}$$

it is $t < 0$ for any $\psi \in [0, 2\pi]$.

Once the described steps are proved, we will show that they ensure the proof of the proposition.

Proof of Step 1. Put $\alpha = -\frac{s}{2}$, $\phi = 1$ and $R = (1+\sigma)(\pi - s)$ into the third equation of (5.8). After some simplifications and a Taylor expansion near $\sigma = 0$, we get

$$\begin{aligned} t &= 2\sigma(\pi - s) - 2\sin s - 2\sin(s - \sigma(\pi - s)) + 8\sin\left(\frac{s}{2}\right)\cos\left(\frac{s}{2} - \frac{\sigma}{2}(\pi - s)\right) \\ &= 2\sigma(\pi - s)\Big(1 + \cos s + 2\sin^2\left(\frac{s}{2}\right)\Big) + o(\sigma(\pi - s)) = 4\sigma(\pi - s) + (\pi - s)o(\sigma), \end{aligned}$$

as $\sigma \to 0$. Then $t > 0$, if σ is smaller than an absolute constant σ_0.

Proof of Step 2. We introduce the more comfortable variables x, β, λ,

$$x = \pi - s, \qquad \phi = 1 + \beta, \qquad \alpha = -s/2 + \lambda.$$

Then $\phi R = (1+\beta)(1+\sigma)x$. Put also $\phi R - x := \delta = (\beta + \sigma + \sigma\beta)x$. Then, starting again from the last line of (5.8), we get

$$\begin{aligned} \frac{t(1+\beta)^2}{2} &= (1+\beta)(\sigma - \beta)x - (1+\beta)^2\sin x - \sin(x + \delta) \\ &\quad + 4(1+\beta)\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2} + \frac{\delta}{2}\right)\cos\lambda. \end{aligned}$$

Expanding the right hand side for σ and δ near 0, we get

$$\begin{aligned} \frac{t(1+\beta)^2}{2} &= O(\sigma) - \beta x - \beta^2 x - \sin x - 2\beta\sin x - \beta^2\sin x \\ &\quad - \sin(x) - \cos(x)\delta + \sin x\frac{\delta^2}{2} + O(\delta^3) \\ &\quad + 4\cos\left(\frac{x}{2}\right)\Big\{(1+\beta)\cos\lambda\Big(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\frac{\delta}{2} - \sin\left(\frac{x}{2}\right)\frac{\delta^2}{8} + O(\delta^3)\Big)\Big\}_1. \end{aligned}$$

Another Taylor expansion at the second order in β, λ, δ, gives

$$\{\dots\}_1 = \sin\left(\frac{x}{2}\right) + \beta\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\frac{\delta}{2} + \cos\left(\frac{x}{2}\right)\beta\frac{\delta}{2} - \sin\left(\frac{x}{2}\right)\frac{\delta^2}{8} - \frac{\lambda^2}{2}\sin\left(\frac{x}{2}\right) + R_1,$$

with the estimate $R_1 \le C_0(|\beta|^3 + |\lambda|^3 + |\delta|^3)$, as soon as $\beta + \lambda + \delta \le \sigma_0$. C_0 and σ_0 are absolute constants. Observe in particular that all these expansions are uniform in the variable $x \in [0, \pi]$. Now recall that $\delta = (\beta + \sigma + \sigma\beta)x = \beta x + O(\sigma)$. Then we can write βx instead of δ and $\beta^2 x^2$ instead of δ^2, making an error of $O(\sigma)$. Then

$$\frac{t(1+\beta)^2}{2} = -\beta^2 \Big\{ \sin x - x\cos x - \frac{x^2}{4}\sin x \Big\} - \lambda^2 \sin x + R_2,$$

with $R_2 \le C_0(\sigma + |\beta|^3 + |\lambda|^3)$, as before, if $\beta + \lambda + \delta \le \sigma_0$.

To conclude the argument, note that the function $x \mapsto \sin x - x\cos x - \frac{x^2}{4}\sin x$ is increasing on $(0, \pi)$ (it has positive derivative). Therefore, since it behaves as $C_1 x^3$, $C_1 > 0$, near 0, it turns out that $\sin x - x\cos x - \frac{x^2}{4}\sin x \ge C_1 x^3$, for any $0 \le x \le \pi$. Then

$$\frac{t(1+\beta)^2}{2} \le -\beta^2 C_1 x^3 - \lambda^2 \sin x + C_2\{\sigma + |\beta|^3 + |\lambda|^3\},$$

provided σ, β, λ are small enough. Now hypothesis (5.4), in term of our variable x, becomes $x \ge \sigma^{1/16}$ and $x \le \pi - \sigma^{1/8}$, so that it is also $x^3 \ge \sigma^{3/16}$ and $\sin x \ge C_3\sigma^{1/8}$. Write $\beta = \sigma^{1/4}\cos\psi$, $\lambda = \sigma^{1/4}\sin\psi$. Then

$$\frac{t(1+\beta)^2}{2} \le -C_1\sigma^{11/16}\cos^2\psi - C_3\sigma^{5/8}\sin^2\psi + C_2\sigma^{3/4}.$$

Now if σ is small enough with respect to the absolute constants C_j's, we have proved that $t < 0$. This ends the proof of *Step 2.*

Now, if the Carnot sphere were convex, then Steps 1 and 2 would give almost immediately inclusion (5.5). This is not the case, but we will show that all of the interesting action takes place in the convex part of the ball's boundary.

We claim that $A(\mu(\psi)) \subset S^+(O, R) \cap \partial B_{co}(O, R)$, where B_{co} denotes the complex envelope, in the Euclidean sense, of $B(O, R)$. By Lemma 2.1, this amounts to showing that $|A_1(\mu(\psi))| \ge \frac{2R}{\pi}$, where we decomposed $A = (A_1; A_2)$. Clearly $A_1(\alpha, \phi) = 2\frac{\sin(\phi R/2)}{\phi}e^{i\alpha}$, where (ϕ, α) are of the form (5.9). This implies $\phi \in [1 - \sigma^{1/4}, 1 + \sigma^{1/4}]$. Moreover, $R = (\pi - s)(1 + \sigma)$ and s satisfies (5.4). Then we have $\phi R/2 \in \,]0, \pi/2[$. Thus, the elementary inequality $\sin(x) \ge \frac{2}{\pi}x$, $x \in [0, \pi/2]$, provides the required lower estimate on $|A_1|$.

