Limiti con Taylor.

① $$\lim_{x\to 0^+} \frac{\log(\cos(x)) - \cos(x)+1}{x^4}$$ $\boxed{-1/8}$

② $$\lim_{x\to 0^+} e^{\sin x}\cdot \frac{e^{\cos(x)}-e}{x^2}$$ $\boxed{-e/2}$

③ $$\lim_{x\to 0^+} \frac{\frac{1}{1-x+x^2} - e^x + \frac{x^2}{2}}{x^3}$$ $\boxed{-7/6}$

Integrali.

① $$\int_{e^2}^{e^3} \frac{1}{\log^2 x - 1}\cdot \frac{dx}{x}$$ $\boxed{\frac{1}{2}\log\frac{2}{3}}$

② $$\int_0^{\pi/8} \sin(2x)\cdot \operatorname{arctg}(\cos(2x))\,dx$$ $\boxed{\frac{\pi}{8} - \frac{1}{2\sqrt{2}}\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4}\log\left(\frac{4}{3}\right)}$

③ $$\int_{-1}^{0} \frac{e^{3x}}{e^{6x}+2\cdot e^{3x}+2}\,dx$$ $\boxed{\frac{1}{3}\left[\operatorname{arctg}(2) - \operatorname{arctg}(1+e^{-3})\right]}$

Studio di funzione.

Studiare $f : [-4\pi, 4\pi] \longrightarrow \mathbb{R}$,

$$f(x) = \sin(x) + |x| .$$

Limiti

(1) $$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \underset{x\to 0}{o}(x^4)$$

e $$\log(1+y) = y - \frac{y^2}{2} + \underset{y\to 0}{o}(y^2)$$

quindi:

$$\log(\cos(x)) = \log\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right)$$

$$= \log\left(1 + \left[-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right]\right)$$

$$= \left[-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right] - \frac{1}{2}\left[-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right]^2 + o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + o(x^4) = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^4}{8} + o(x^4)$$

e

$$\frac{\log(\cos(x)) - \cos(x) + 1}{x^4} = \frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^4}{8} + o(x^4) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\right) + 1}{x^4}$$

$$= \frac{-x^4/8 + o(x^4)}{x^4} = -\frac{1}{8} + o(1) \underset{x\to 0}{\longrightarrow} -\frac{1}{8}$$

(2) $\underset{x\to 0}{\sin(x) \to 0}$ e $\underset{x\to 0}{\cos(x) \to 1}$, quindi

$$e^{\sin(x)} \cdot \frac{e^{\cos(x)} - e}{x^2} \underset{x\to 0}{\sim} e^0 \cdot \frac{e^{1 - x^2/2 + o(x^2)} - e}{x^2}$$

$$= 1 \cdot \frac{e \cdot e^{-x^2/2 + o(x^2)} - e}{x^2} = 1 \cdot e \cdot \frac{e^{-\frac{x^2}{2} + o(x^2)} - 1}{x^2}$$

$$= e \cdot \frac{1 + \left(-\frac{x^2}{2} + o(x^2)\right) + o(x^2) - 1}{x^2} = e \cdot \frac{-\frac{x^2}{2} + o(x^2)}{x^2} \underset{x\to 0}{\longrightarrow} -\frac{e}{2}$$

(3) $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \underset{y \to 0}{o}(y^3)$, quindi

$$\frac{1}{1-x+x^2} = \frac{1}{1-(x-x^2)} = 1 + (x-x^2) + (x-x^2)^2 + (x-x^2)^3 + \underset{x \to 0}{o}(x^3)$$

$$= 1 + x - x^2 + x^2 - 2x^3 + o(x^3) + x^3 + o(x^3)$$

$$= 1 + x - x^3 + o(x^3) \quad e$$

$$\frac{\frac{1}{1-x+x^2} - e^x + \frac{x^2}{2}}{x^3} = \frac{\left(1 + x - x^3 + o(x^3)\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)\right) + \frac{x^2}{2}}{x^3}$$

$$= \frac{-x^3 - x^3/3! + o(x^3)}{x^3} \xrightarrow[x \to 0]{} -1 - \frac{1}{3!} = -1 - \frac{1}{6} = -\frac{7}{6}$$

Integrali.

(1) $$\int_{e^2}^{e^3} \frac{1}{\log^2 x - 1} \frac{dx}{x} \underset{A}{=} \int_2^3 \frac{dy}{1-y^2} = \int_2^3 \frac{dy}{(1-y)(1+y)} = I$$

(A) Sostituisco $y = \log(x)$,
$dy = \frac{d}{dx} \log(x) \cdot dx = \frac{1}{x} dx$
e $x = e^2 \Leftrightarrow y = \log(e^2) = 2$,
$x = e^3 \Leftrightarrow y = \log(e^3) = 3$

Cerco $A, B \in \mathbb{R}$:

$$\frac{A}{1-y} + \frac{B}{1+y} = \frac{1}{(1-y)(1+y)}$$

$$\frac{A(1+y) + B(1-y)}{(1-y)\cdot(1+y)} = \frac{A + B + (A-B)\cdot y}{(1-y)\cdot(1+y)} ; \begin{cases} A + B = 1 \\ A - B = 0 \end{cases} \quad A = B = \frac{1}{2}$$

