INVARIANCE OF CAPACITY UNDER QUASISYMMETRIC MAPS OF THE CIRCLE: AN EASY PROOF

NICOLA ARCOZZI AND RICHARD ROCHBERG

Abstract. We give a direct, combinatorial proof that the logarithmic capacity is essentially invariant under quasisymmetric maps of the circle.

It is a known fact that logarithmic capacity of closed sets is essentially invariant under quasisymmetric maps of the unit circle. An orientation preserving homeomorphism of the unit disc, identified with an increasing homeomorphism $\varphi : [0,1] \to [0,1]$ via the map $e^{2\pi it} \mapsto e^{2\pi i\varphi(t)+\alpha}$, for some real α, is *quasisymmetric* if

$$\frac{1}{M} \le \frac{\varphi(x+t)-\varphi(x)}{\varphi(x)-\varphi(x-t)} \le M$$

for some fixed $M > 1$. The logarithmic capacity of a closed subset of the unit circle, identified with a subset E of the unit interval, is comparable with its Bessel $(2,1/2)$-capacity $\mathrm{Cap}(E)$. Given a positive, Borel measure μ on $[0,1]$, let

$$\mathcal{E}(\mu) = \int_0^1 K\mu(x)^2 dx = \int_0^1 \left(\int_0^1 \frac{d\mu(y)}{|x-y|^{1/2}}\right)^2 dx$$

be its energy. Here differences are taken modulus integers. Then,

$$\mathrm{Cap}(E) = \inf\left\{\frac{\mu(E)^2}{\mathcal{E}(\mu)} : \ \mathrm{supp}(\mu) \subseteq E\right\}.$$

Theorem 1 (Corollary of a Theorem of Beurling and Ahlfors [BA].)**.** *There is a constant $C(M)$ depending on M only such that*

$$\frac{1}{C(M)} \le \frac{Cap(\varphi^{-1}(E))}{Cap(E)} \le C(M)$$

holds for all closed subsets E of $[0,1]$.

The indirect proof goes as follows. A quasisymmetric map φ of the circle extend to a quasiconformal map f of the unit disc [BA]. Such maps leave set capacities essentially invariant ([A]).

We will show that Theorem 1 can be rephrased as the "stable" version of Benjamini and Peres' result about the equivalence of classical and (a notion of) discrete capacity. An excellent exposition of potential theory in the generality we need in this note (and more) is in [AH], Chapter 2.

Let T be the usual rooted dyadic tree. We can think T to be the tree of the labels (n,k) for the dyadic subintervals $J^0_{(n,k)} = [(k-1)/2^n, k/2^n]$ $(n \ge 0,\ 1 \le k \le 2^n)$. An edge of the

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tree joins labels (n,k) and $(n+1,k')$ if $J^0_{(n+1,k')}\subset J^0_{(n,k)}$. The *root* ?? of T is $o:(0,1)$. If $\alpha=(n,k)$, we set $n=d(\alpha)$ to be the *level* of α. Consider a different collection $\mathcal{J}$ of closed intervals $J_\alpha\subseteq[0,1]$ $(\alpha\in T)$ such that

- $\mathcal{J}_o=[0,1]$;
- $\mathcal{J}_{\alpha_+}\cup\mathcal{J}_{\alpha_-}=\mathcal{J}_\alpha$, where $\alpha_\pm$, the *children* of α, are labels for the two halfs of $\mathcal{J}^0_\alpha$, the dyadic interval labeled by α.

Clearly, $\mathcal{J}$and $\mathcal{J}^0$ have the same combinatorial properties, a fact which will be used below to quickly introduce notation.

A half-infinite geodesic starting at the root is a sequence $\{\zeta_n:\ge 0\}$ in T, where $\zeta_0=o$ and $\zeta_{n+1}\in\{\zeta_n\}$. Given $\zeta\neq\xi$ in ∂T, let $\zeta\wedge\xi\in T$ be the element of T common to both ζ and ξ having the greatest level. The function $\rho(\zeta,\xi):=2^{-d(\xi\wedge\zeta)}$ defines a distance on ∂T.

Given a closed subset subset E of $[0,1]$, we identify it with a subset of T's boundary ∂T in the usual way: if $\Lambda_{\mathcal{J}}:\partial T\to[0,1]$ maps the geodesic $\zeta=\{\zeta_n\}_{n\ge0}\in\partial T$ to the point $\Lambda_{\mathcal{J}}(\zeta)=\cap_{n\ge0}J_{\zeta_n}$. The map $\Lambda_{\mathcal{J}}$ is clearly continuous. In the case of $\mathcal{J}=\mathcal{J}^0$, it is a contraction: $|\Lambda_{\mathcal{J}^0}(\xi)-\Lambda_{\mathcal{J}^0}(\xi)|\le\rho(\zeta,\xi)$.