Let $\mu^* = (\mu_1^*; \mu_2^*) := P \cdot A(\mu)$. After a translation, the fact that the curve $A(\mu)$ lies in the convex part of $S^+(O, R)$ implies that $\Omega' = B_{co}(P, R) \cap \{t = 0\}$ is a convex set contained inside the curve μ_1^*. *A fortiori,* $\Omega \subset \Omega'$ is contained inside μ_1^*.

To finish the proof, we have to show the inclusion in (5.5). Since Ω is contained inside μ_1^*, it suffices to prove that

$$|\mu_1^*(\psi) - 2| \le C_0\sigma^{1/4}, \ \psi \in [0, 2\pi], \tag{5.10}$$

for some absolute $C_0 > 0$. After a translation, this amounts to showing that $|A_1(\mu(\psi)) - A_1(-\frac{s}{2}, 1)| \le C_0\sigma^{1/4}$. The latter follows from the definition (5.9) of μ and the elementary estimate $|D_{Euc}A_1(\alpha,\phi)| \le CR \le C$ for the Euclidean derivative's norm of A_1. $\square$

5.1 Points on the $t-$axis

Next we analyze the position of points of the form $f(0;t)$. Our result is the following

Theorem 5.2 *There are ε_0 and $C>0$ such that, if $(z,t)\mapsto f(z;t)=(\zeta(z;t);\tau(z;t))$ is $(1+\varepsilon)-$BiLipschitz on $\mathbb{H}$ and satisfies $f(0)=0$, then*

$$|\zeta(0,0,t)| \le C\varepsilon_4|t|^{1/2} \quad \textit{and} \quad \big|t-|\tau(0,0,t)|\big| \le C\varepsilon_4|t|, \quad \forall \quad t\in\mathbb{R}. \tag{5.11}$$

The proof of Theorem 5.2 involves only values of f on the set $K := \{t=0\}\bigcup\{$ $t-$axis$\}$. This does not contain information enough to precise the sign of $\tau(0,0,t)$ (the map $f(x,y,t)=(x,y,-t)$, which is far from being an isometry in $\mathbb{H}$, is an isometry while restricted to K). In order to precise the sign of $\tau(0,0,t)$, we need to take into account values of f outside the set K. This is done in the following proposition.

Proposition 5.3 *If f is $(1+\varepsilon)-$biLipschitz, $\varepsilon\le\varepsilon_0$, $f(0)=0$ and* $\det(A)=+1$ *in* (5.1) *with $R=1$, then the set $\{t<0\}$ goes into the set* below *the image $f(\{t=0\})$.*

The proof of Proposition 5.3 is given at the end of the section. But note that putting Theorem 5.2 together with Proposition 5.3, we get

$$\begin{aligned} &|\zeta(0,0,t)| \le C\varepsilon_4|t|^{1/2} \quad \text{and} \quad \big|t-\tau(0,0,t)\big| \le C\varepsilon_4|t|, \quad \forall \quad t\in\mathbb{R}, \\ &\text{which implies} \quad d(f(0;t),(0;t)) \le C\varepsilon_5|t|^{1/2}, \quad \forall t\in\mathbb{R}, \end{aligned} \tag{5.12}$$

for any $(1+\varepsilon)-$biLipschitz map such that $f(0)=0$, after eventually applying a rigid motion $(x,y,t)\mapsto(x,-y,-t)$.

Proof of Theorem 5.2. Since the statement is dilation invariant, we prove it for the point $(0;2\pi)$. By Theorem B with $R=2$, we may assume that (5.1) holds with $A=I$ for $R=2$. Write $f(0;2\pi)=(\xi,0,\tau)$, $\xi>0$ (the proof which follows can be easily modified to cover the general case $f(0,0,2\pi)=(\xi,\eta,\tau)$). Note that $(0,0,2\pi)=\gamma(2\pi)$, where γ is one among the geodesics in (5.2), with $q=1$, see (5.3). Moreover, we know that the distance of the point $(0;2\pi)$ from the plane $t=0$ is realized by all the points of the form $(2e^{i\theta};0)$ and its value is π.

The idea of the proof is the following. We will choose $\theta=\pi/2$ and $\theta=\frac{3}{2}\pi$. By the biLipschitz property we will show that the point $(\xi,0,\tau)$ should have distance $\pi+o(\text{power of }\varepsilon)$ from both the points $(0,2,0)$ and $(0,-2,0)$. These information, together with the the one about the distance of $(\xi,0,\tau)$ from the origin, $d(f(0;2\pi),(0;0)) = (1+b\varepsilon)d((0;2\pi),(0;0))$, will give a rigid estimate of the position of the point $(\xi,0,\tau)$.

Take $\theta = \frac{\pi}{2}$. By the triangle inequality and the biLipschitz property we have

$$\begin{aligned} d\big((\xi, 0, \tau), (0, 2, 0)\big) &= d\big(f(0; 2\pi), (0, 2, 0)\big) \\ &= d\big(f(0; 2\pi), f(0, 2, 0)\big) + bd\big(f(0, 2, 0), (0, 2, 0)\big) \\ &= \pi(1 + b\varepsilon) + b\varepsilon_4 = \pi + b\varepsilon_4, \end{aligned} \tag{5.13}$$

where we used (5.1), which holds for $R = 2$. The same computation for the opposite point $(0, -2, 0)$ shows that

$$d\big((\xi, 0, \tau), (0, -2, 0)\big) = \pi(1 + b\varepsilon_4). \tag{5.14}$$

Write again (5.13) and (5.14) using the group law and recalling that the distance from the origin satisfies $d((0; 0), (z; t)) := d_0(z; t) = d_0(|z|; |t|)$ for any $(z; t) \in \mathbb{H}$. This gives

$$d_0\left(\sqrt{\xi^2 + 4}, \tau - 2\xi\right) = \pi(1 + b\varepsilon_4) = d_0\left(\sqrt{\xi^2 + 4}, \tau + 2\xi\right). \tag{5.15}$$

Denote now $\varrho = \sqrt{\xi^2 + 4}$, $\tau_- = \tau - 2\xi$ and $\tau_+ = \tau + 2\xi$. The equivalences in (5.15) can be written as

$$d_0(\varrho; \tau_+) = \pi(1 + b\varepsilon_4) \quad \text{and} \quad d_0(\varrho; \tau_-) = \pi(1 + b\varepsilon_4). \tag{5.16}$$

