Long time behavior of Riemannian mean curvature flow of graphs

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In this paper we consider long time behavior of a mean curvature flow of non parametric surface in $\mathbb{R}^n$, with respect to a conformal Riemannian metric. We impose zero boundary value, and we prove that the solution tends to 0 exponentially fast as $t \to \infty$. Its normalization $\frac{u}{\sup u}$ tends to the first eigenfunction of the associated linearized problem.

1. INTRODUCTION

In this paper we consider long time behavior of a mean curvature flow of non parametric surface in $\mathbb{R}^n$, with respect to a conformal Riemannian metric. Let Ω be a convex domain of $\mathbb{R}^n$ and let h a function of class $C^2(\Omega)$, such that $h \geq$costant> 0. If we set $g = h^{(n-1)/2}$, the mean curvature of the graph of u with respect to the conformal metric $(h\delta_{ij})$ is defined as follows

$$H_g(u) = \frac{1}{g^{n/(n-1)}} div \left(\frac{gDu}{\sqrt{1 + |Du|^2}} \right), \quad \varepsilon > 0$$

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and the associated nonlinear evolution equation

$$\begin{cases} u_t = g^{n/(n-1)}\sqrt{1+|Du|^2}\,H_g(u) & in\ \Omega\times[0,+\infty[\\ u=0 & on\ \partial\Omega\times[0,+\infty[\\ u(x,0)=u_0(x) & in\ \Omega, \end{cases} \tag{1}$$

where Du is the spatial gradient, u_t is the partial derivative with respect to t and u_0 is a smooth function.

The flow of a graph by the curvature of its level sets has been intensively used as a model for image recognition (see for example [26], [28], [2], [19] [25] for level set evolution in Euclidean metric [3], [16] and [29] in a Riemann setting). Problem (1) has been proposed in [27] for segmentation of a given image I_0. The process of vision is considered a subjective process, in which the human visual system completes informations that are not present in the given image. An initial function, called the point of view surface $u_0 : \Omega \to \mathbb{R}$, contains the dependence of the observer, and it is evolved by mean curvature flow with respect to a Riemannian metric $h\delta_{ij}$ induced by I_0. The reconstructed image is the normalization $\frac{u}{\sup u}$ of the solution, hence we study the asymptotic behavior of this quotient.

Problem (1) has been studied mainly in the Euclidean case, when $g = 1$. It has been proved in [18] that for general Dirichlet boundary data, a smooth solution does not exists. If Ω is convex, on the contrary it is well known that the solution exists and is defined on all $\Omega \times [0, \infty[$ (see [18], Huisken [14] for time dependent boundary conditions and Ecker and Huisken [6]). Besides the solutions of the parabolic boundary value problem, tend to the solution of the associated elliptic prescribed mean curvature equation. Analogous arguments ensure in our contest that the solution of (1) exists for all instant of time, and asymptotically tends to 0, since the boundary datum is 0. We also refer to by Evans and Spruck in [7]-[10], Giga and Goto [11], Chen, Giga and Goto [4], Huisken [12]-[13], Ilmannen [15], Soner [30], Bellettini and Paolini [1] for motions by curvature of compact surfaces and [31] for asymptotic behavior of graph evolving by curvature of its level set. More recently the normalized solution of the parabolic equation has been studied in order to give a week definition of solution of the prescribed mean curvature equation, in case that the classical one does not exists, see Oliker and Ural'tseva in [21]-[24] for problem (1) with $g = 1$, and non convex set Ω, Lichnewski and Temam [17], Marcellini and Miller [20] for an other flow, always defined in terms of Euclidean curvature.

Here we assume that Ω is convex, because the domain of an image is in general a square, and study the normalized solution of (1), with a technique inspired form the idea in [22]. We show that the normalized solution approach exponentially the first eigenfunction of the operator $L_g(u) = div(gDu)$ with Dirichlet boundary data in Ω.

Precisely if $u : \Omega \times [0, +\infty[\to \mathbb{R}$ denotes the unique solution of (1), our main theorem is

Theorem 1.1. *Let ϕ_1 be the first eigenfunction of the linear operator*

$$L_g u = div(gDu) \tag{2}$$

in Ω with Dirichlet boundary data u_0, and let $\lambda_1(> 0)$ be the corresponding eigenvalue. Then, there exist constants $c, c_1 > 0$ and $\nu > 0$ depending on u_0 such that

$$\sup_{x\in\Omega} |\exp(\lambda_1 t)u(x,t) - c\phi_1(x)| \leq c_1 \exp(-\nu t) \tag{3}$$

for sufficiently large t.

Remark 1. 1. The same result is also true with the same proof if the conformal metric g is substituted with any other Riemannian metric h_{ij}, induced by an image I_0. This means that h_{ij} is direct sum of a $n-1 \times n-1$ and a 1×1 matrix.

The paper is organized as follows: in section 2 we consider the linearized problem and we study the asymptotic behaviour of the solutions, by means of a Moser technique. In section 3, after proving that the solution of problem (1) exists for every $t > 0$, we give some asymptotic estimates for its gradient and we prove Theorem 1.1.