Quindi

$$I = \int_2^3 \left(\frac{1}{2} \frac{1}{1-y} + \frac{1}{2} \frac{1}{1+y} \right) dy = \frac{1}{2} \int_2^3 \frac{dy}{1-y} + \frac{1}{2} \int_2^3 \frac{dy}{1+y}$$

$$= \frac{1}{2} \left(-\log|1-y| + \log|1+y| \right)_{y=2}^{y=3} = \frac{1}{2} \left(\log\left|\frac{1+y}{1-y}\right| \right)_{y=2}^{y=3} = \frac{1}{2} \cdot \left(\log 2 - \log \right.$$

$$= \frac{1}{2} \log(2/3)$$

(2) $\displaystyle\int_0^{\pi/8} \sin(2x)\cdot \operatorname{arctg}(\cos(2x))\cdot dx \overset{\uparrow}{=}$

Sostituisco $y = \cos(2x)$,
$dy = -2\cdot\sin(2x)\cdot dx$,
$x = 0 \iff y = \cos(2\cdot 0) = 1$
$x = \pi/8 \iff y = \cos\left(2\cdot\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}}$

$$= -\frac{1}{2}\int_1^{1/\sqrt{2}} \operatorname{arctg}(y)\, dy$$

$$= \frac{1}{2}\cdot\int_{1/\sqrt{2}}^{1} \operatorname{arctg}(y)\, dy \overset{\uparrow}{=} \frac{1}{2}\left\{\left[y\cdot\operatorname{arctg}(y)\right]_{y=1/\sqrt{2}}^{y=1} - \int_{1/\sqrt{2}}^{1}\frac{y}{1+y^2}\,dy\right\}$$

Integro per parti

$$= \frac{1}{2}\cdot\left\{\left(\operatorname{arctg}(1) - \frac{1}{\sqrt{2}}\cdot\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right)\right) - \frac{1}{2}\cdot\left[\log(1+y^2)\right]_{y=1/\sqrt{2}}^{y=1}\right\}$$

$$= \frac{1}{2}\cdot\left\{\operatorname{arctg}(1) - \frac{1}{\sqrt{2}}\cdot\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\log 2 - \log\left(1+\frac{1}{2}\right)\right)\right\}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2\sqrt{2}}\cdot\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4}\log\frac{2}{3/2}$$

$$= \frac{\pi}{8} - \frac{1}{2\sqrt{2}}\cdot\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4}\cdot\log\left(\frac{4}{3}\right)$$

(3) $\displaystyle\int_{-1}^{0}\frac{e^{3x}}{e^{6x}+2\cdot e^{3x}+2}\,dx \overset{\uparrow}{=} \frac{1}{3}\int_{e^{-3}}^{1}\frac{dy}{y^2+2y+2}$,

Sostituisco $y = e^{3x}$; $dy = 3\cdot e^{3x}\,dx$
$x = -1 \iff y = e^{-3}$
$x = 0 \iff y = 1$

usando che $e^{6x} = (e^{3x})^2 = y^2$.

Il discriminante di

$P(y) = y^2 + 2y + 2$ è $\Delta = (-2)^2 - 4\cdot 2 = -4 < 0$.

Scrivo $y^2+2y+2 = y^2+2y+1-1+2 = (y+1)^2+1$

così che

$$\frac{1}{3}\int_{e^{-3}}^{1} \frac{dy}{y^2+2y+2} = \frac{1}{3}\int_{e^{-3}}^{1} \frac{dy}{(1+y)^2+1} =$$

$$= \frac{1}{3}\cdot\Big[\operatorname{arctg}(1+y)\Big]_{y=e^{-3}}^{y=1} = \frac{1}{3}\cdot\Big[\operatorname{arctg}(2)-\operatorname{arctg}(1+e^{-3})\Big].$$

<u>Studio di funzione.</u> Per definizione,

dominio $(f) = [-4\pi, 4\pi]$ e $f \in C([-4\pi, 4\pi])$.

Calcolo $f(-4\pi) = 4\pi = f(4\pi)$.

Se $x \neq 0$, $\exists\, f'(x) = \cos(x) + \operatorname{sgn}(x)$.

$$f'(0^+) = \lim_{x\to 0^+} f'(x) = 1+1 = 2$$
$$\neq$$
$$f'(0^-) = \lim_{x\to 0^-} f'(x) = 1-1 = 0$$

f non è derivabile in $x = 0$.

Ho che
$$f'(x) = \begin{cases} 1+\cos(x) & \text{se } x>0 \\ -1+\cos(x) & \text{se } x<0 \end{cases}$$

quindi: $f'(x) \geq 0 \quad \forall x \in]0, 4\pi]$

e $f'(x) \leq 0 \quad \forall x \in [-4\pi, 0[$

(~~Infatti~~ Possiamo anche vedere che $f'(x) = 0 \iff x = \pi, 3\pi, -2\pi, -4\pi$).

Studio anche la convessità: $x \neq 0 \Rightarrow$

$\Rightarrow f''(x) = -\sin(x) \geq 0 \iff x \in [-3\pi, -2\pi] \cup [-\pi, 0[\cup$
$\cup [\pi, 2\pi] \cup [3\pi, 4\pi]$.

- f è continua in $[-4\pi, 4\pi]$ e derivabile in $[-4\pi, 0[\cup]0, 4\pi] = [-4\pi, 4\pi] \setminus \{0\}$
- f è crescente in $[0, 4\pi]$ e decrescente in $[-4\pi, 0]$.
- f è convessa in $[-3\pi, -2\pi]$ e $[-\pi, 0]$ e $[\pi, 2\pi]$ e $[3\pi, 4\pi]$

  f è concava in $[-4\pi, -3\pi]$ e $[-2\pi, -\pi]$ e $[0, \pi]$ e $[2\pi, 3\pi]$.

Grafico di f (usando $f(0) = 0$):