Next we prove that $|\tau| \geq \tau_0$ for some small but absolute constant $\tau_0 > 0$, uniformly for small ε. In order to get this properties we add to (5.15) (or the equivalent (5.16)) the third information given by the biLipschitz property

$$d_0(\xi, 0, \tau) = d_0\big(f(0; 2\pi)\big) = (1 + b\varepsilon)d_0(0; 2\pi) = (1 + b\varepsilon)\,\pi\sqrt{2} \tag{5.17}$$

(see (2.8), for the last equality). By (5.15), since the ball $B(0, r)$ is contained in the cylinder $\{|z| < r\}$ for all $r > 0$, we have $\xi^2 + 4 \leq \pi^2(1 + C\varepsilon_4)$. Since $\pi^2 - 4 < 2\pi$, this gives for small ε the estimate

$$\xi^2 \leq 2\pi, \quad \text{that is} \quad \xi \leq \sqrt{2\pi}. \tag{5.18}$$

It is immediate to see (note that $\pi\sqrt{2} > \sqrt{2\pi}$) that (5.18) and (5.17) together imply

$$|\tau| \geq \tau_0, \tag{5.19}$$

for some absolute constant τ_0.

Now, given $C_0 > 0$ introduce the annulus $A_\varepsilon := B\left(0, \pi(1 + C_0\varepsilon_4)\right) \setminus B\left(0, \pi(1 - C_0\varepsilon_4)\right)$, and let $A_\varepsilon^+ = A_\varepsilon \cap \{t > 0\}$, $A_\varepsilon^- = A_\varepsilon \cap \{t < 0\}$. By (5.16), we may choose an absolute constant $C_0 > 0$ such that $(\varrho; \tau_+), (\varrho; \tau_-) \in A_\varepsilon$. Note that (5.19) says that if ε is small enough, it can not happen that $(\varrho; \tau_+) \in A_\varepsilon^+$ and $(\varrho; \tau_-) \in A_\varepsilon^-$. Thus it should be $(\varrho; \tau_+), (\varrho; \tau_-) \in A_\varepsilon^+$ or $(\varrho; \tau_+), (\varrho; \tau_-) \in A_\varepsilon^-$.

Rescaling Lemma 5.4 (with $\sigma = C_0\varepsilon_4$) from the unit radius to the radius π, we get the estimate $\tau_+ - \tau_- \le C\varepsilon_4$, which by the definition of τ_+, τ_- gives $4\xi \le C\varepsilon_4$, that is $\xi \le C\varepsilon_4$. This ends the proof of the first inequality in (5.11).

In order to prove the second inequality in (5.11), recall that, by (5.16),

$$\left(\sqrt{\xi^2+4}, \tau - 2\xi\right) \in S\left(0, \pi(1+b\varepsilon_4)\right) \quad \text{and } \xi \le C\varepsilon_4.$$

Inserting these information into equation (2.11) of the sphere of radius $\pi(1+b\varepsilon_4)$, we get

$$\left|\frac{\tau + b\varepsilon_4}{\pi^2(1+b\varepsilon_4)^2}\right| = u\left(\frac{\sqrt{4+b\varepsilon_3}}{\pi(1+b\varepsilon_4)}\right). \tag{5.20}$$

Assume first that the quantity inside the absolute value is positive. Recall that, by (2.9), $u(w) = \frac{2}{\pi} + O(w - \frac{2}{\pi})^2$ as $w \to \frac{2}{\pi}$. After a short manipulation (5.20) becomes $\tau = 2\pi + b\varepsilon_4$, which is the required estimate. If instead the number in the absolute value in the left hand side of (5.20) is negative, then we get $\tau = -2\pi + b\varepsilon_4$. Ultimately, the second estimate of (5.11) holds and the proof of the theorem is finished. $\square$

Lemma 5.4 *If $\tau_0 > 0$ is given, then there exists σ_0 and $C_0 > 0$ such that, for any $\sigma < \sigma_0$, given any pair of point $(\varrho; \tau_-), (\varrho; \tau_+) \in B(0, 1+\sigma) \setminus B(0, 1-\sigma)$, $\tau_- \le \tau_+$ with $\tau_+ \ge \tau_0$ (or $\tau_- \le -\tau_0$), then $\tau_+ - \tau_- \le C_0\sigma$.*

Proof. The proof of the lemma can be rather easily obtained by means of the properties of the unit ball described in Section 2.

Proof of Proposition 5.3. We prove that the point $P = (0, 0, -2\pi)$ goes into a point below the image of $\{t = 0\}$. This will be enough to prove the theorem. Moreover we may assume without loss of generality that (5.1) holds with $A = I_2$, the 2×2 identity matrix, and $R = 2$.

In the proof of this proposition, which is qualitative, $o(1)$ always denotes (scalar or vector) functions such that $|o(1)| \le C\varepsilon^k$ for some absolute but unimportant positive constants C and k, which may change at each occurrence.

By (5.11) we know that it should be $\zeta(0, 0, -2\pi) = o(1)$ and $|\tau(0, 0, -2\pi)| = 2\pi + o(1)$. Assume by contradiction that $\tau(0, 0, -2\pi) = +2\pi + o(1)$. Consider the geodesic (5.2) with $\alpha = 0$ and $q = 1$, which has the form

$$\gamma(s) = (1 - \cos s, \sin s, 2(s - \pi - \sin s)). \tag{5.21}$$

Note that $P = \gamma(0)$. Write $Q = \gamma(\pi) = (2, 0, 0)$ and note also that $\gamma(2\pi) = -P = (0, 0, 2\pi)$. Take now the intermediate point $M = (\gamma(\pi/2)) = (1 + i; -(\pi + 2))$. Our assumption $\tau(0; -2\pi) = +2\pi + o(1)$ implies also $\tau(0; -(\pi + 2)) = +\pi + 2 + o(1)$.