Theorem 2. *If $\mathcal{J}$ satisfies the quasi-symmetry condition*

$$\frac{1}{M}\le\frac{|J_\alpha|}{|J_\beta|}\le M \tag{1}$$

whenever J_α and J_β are intervals in $\mathcal{J}$ such that $d(\alpha)=d(\beta)$, and J_α and J_β are adjacent **as intervals in** $[0,1]$ *(modulus one), then $Cap_T(\Lambda^{-1}_{\mathcal{J}}(E))\approx Cap(E)$.*

Requiring the condition that J_α and J_β adjacent in $[0,1]$ is more than just asking the same to hold for $\alpha=\gamma_-$ and $\beta=\gamma_+$: the continuous topology of $[0,1]$ plays a rôle here. The tree capacity Cap_T is the (linear) one naturally defined on the (unweighted) dyadic tree T. Given $h:T\to[0,\infty)$ and $\zeta=\{\zeta_n:\ n\ge0\}$ in ∂T, let

$$Ih(\zeta)=\sum_{n=0}^{\infty}h(\zeta_n).$$

For $F\subseteq\partial T$,

$$\mathrm{Cap}_T(F):=\inf\{\|h\|^2_{\ell^2(T)}:h\ge0,\quad Ih\ge1\text{ on }F\}.$$

In [BP] Benjamini and Peres show that $\mathrm{Cap}(E)\approx\mathrm{Cap}_T(\Lambda^{-1}_{\mathcal{J}^0}(E))$. The same result, in a more general setting, is proved in [ARSW], from which we pick the notation we use in this note.

We first show that Theorem 2 implies Theorem 1. Let φ be quasisymmetric and let $J_\alpha=\varphi(J^0_\alpha)$. Then $\mathcal{J}$ satisfies the hypothesis of Theorem 2 and $\Lambda_{\mathcal{J}}=\varphi\circ\Lambda_{\mathcal{J}^0}$. If E is a closed subset of $[0,1]$,

$$\mathrm{Cap}(\varphi^{-1}(E))\approx\mathrm{Cap}_T(\Lambda^{-1}_{\mathcal{J}^0}(\varphi^{-1}(E)))=\mathrm{Cap}_T(\Lambda^{-1}_{\mathcal{J}}(E))\approx\mathrm{Cap}(E).$$

We now prove Theorem 2. Let $\mu\ge0$ be an atomless Borel masure on $[0,1]$ (atoms make energy infinite), which we may identify with a measure μ^* on ∂T: $\mu^*(\Lambda^{-1}_{\mathcal{J}}(J_\alpha)):=\mu(J_\alpha)$ defines μ^* uniquely. Consider the Bessel potential

$$K\mu(x)=\int\frac{d\mu(y)}{|x-y|^{1/2}}.$$

For $x \in [0,1]$, let $P_0(x) = \{\alpha \in T : \ x \in J_\alpha\}$ and let

$$P_1(x) = \cup_{\alpha\in P_0(x)}\{\beta : \ d(\alpha) = d(\beta) \text{ and } d_G(\alpha,\beta) \leq 3\}.$$

and

$$P_2(x) = \cup_{\alpha\in P_0(x)}\{\beta : \ d(\alpha) = d(\beta) \text{ and } 2 \leq d_G(\alpha,\beta) \leq 3\}.$$

Here, d_G is a *graph distance* which takes into account the adjacency relations of the J_α's in $[0,1]$:

$$\min\{|z-y| : \ z \in J^0_\alpha, \ y \in J^0_\beta\} = [d_G(\alpha,\beta) - 1]2^{-d(\alpha)}$$

if $d(\alpha) = d(\beta)$. In words, moving from J_α to J_β across adjacent intervals at the same level, we have to make $d_G(\alpha,\beta)$ steps. The basic properties of $P_2(x)$ we need are:

(i) $\sum_{\alpha\in P_2(x)} \chi_{\mathcal{J}_\alpha} \approx 1$;

(ii) $\min\{|y-x| : \ y \in J_\alpha, \ \alpha \in P_2(x)\} \approx \max\{|y-x| : \ y \in J_\alpha, \ \alpha \in P_2(x)\} \approx |J_\alpha|$.

The proofs are easy and they are left to the reader. Note that the (i) is purely combinatoric, while (ii) relies on the metric hypothesis (1): adjacent intervals have comparable length. Using (i), then (ii), we have

$$\begin{aligned} K\mu(x) &\approx \sum_{\alpha\in P_2(x)} \int_{J_\alpha} \frac{d\mu(y)}{|x-y|^{1/2}} \\ &\approx \int_0^1 \sum_{\alpha\in P_2(x)} |J_\alpha|^{-1/2}\chi(y \in J_\alpha)d\mu(y) \\ &= \sum_{\alpha\in P_2(x)} |J_\alpha|^{-1/2}\mu(J_\alpha). \end{aligned}$$

Things are more easily visualized if we replace P_2 by P_1. Trivially,

$$\sum_{\alpha\in P_2(x)} |J_\alpha|^{-1/2}\mu(J_\alpha) \leq \sum_{\alpha\in P_1(x)} |J_\alpha|^{-1/2}\mu(J_\alpha).$$

In the other direction, observe that, if $\alpha \in P_1(x)$, then

$$J_\alpha = \cup_{\beta\in P_2(x), \ d(\beta)\geq d(\alpha)} J_\beta, \tag{2}$$

and, for each α in $P_1(x)$, there are boundedly many β's in $P_2(x)$ such that $d(\beta) \geq d(\alpha)$ and $J_\beta \cap J_\alpha \neq \emptyset$. The second assertion is obvious. For the first one, given $\alpha \in P_1(\alpha) \setminus P_2(\alpha)$, it is easy to see that J_α can be decomposed as the union of J_β's as in (2).