Moreover, (5.11) gives also $\zeta(0;-(\pi+2)) = o(1)$. Our knowledge on global quasigeodesics, applied to the quasigeodesic $\lambda(s) := f\big(\frac{1+i}{\sqrt{2}}s; -(\pi+2)\big)$, $s \in \mathbb{R}$ (note that $\lambda(0) = f(0;-(\pi+2))$ and $\lambda(\sqrt{2}) = f(M)$), tells us that it should be

$$f(M) = ((1+i)e^{i\theta}; \pi+2) + o(1), \tag{5.22}$$

for some $\theta \in [0, 2\pi]$. Furthermore, we may also assert that, by (5.1), with the matrix $A = I_2$, $f(Q) = Q + o(1)$. Then, the triangle inequality and the biLipschitz property give

$$d(f(M), Q) = d(f(M), f(Q)) + o(1) = d(M, Q) + o(1) = \frac{\pi}{2} + o(1). \tag{5.23}$$

To get some information on θ, we use Proposition 5.1. Indeed, since both (5.22) and (5.23) hold, it must be

$$f(M) = \gamma\left(\frac{3}{2}\pi\right) + o(1) = (1, -1, \pi+2) + o(1), \tag{5.24}$$

where γ is defined in (5.21). To see (5.24), consider the geodesic γ restricted to $[\pi/2, 2\pi + \pi/2]$. Since $\gamma(\pi) = Q$, $f(M) + o(1) \in \{t = \pi + 2\}$, $\gamma(3/2\pi) \in \{t = -\pi - 2\}$, $f(M) + o(1) \in B(Q, \pi/2 + o(1))$, Proposition 5.1 says that $d(f(M), \gamma(3/2\pi)) = o(1)$, hence (5.24) holds.

Finally use the biLipschitz property $d\big(f(M), f(1,1,0)\big) = d\big(M, (1,1,0)\big) + o(1)$, that is $d\big((1,-1,\pi+2), (1,1,0)\big) = d\big((1,1,-(\pi+2)), (1,1,0)\big) + o(1)$. Translating in term of the distance from the origin d_0,

$$d_0\big(0,-2,\pi-2\big) = d_0\big(0,0,-(\pi+2)\big) + o(1) \quad \Rightarrow d_0\big(2;\pi-2\big) = \sqrt{\pi(\pi+2)} + o(1). \tag{5.25}$$

We have concluded that the point $(2;\pi-2)$ has distance from the origin $\sqrt{\pi(\pi+2)} + o(1)$. The latter number is greater than 4 if $o(1)$ is small enough. But the ball of radius 4 contains the rectangle $\left[0, \frac{8}{\pi}\right] \times \left[0, \frac{16}{\pi}\right]$ (see Section 2). The point $(2;\pi-2)$ is strictly inside the mentioned rectangle. This is in contradiction with (5.25). ☐

5.2 Image of points outside the $t-$axis.

Theorem 5.5 *There are ε_0 and C_0 absolute constant such that, if f is $(1+\varepsilon)-$biLipschitz, $\varepsilon \le \varepsilon_0$, $f(0) = 0$ and $A = I_2$ in Theorem B at the scale $R > 0$, then*

$$d\big(f(z;t), (z;t)\big) \le C\varepsilon_{11}R,$$

for any $(z;t)$ s.t. $[|z|^2 + |t|]^{1/2} \le C_0 R$.

Proof. We prove the statement for $R = 1$. Consider a point $P = (z;t)$, outside the set $\{t = 0\} \bigcup \{t - \text{axis}\}$. To locate quantitatively the position of $f(P)$, we will use

Proposition 5.1. Therefore it is convenient to think the point P in the form $P = \gamma(s)$, where γ is the geodesic in (5.2), for some $q > 0$ and $0 < s < \pi q$. We may also choose $\alpha = 0$. The choice $R = 1$ means $q \approx d_O(z;t) \le C_0$, C_0 absolute.

Roughly speaking, if the point is near $z = 0$ or near the $t-$axis, we will get the required estimate by means of the previous results and of a trivial estimate (triangle inequality). If instead the point is "far" from $\{t = 0\} \bigcup \{t - \text{axis}\}$, then we will invoke Proposition 5.1.

To be more precise, distinguish the following cases:

Case A.1: $0 \le q \le \varepsilon_{11}$ ($(z;t)$ close the origin).
Case A.2: $\varepsilon_{11} \le q \le C_0$ and $0 \le \frac{s}{q} \le \varepsilon_9$ ($(z;t)$ close the $t-$axis).
Case A.3: $\varepsilon_{11} \le q \le C_0$ and $\pi - \varepsilon_{10} \le \frac{s}{q} \le \pi$ ($(z;t)$ close to the plane $\{t = 0\}$).
Case B: $\varepsilon_{11} \le q \le C_0$ and $\varepsilon_9 \le \frac{s}{q} \le \pi - \varepsilon_{10}$ ($(z;t)$ far from $\{t = 0\} \bigcup t - \text{axis}$).

We discuss first the cases A, which all will be treated by the triangle inequality.

Case A.1. Recall that $f(0) = 0$ and $\gamma(s) = (z;t)$. The triangle inequality gives

$$\begin{aligned} d\big(f(z;t),(z;t)\big) &\le d\big(f(z;t),(0;0)\big) + d\big((0;0),(z;t)\big) \\ &\le (2+\varepsilon) d\big((z;t),(0,0)\big) \le C\varepsilon_{11}. \end{aligned}$$

Case A.2. We have $0 \le \frac{s}{q} \le \varepsilon_9$. Then

$$\begin{aligned} d\big(f(z;t),(z;t)\big) &\le d\big(f(z;t),f(0;t)\big) + d\big(f(0;t),(0;t)\big) + d\big((0;t),(z;t)\big) \\ &\le (2+\varepsilon) d\big((z,t),(0;t))\big) + C\varepsilon_5 \le C|z| + C\varepsilon_5, \end{aligned}$$

where we used biLipschitz property, triangle inequality and (5.12). Moreover, since $\frac{s}{q} \le \varepsilon_9$, (5.2) gives, for small ε, $|z| \le C\varepsilon_9$. Therefore the right-hand side can be estimated by $C\varepsilon_9$ which is clearly smaller than $C\varepsilon_{11}$.

Case A.3. Use the triangle inequality and (3.6).

$$\begin{aligned} d\big(f(z;t),(z;t)\big) &\le d\big(f(z;t),f(z;0)\big) + d\big(f(z;0),(z;0)\big) + d\big((z;0),(z;t)\big) \\ &\le (2+\varepsilon) d\big((z;0),(z;t)\big) + C\varepsilon_3 \le (2+\varepsilon)\sqrt{\pi t} + C\varepsilon_3. \end{aligned}$$

Since $\pi \ge \frac{s}{q} \ge \pi - \varepsilon_{10}$, we have, by (5.2), if ε is small enough, $|t| \le \varepsilon_{10}$. Therefore the last line can be estimated by $C\varepsilon_{11}$.