Then,

$$\begin{aligned} \sum_{\alpha\in P_1(x)} |J_\alpha|^{-1/2}\mu(J_\alpha) &\lesssim \sum_{\alpha\in P_1(x)} \sum_{\beta\in P_2(x), \ d(\beta)\geq d(\alpha)} \mu(J_\beta) \\ &= \sum_{\beta\in P_2(x)} \mu(J_\beta) \sum_{\alpha\in P_1(x), \ d(\alpha)\geq d(\beta)} |J_\alpha|^{-1/2} \\ &\approx \sum_{\beta\in P_2(x)} |J_\beta|^{-1/2}\mu(J_\beta). \end{aligned}$$

The energy becomes

$$\mathcal{E}(\mu) = \int_0^1 K\mu(x)^2 dx$$

$$\begin{aligned}
&\approx \int_0^1 \left(\int_0^1 \sum_{\alpha\in P_1(x)} |J_\alpha|^{-1/2}\chi(y\in J_\alpha)d\mu(y)\right)^2 dx \\
&= \int_0^1 d\mu(y)\int_0^1 d\mu(z)H(y,z),
\end{aligned}$$

where

$$H(y,z) = \sum_{J_\alpha\ni y, J_\beta\ni z} |J_\alpha|^{-1/2}|J_\beta|^{-1/2}\int_0^1 \chi(\alpha,\beta\in P_1(x))dx. \tag{3}$$

The kernel $H(y,z)$ is estimated from above and below by a purely combinatorial quantity. Let $d(y\tilde{\wedge}z) = n\in\mathbb{N}$ be the greatest integer such that there are elements γ_1,γ_2 at level n with $y\in J_{\gamma_1},\ z\in J_{\gamma_2}$ and $J_{\gamma_1}\cap J_{\gamma_2}\neq\emptyset$ (γ_1 and γ_2 either coincide or they label adjacent intervals in $\mathcal{J}$). After considering a handful of geometric series (in which hypothesis (1) is crucial), it is easily verified that

$$H(y,z)\approx d(y\tilde{\wedge}z)+1. \tag{4}$$

Being the quantity $d(y\tilde{\wedge}z)$ purely combinatorial, it is the same for $\mathcal{J}$ and $\mathcal{J}^0$, and this remarks by itself proves Theorem 1, without passing through Theorem 2. However, we take a different route. It is proved in [BP] (and, in greater generality, in [ARSW]) that

$$\int_0^1 d\mu(y)\int_0^1 d\mu(y)[d(y\tilde{\wedge}z)+1]\approx\mathcal{E}_T(\Lambda^*_{\mathcal{J}}\mu),$$

where $\mathcal{E}_T(\cdot)$ is the energy associated with the tree capacity Cap_T:

$$\mathcal{E}_T(\nu) := \int_{\partial T} d\nu(\zeta)\int_{\partial T} d\nu(\xi)[d(\zeta\wedge\xi)+1].$$

Equivalence of energies, $\mathcal{E}(\mu)\approx\mathcal{E}_T(\Lambda^*_{\mathcal{J}}\mu)$, easily implies the equivalence of capacities in Theorem 2.

We are left with the two-sided estimate for $H(y,z)$.

The proof of Theorems 1 and 2 we have presented here is of a combinatorial nature, it is then to be expected that it can be extended to a more general context. We think that a a general statement can be proved for quasisymmetric maps between Ahlfors regular spaces, using some technical tools contained in [ARSW]. We plan to return on this issue in an other article. It would also be interesting to see is there are any relations between this approach to quasisymmetric maps in one dimension and the interesting circle of ideas outlined in [NS].

A downside of the approach we take here (which was first considered, we like to stress, in [BP]) is that we are able to deal with logarithmic capacity of subsets of the unit circle only. Is there a result like Theorem 2, relating logarithmic and tree capacities, that can applied to more general closed subsets of the complex plane? And a last question: can one find estimates for the capacity of *condensers,* rather than *sets,* showing that some features of classical potential theory in the complex plane and of potential theory on trees are essentially equivalent?

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Dipartimento di Matematica, Universita di Bologna, 40127 Bologna, ITALY
E-mail address: arcozzi@dm.unibo.it

Department of Mathematics, Washington University, St. Louis, MO 63130, U.S.A
E-mail address: rr@math.wustl.edu