Case B. Write again $(z;t) = \gamma(s)$ and, as usual $f = (\zeta;\tau)$. The key point is to show that, since, by hypothesis, (5.1) holds with $A = I$ and $R = 1$, then

$$\zeta(z;t) = z + b\varepsilon_8 \quad \text{and} \quad \tau(z;t) = t + b\varepsilon_4, \tag{5.26}$$

for any $(z;t)$ such that $|z|^2 + |t| \le C_0$ and Case B holds.

To prove (5.26), recall that we know that $f(0;t) = (b\varepsilon_4; t + b\varepsilon_4)$, by Theorem 5.2. Then, by our result on the image of a horizontal plane, Theorem B,

$$f(0;t)^{-1} \cdot f(z;t) = \big(z(1 + b\varepsilon_1)e^{i\beta}; b\varepsilon_2\big),$$

for some $\beta \in [0, 2\pi]$. Therefore, writing $f(z;t) = f(0;t) \cdot \big(z(1+b\varepsilon_1)e^{i\beta}; b\varepsilon_2\big)$, it turns out that $\tau(z;t) = t + b\varepsilon_4$. This is the second equality in (5.26).

In order to get the first one, we need to locate $\zeta(z;t)$ with the help of Proposition 5.1. This will provide information on the angle β. Recall first that, if $Q = \gamma(\pi q)$, then $d(f(Q), Q) \le C\varepsilon_4$. The triangle inequality, the biLipschitz assumption and the (already proved) second equation of (5.26) give

$$\begin{aligned} d\big((\zeta(z;t);t), Q\big) &\le d\big((\zeta(z;t);t), f(z;t)\big) + d\big(f(z;t), f(Q)\big) + d\big(f(Q), Q\big) \\ &\le C\varepsilon_5 + (1+\varepsilon)d((z;t), Q) + C\varepsilon_4 \le d((z;t), Q) + C\varepsilon_5. \end{aligned} \tag{5.27}$$

To write (5.27) in a more suitable form for the application of Proposition 5.1, recall that (Case B) we are assuming $q \ge \varepsilon_{11}$ and $\pi - \frac{s}{q} \ge \varepsilon_{10}$. Then, since $(z;t) = \gamma(s)$ and $Q = (\gamma(\pi q)$, we have

$$d((z;t), Q) = \pi q - s = q\Big(\pi - \frac{s}{q}\Big) \ge \varepsilon_{11}\varepsilon_{10} \ge \varepsilon_6.$$

Then $\varepsilon_5 = \varepsilon_6^2 \le \varepsilon_6 d((z;t), Q)$. Thus (5.27) takes the more dilation invariant form

$$d\big((\zeta(z,t);t), Q\big) \le d((z;t), Q)\{1 + C\varepsilon_6\}. \tag{5.28}$$

Looking at (5.28) and recalling $\zeta(z;t) = ze^{i\beta} + b\varepsilon_1$, we get by triangle inequality that

$$\begin{aligned} &Q \in B((ze^{i\beta};t), d((z;t), Q)(1 + C\varepsilon_6)), \text{ i.e.} \\ &\mathcal{R}_{-\beta}Q \in B((z;t), d((z;t), Q)(1 + C\varepsilon_6)), \end{aligned}$$

where $\mathcal{R}_{-\beta}(w;s) = (e^{-i\beta}w; s)$. This, together with the assumption of Case B, which provides (5.4) with $\sigma = C\varepsilon_6$, enables us to apply apply (5.5) with $\sigma = C\varepsilon_6$, which reads $|R_{-\beta}Q - Q| \le Cq\varepsilon_6^{1/4} = Cq\varepsilon_8$. Dividing both members by q gives $|e^{i\beta} - 1| \le C\varepsilon_8$, hence $\big|\zeta(z;t) - z\big| \le C\varepsilon_8$. Thus (5.26) is proved.

Finally, write $f(z;t)$ in the form given by (5.26). Then

$$d\big(f(z;t), (z;t)\big) = d_0\left((-z;-t) \cdot \big(z + b\varepsilon_8; t + b\varepsilon_4\big)\right) \le C\varepsilon_9 \le C\varepsilon_{11},$$

as desired. The proof of the theorem is concluded. □

6 Approximation of derivatives

We prove here the following theorem:

Theorem 6.1 *There are universal constants C and ε_0 such that the following statement holds. Let f be $(1+\varepsilon)-$biLipschitz with $\varepsilon \le \varepsilon_0$, $f(0) = 0$. Assume that for some $r > 0, k \in \mathbb{N}$,*

$$d\big(f(z;t), (z;t)\big) \le C\varepsilon_k r, \quad (z;t) \in B(0,r),$$

Denote by Jf the Jacobian matrix of f in the sense of Pansu. Then,

$$\int_{B(0,C^{-1}r)} \|Jf(x,y,t) - I\| dxdydt \leq C\varepsilon_{k+1} r^4. \tag{6.1}$$

In estimate behind, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\|\cdot\|$ denotes the norm of a 2×2 matrix.

By the elementary properties of Pansu derivative, see Section 2, if a given map $f = (\xi, \eta, \tau)$ is Pansu-differentiable at a point P and $Jf = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then we have

$$\frac{d}{ds}\xi\big(e^{sX}(P)\big)\Big|_{s=0} = X\xi(P) = \Big(Df\big(P\big)(1,0,0)\big)\Big)_1 = \alpha(P). \tag{6.2}$$

Here we used the notation $(x,y,t)_1 = x$ for the first component and $s \mapsto e^{sX}(P)$ denotes the integral curve of X emanating from P at $s = 0$. An analogous formula holds for Y.

Proof of Theorem 6.1. It is not restrictive to choose $r = 1$. First we show that if f is $(1+\varepsilon)-$biLipschitz, then the Jacobian matrix Jf satisfies

$$Jf(P)^T Jf(P) = 1 + b\varepsilon, \qquad \text{for a.e. } P \in \mathbb{H}. \tag{6.3}$$

In particular its diagonal components, whose sum plays the role of the divergence that appears in John proof, satisfy the estimate

$$|\alpha(P)| \leq 1 + C\varepsilon, \qquad |\delta(P)| \leq 1 + C\varepsilon, \qquad \text{for a.e. } P. \tag{6.4}$$

To prove (6.3), recall that for a.e. P there are $\alpha, \beta, \gamma, \delta$ s.t.

$$\big(\alpha u+\beta v, \gamma u+\delta v, (\alpha\delta-\beta\gamma)w\big) = Df(P)(u,v,w) = \lim_{\sigma\to 0} \delta_{1/\sigma}\big\{f(P)^{-1}\cdot f\big(P\cdot\delta_\sigma(u,v,w)\big)\big\}$$

Then, letting as usual d_0 to indicate the distance from the origin

$$\begin{aligned} d_0\big(\alpha u + \beta v, \gamma u + \delta v, (\alpha\delta - \beta\gamma)w\big) &= \lim_{\sigma\to 0} d_0\left(\delta_{1/\sigma}\big\{f(P)^{-1}\cdot f\big(P\cdot\delta_\sigma(u,v,w)\big)\big\}\right) \\ &= (1+b\varepsilon)d_0(u,v,w), \end{aligned}$$

by the biLipschitz property. Taking $w = 0$ we get, at any differentiability point P,

$$\left|\begin{pmatrix} u \\ v \end{pmatrix}\right|(1+b\varepsilon) = d_0(u,v,0)(1+b\varepsilon) = d_0\big(\alpha u + \beta v, \gamma u + \delta v, 0\big) = \left|Jf(P)\begin{pmatrix} u \\ v \end{pmatrix}\right|, \tag{6.5}$$

for all $(u,v) \in \mathbb{R}^2$. Equality (6.3) then follows by simple considerations of linear algebra in the Euclidean plane. Finally, (6.4) follows immediately from (6.3).

Our next task is to follow John's argument, starting from a one dimensional estimate and integrating it by Fubini Theorem.

Consider the set $\Omega_1 = \{e^{sX}(0,y,t) : |y|,|t|,|s| \le 1\}$. Here and in the following we denote by $e^{sX}(0,y,t)$ the integral curve of X starting at $s=0$ from the point $(0,y,t)$. The map $(s,y,t) \mapsto e^{sX}(0,y,t) = (s,y,t+2ys)$ is volume preserving. By the change of variable formula and Fubini Theorem, we get the formula

$$\int_{\Omega_1} g = \int_{|y|,|t|\le 1} dydt \int_{-1}^{1} ds\, g(e^{sX}(0,y,t)), \tag{6.6}$$

for any function g. By Pansu Theorem, there is $\Sigma \subset [-1,1]\times[-1,1]$ of full 2-dimensional measure such that, given any $(y,t)\in\Sigma$, the map f is Pansu differentiable at the point $e^{sX}(0,y,t)$ for a.e. $s\in[-1,1]$.

Introduce the function

$$\Phi(x,y,t) = f(x,y,t)^{-1}\cdot(x,y,t) := \big(u(x,y,t), v(x,y,t), w(x,y,t)\big).$$

Note immediately that $u(x,y,t) = x - \xi(x,y,t)$, $v(x,y,t) = y - \eta(x,y,t)$. By the formula for Jf in Section 2, at any point $P=(x,y,t)$ where f is differentiable and Df is a homomorphism we have then

$$J\Phi(P) = I - Jf(P). \tag{6.7}$$

Now, for any $(y,t)\in\Sigma$ and $s\in[-1,1]$, we have

$$\frac{d}{ds}u(e^{sX}(0,y,t)) = \frac{d}{ds}\Big(s - \xi(e^{sX}(0,y,t))\Big) = 1 - \big(Df(e^{sX}(0,y,t))(1,0,0)\big)_1 \ge -\varepsilon.$$

Here we used (6.2) and (6.4). Therefore $\frac{d}{ds}u(e^{sX}(0,y,t)) + \varepsilon \ge 0$.

Now we are ready to integrate: take $(y,t)\in\Sigma$. Then

$$\begin{aligned}\int_{-1}^{1}\left|\frac{d}{ds}u(e^{sX}(0,y,t))\right| ds &\le \int_{-1}^{1}\Big\{\Big|\frac{d}{ds}u(e^{sX}(0,y,t)) + \varepsilon\Big| + \varepsilon\Big\} ds\\ &= \int_{-1}^{1}\Big\{\frac{d}{ds}u(e^{sX}(0,y,t)) + 2\varepsilon\Big\} ds = u(e^{X}(0,y,t)) - u(e^{-X}(0,y,t)) + 4\varepsilon.\end{aligned}$$

By hypothesis $d(f(P),P) \le C\varepsilon_k$. Since $(f(P)^{-1}\circ P)_1 = u(P)$, the first two terms can be estimated by $C\varepsilon_k$. Let α_Φ be the entry in the left upper corner of $J\Phi$. Thus

$$\int_{-1}^{1} |\alpha_\Phi(e^{sX}(0,y,t))| ds \le C\varepsilon_k, \quad \text{for a.e. } (y,t)\in[-1,1]^2$$

Integrating over $[-1,1]^2$, see (6.6), $\int_{\Omega_1}|\alpha_\Phi| \le C\varepsilon_k$. The same argument can be used with the field Y instead of X. Then we get, for a suitable $\Omega\subset\Omega_1$,

$$\int_\Omega |\alpha_\Phi| + |\delta_\Phi| \le C\varepsilon_k.$$

The estimate of the trace of $J\Phi$ is accomplished.

The remaining part of the proof can be concluded exactly as in John's paper (see [J, p. 407]). We just sketch it. Recall that $J\Phi = I - Jf$. Put $U = J\Phi + (J\Phi)^T$ and $V = (J\Phi)^T J\Phi$. Thus $\|V - U\| = \|(Jf)^T Jf - I\| \le C\varepsilon$, by (6.3). Ultimately

$$\int_\Omega \|J\Phi\|^2 = \int_\Omega \|V\| \le \int_\Omega |\mathrm{tr}(V)| \le C \int_\Omega |\mathrm{tr}(U)| \le C\varepsilon_k.$$

To conclude the proof it suffices to apply Hölder inequality $\int_\Omega \|J\Phi\|^2 \le C\big(\int_\Omega \|J\Phi\|^2\big)^{1/2}$. The proof is concluded. $\square$

Theorem 6.1 says that Jf belongs to $BMO(\mathbb{H})$. In the Euclidean case, the John-Nirenberg inequality [JN] allowed John to deduce an local exponential integrability result for the Jacobian of a biLipschitz map. In the context of the Heisenberg group the same conclusion holds, due to the far-reaching generalization of the John-Nirenberg inequality due to Buckley [Bu].

Corollary 6.2 *There exist constants $\varepsilon_0, C > 0$ such that, if $\varepsilon \le \varepsilon_0$, f is $(1+\varepsilon)$-biLipschitz on $\mathbb{H}$ and B is a ball in $\mathbb{H}$, then*

$$\frac{1}{|B|}\int_B \exp\left(\frac{\|Jf(Q) - (Jf)_B\|}{C\varepsilon_{12}}\right) dQ \le 2. \tag{6.8}$$

In (6.8), $(Jf)_B$ is the average of Jf on B and the constant 2 on the right hand side could be replaced by any constant $\lambda > 1$, changing the value of C.

7 Examples

In this section we discuss some examples.

Example 7.1 We show here that, in Theorems C and D, the powers of ε on the right hand side of the inequalities can not be improved to be ε^1, but have to be at least $\varepsilon^{1/2}$. Consider the dilation $\delta_{1+\varepsilon} = f$, which is $(1+\varepsilon)$-biLipschitz in reason of the homogeneity of the distance function. It is obvious that $d(f(0;1),(0;1)) = c\sqrt{\varepsilon}d((0;0),(0;1))$ for some explicit absolute $c > 0$. This shows that the estimate of Theorem C can not hold with ε^1 in the right-hand side.

Next, we show that for any isometry Γ of $\mathbb{H}$ there is a point P such that $d(P,O) \le \sqrt{\pi}$ and $d(f(P),\Gamma(P)) \ge c\sqrt{\varepsilon}$ for some absolute $c > 0$. By the proof of the isometries' classification in the Appendix, any isometry Γ of $\mathbb{H}$ can be written as $\Gamma = L_{(w,s)} \circ \mathcal{R}_\theta \circ J^m$, for some $\theta \in \mathbb{R}$, $(w,s) \in \mathbb{H}$ and $m \in \{0,1\}$. We assume $m = 0$, the other case being similar. We have $\Gamma(0;0) = (w;s)$ and $\Gamma(0;1) = (w;s+1)$, hence $A := d(f(0;0),\Gamma(0;0)) = d_0(w;s)$ and $B := d(f(0;1),(\Gamma(0;1))) = d_0(w,s-(2\varepsilon+\varepsilon^2))$. From the geodesic equations, (2.6) or (2.7), we deduce that, for fixed w, $\max\{A,B\}$ is minimized when $s = \frac{1}{2}(2\varepsilon + \varepsilon^2)$, hence $A = B$. Keeping $s = \frac{1}{2}(2\varepsilon + \varepsilon^2)$ fixed, it is easy to see that A is minimized when $|w| = \sqrt{2s/\pi}$ and $A = \sqrt{\pi s/2} \sim \sqrt{\pi\varepsilon/2}$, for small ε.

Next we briefly describes two procedures for producing contact maps, devised respectively by Korányi and Reimann [KR1] and by Capogna and Tang [CT].

Before introducing the procedures we observe the following standard fact. Let $f : \mathbb{H} \to \mathbb{H}$ be a differentiable map and assume that f is contact. Denote by Jf the invariant components of its Jacobian, see formula (2.5). Then, letting

$$L = \sup_{p\in\mathbb{H}} \|Jf(P)\|, \tag{7.9}$$

the map f il $L-$Lipschitz. Here $\|\cdot\|$ is the operator norm of the matrix, acting on Euclidean $\mathbb{R}^2$. We omit the standard proof.

Example 7.2 (Korányi and Reimann type maps.) In [KR1], Korányi and Reimann show how to produce quasiconformal maps as flows of a suitable vector field.

Consider a function $p : \mathbb{H} \to \mathbb{R}$, say C^2-smooth. Define the vector field

$$v = -\frac{1}{4}(Yp)X + \frac{1}{4}(Xp)Y + pT, \tag{7.10}$$

Denote by $f_s(P)$, $P \in \mathbb{H}$, the solution of the Cauchy problem $\frac{d}{ds} f_s(P) = v(f_s(P))$, $f_0(P) = P$. It is known that such a vector field generates a contact flow. The differential of the map f_s at $P \in \mathbb{H}$ sends $\mathcal{H}_P$ into $\mathcal{H}_{f(P)}$. See [KR1, Theorem 5, p.331].

It is not difficult to check, by a slight modification of the argument in [KR1], that a condition on p which ensures the biLipschitz property is an estimate on the form

$$\sup_{\mathbb{R}^3}\{|X^2p| + |Y^2p| + |XYp| + |YXp|\} = C_0 < \infty. \tag{7.11}$$

In that hypothesis, for all $s \in \mathbb{R}$ the map f_s is biLipschitz and the biLipschitz constant is controlled by $L = e^{C_1|s|}$.

Note that, in order to obtain an estimate on the Lipschitz constant, we assume (7.11), which is slightly stronger than the one in [KR1], which involves only a bound on $\sup |Z^2p|$, $Z = X - iY$.

Example 7.3 (Maps which preserve vertical lines.) We follow [CT], [BHT]. Consider a nonsingular contact map $f : \mathbb{H} \to \mathbb{H}$ of the form $f(z;t) = (\zeta;\tau) = (\zeta(z);\tau(z;t))$. We say that f if *vlp, vertical lines preserving.* If f is vlp, then $Jf(P)$ coincides with the Euclidean Jacobian of w. If f as above is C^2, a standard calculation shows that f is vlp if and only if $\det J\zeta(P) = \gamma$ is constant, $\gamma \neq 0$, and $\tau = B(z) + \gamma t$, where B can be recovered from its Euclidean differential, $1/2dB = (xdy - ydx) - \gamma(udv - vdu)$, where $x + iy = z$ and $u + iv = \zeta$.

By (7.9), we have that the biLipschitz norm of the vlp map $f = (\zeta;\tau)$ equals the *Euclidean* biLipschitz norm of the map $w : \mathbb{R}^2 \to \mathbb{R}^2$. Observe that the dilation $\delta_{1+\varepsilon}$ considered above is vlp. Other interesting examples of vlp maps arise when we consider ζ to be one of the spiral-like plane maps in [GM]. By lifting their maps to $\mathbb{H}$ we obtain, for $k \leq 0$, the vlp maps:

$$S_k(z,t) = (ze^{ik\log|z|}, t - k|z|^2).$$

By the results in [GM], S_k is α-biLipschits, with $\alpha = \frac{|k|+\sqrt{|k|+4}}{2} = 1 + \frac{5}{8}|k| + o(k)$. The image of the plane $\{t = 0\}$ under S_k is the cone $\{(w,s) : \ s = |k||w|^2\}$, hence this class of examples dos not say anything new on the power of ε in Corollary 3.3.

Appendix: the case $\varepsilon = 0$

We show here that any isometry of $\mathbb{H}$ which fixes the origin has the form $f(z;t) = J^m\mathcal{R}_\theta$ for some $\theta \in \mathbb{R}$ and $m \in \{0,1\}$, see Section 2. There are at least two proofs in the literature. The first is by noting that isometries are 1-Quasiconformal maps and that the latter are described in [KR1] and [C1]. The second consists in analyzing the geometry of the group $\mathbb{H}$ at the level of its Lie algebra [Ki].

The simple proof we provide below relies on properties of the distance d alone. We were interested in finding such a proof to have a clue at how to investigate biLipschitz mappings of $\mathbb{H}$ from a purely metric point of view.

Let $f : \mathbb{H} \to \mathbb{H}$ an isometry such that $f(0) = 0$. Consider the geodesic $\gamma(s) = (s,0,0)$, $s \in \mathbb{R}$. This is a globally minimizing geodesic and it is sent by the isometry f to another globally minimizing geodesic, which has the form $f(s,0,0) = (sv;0)$, $v_1^2 + v_2^2 = 1$. Up to a rotation we may choose $v = (1,0)$. In other words we may assume that f is the identity on the line $y = t = 0$. The same argument shows that the image of the plane $\{t = 0\}$ is the plane itself.

Next, look at the set $S = \{(x,y,0) : x^2+y^2 = 1\}$. Since rotations are isometries and $(-1,0;0)$, O, $(1,0;0)$ are collinear, $f(S) = S$, $f(1,0,0) = (1,0,0)$ and $f(-1,0,0) = (-1,0,0)$. Up to a transformation of the form $(x,y,t) \mapsto (x,-y,-t)$ we can assume that $S^+ := S \cap \{y > 0\}$ go onto itself. We claim that f is the identity on S. Suppose there is a point $e^{i\theta_0}$, $\theta_0 \in]0,\pi[$ which is not sent onto itself by f, say $f(e^{i\theta_0}) = e^{i\theta_1}$ with $\theta_1 > \theta_0$ (the opposite case can be treated in the same way). Inductively $e^{i\theta_{n+1}} = f(e^{i\theta_n})$. Since the map is one-to-one, the sequence of angles is strictly increasing: $\theta_0 < \theta_1 < \cdots < \theta_n < \cdots$. Either there is θ_k such that $\theta_k < \pi$ and $\theta_{k+1} > \pi$, but this contradicts the fact that $f(\Sigma^+) \subset \Sigma^+$, or the sequence $(\theta_n)_{n\geq 0}$ is infinite. Since f is an isometry,

$$d(e^{i\theta_1}, e^{i\theta_0}) = d(e^{i\theta_2}, e^{i\theta_1}) = \cdots = d(e^{i\theta_n}, e^{i\theta_{n-1}}) = \cdots$$

Now $\theta_n \to \bar\theta \le \pi$, which implies $d(e^{i\theta_n}, e^{i\theta_{n-1}}) \to 0$, as $n \to \infty$. But this contradicts the fact that $d(e^{i\theta_n}, e^{i\theta_{n-1}})$ is the same for all n.

¿From the above it follows that, up to the application of a map J, the map f, when restricted to the plane $t = 0$, is the identity.

Next consider the plane $t = \bar t$, where $\bar t > 0$. This plane is sent in a left translate of the plane $t = 0$. But the only left translated of $t = 0$ which do not intersects the plane $t = 0$ itself (this would violate the injectivity of f) are of the form t =constant. Therefore $f(\{t = \bar t\}) = \{t = \widetilde{t}\}$, for a suitable $\widetilde{t} \neq 0$.

Now we claim that $f(0, 0, \bar t) = (0, 0, \tilde t)$. This follows from the fact that, since $(0, 0, \bar t)$ is the unique point of the plane $t = \bar t$ which can be connected through geodesics lying in the plane $t = \bar t$ to any other point $(z; \bar t)$, $z \in \mathbb{C}$, the same happens to its image. Thus, $f(0, 0, \bar t) = (0, 0, \tilde t)$. Formula (2.8) tells also that it must be $\tilde t = \pm \bar t$.

Assume first that $\widetilde{t} = \bar t$ (the opposite case will be discussed later). The image of the global geodesic $(s, 0, \bar t)$, $s \in \mathbb{R}$, is of the form $f(s, 0, \bar t) = (sv; \bar t)$, where $|v| = 1$. To recognize that $v = (1, 0)$, observe that

$$d((s, 0, \bar t), (s, 0, 0)) = d(f(s, 0, \bar t), f(s, 0, 0)) = d((vs; \bar t), (s, 0, 0)) \quad \forall\, s \in \mathbb{R}.$$

After a left translation (write as usual d_0 for the distance from the origin),

$$d_0(0, 0, \bar t) = d_0((-1 + v_1)s, v_2 s, \bar t + 2v_2 s^2) \quad \forall\, s \in \mathbb{R}.$$

But this can hold only if $v_2 = 0$ and $v_1 = 1$ (otherwise the point $((-1+v_1)s, v_2 s, \bar t + v_2 s^2)$ would go to infinity, as $s \to \infty$).

Then, if $f(0, 0, \bar t) = (0, 0, \bar t)$, then f is the identity on $t = \bar t$. The same argument can be repeated at any quote $t = t^*$, $t^* \in \mathbb{R}$ and the proof is finished.

Finally, consider the case $f(0, 0, \bar t) = (0, 0, -\bar t)$. Arguing as before, write $f(s, 0, \bar t) = (vs; -\bar t)$, $s \in \mathbb{R}$. Then

$$\begin{aligned} d((s, 0, \bar t), (s, 0, 0)) &= d(f(s, 0, \bar t), f(s, 0, 0)) = d((vs; -\bar t), (s, 0, 0)) \\ &= d_0\big((v_1 - 1)s, v_2 s, -\bar t + 2v_2 s^2\big), \quad s \in \mathbb{R}, \end{aligned}$$

which implies $v_1 = 1$ and $v_2 = 0$. But now we discover that in the latter case, f cannot be an isometry. Without loss of generality suppose $\bar t = 1$ and choose $(z; \bar t) = (1, 0, 1)$. This gives

$$d((1, 0, 1), (1, 1, 0)) = d(f(1, 0, 1), f(1, 1, 0)) = d((1, 0, -1), (1, 1, 0)),$$

which implies $d_0(0, 1, 1) = d_0(0, 1, 3)$, a false equality. □

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