2. THE LINEARIZED PROBLEM: ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

In this section we prove some Moser's a priori estimates for smooth solutions of the linearized problem

$$\begin{cases} z_t - div(gDz) \leq |f| & in\ \Omega \times [t_0, T] \\ z \leq k^* & on\ \partial\Omega \times [t_0, T]. \end{cases} \tag{4}$$

In particular, if f decays exponentially in L^2 or in L^∞ norm also the solution is proved to decay in the same norm.
In order to apply this technique, we recall the usual definition of dyadic balls: let $x_0 \in \Omega$, $t_0, \sigma \geq 0$ and T such that $t_0 + \sigma < T$. Let $B(\rho)$ be the ball of radius ϱ with center x_0. Denote by $G(\varrho, \sigma)$ the set of nonnegative functions ξ such that

$$\xi(x,t) \equiv \omega(x)\Xi(t),\ \omega \in Lip(\Omega \cap B(\varrho)),\ \ \Xi \in Lip[t_0, T];$$

$$\omega = 1 \text{ on } B(\varrho/2) \text{ and } supp\, \omega \subset \overline{B}(\varrho);$$

$$\Xi = 1 \text{ on } [t_0 + \sigma, T] \text{ and } \Xi = 0 \text{ for } t = t_0.$$

LEMMA 2.1. *If z is a smooth solution of*

$$\begin{cases} z_t - div(gDz) \le |f| & in\ \Omega \times [t_0, T] \\ z \le k^* & on\ \partial\Omega \times [t_0, T], \end{cases} \tag{5}$$

for some positive constant k^*, then for every $k \ge k^*$ and for every $\xi \in G(\varrho, \sigma)$ we have

$$\sup_{[t_0,T]} \int_{A_k} \frac{1}{2}(z-k)_+ \xi^2 dx + \int_{t_0}^T \int_{A_k} g|Dz|^2 \xi^2 dxdt \le \tag{6}$$

$$\le \int_{t_0}^T \int_{A_k} (z-k)^2 (g|D\xi|^2 + \xi^2 + |\xi\xi_t|) dxdt + \int_{t_0}^T \int_{A_k} f^2 \xi^2 dxdt$$

where

$$A_k(t) = \{x \in B(\varrho) \cap \Omega \mid z(x,t) > k\}$$

and $G(\varrho, \sigma)$ is the set of nonnegative functions defined above.

Proof Let $t_1 \in [t_0 + \sigma, T]$. Multiplying the first inequality in (5) by $(z-k)_+ \xi^2$, integrating the result and using the fact that $z - k \le 0$ on $\partial\Omega$, we obtain

$$\int_{t_0}^{t_1} \int_\Omega \partial_t((z-k)_+^2) \frac{\xi^2}{2} dxdt - \int_{t_0}^{t_1} \int_\Omega div(gD(z-k)_+)(z-k)_+ \xi^2 dxdt \le$$

$$\le \int_{t_0}^{t_1} \int_\Omega |f|(z-k)_+ \xi^2 dxdt.$$

This expression can be treated in a standard way to obtain the desired inequality, choosing $t_1 = T$.

LEMMA 2.2. *Let z be a smooth solution of (5) and $\sigma > 0$ such that $t_0 + \sigma < T$. Then for every $(x,t) \in B(\varrho/2) \times [t_0 + \sigma, T]$ we have*

$$z(x,t) \le 2 \max\{k^*, \sup_{\widetilde{Q}} |f|, c||(z-k^*)_+||_{L^2(\widetilde{Q})}\} \tag{7}$$

where $\widetilde{Q} = (\overline{B}(\varrho) \cap \overline{\Omega}) \times [t_0, T]$ and

$$c = \left(\frac{c_1}{\rho^2 \sigma} \right)^{\frac{n}{4} + \frac{1}{2}},$$

where c_1 is a positive constant independent of ρ and σ.
Moreover, if $z \geq 0$, $f \equiv 0$ and p such that $0 < p < 2$ then

$$z^p(x,t) \leq 2\max\{(k^*)^p, c||(z-k^*)_+||_{L^p(\widetilde{Q})}\} \tag{8}$$

in $B(\varrho/2) \times [t_0+\sigma, T]$.

Proof The proof of estimate (7) can be carried out as the analogous assertion in Lemma 4.1 in [22].
The second part of the proof follows for the fist one with an argument similar to the one contained in [5]. Indeed, if

$$B_{\varrho,\sigma} = B_\varrho \times [t_o+\sigma, T]$$

from the first part of proof , we have

$$\sup_{B_{\varrho,\sigma}} z^2 \leq (k^*)^2 + \frac{c}{(\varrho_1-\varrho)^{2M}(\sigma-\sigma_1)^M}\int_{B_{\varrho_1,\sigma_1}}(z-k^*)^2 d\eta, \tag{9}$$

where $\varrho < \varrho_1$, $\sigma_1 < \sigma$ and M is a positive constant. If $0 < p < 2$ we have

$$\sup_{B_{\frac{\varrho}{2},\sigma}} z^p \leq \left((k^*)^2 + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{2}{3}\varrho,\frac{\sigma}{2}}}(z-k^*)^2 d\eta\right)^{\frac{p}{2}}.$$

Then

$$\sup_{B_{\frac{\varrho}{2},\sigma}} z^p \leq \left(J\left(\frac{2}{3}\right)\right)^{\frac{p}{2}}\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right) \tag{10}$$

where the function J is defined by

$$J(s) = \left((k^*)^2 + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{s\varrho,(1-s)\sigma}}(z-k^*)^2 d\eta\right)\cdot$$

$$\cdot\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{2}{3}\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right)^{-\frac{2}{p}},$$

for every s such that $\frac{1}{3} \leq s \leq \frac{2}{3}$.
We will prove that $J\left(\frac{2}{3}\right)$ is above bounded by a constant independent of ϱ and σ. Now, for every t, s such that $\frac{1}{3} \leq s < t \leq \frac{2}{3}$ we have

$$J(s) \leq$$

$$\leq \frac{(k^*)^{2-p}(k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\sup_{B_{s\varrho,(1-s)\sigma}}(z-k^*)^{2-p}\int_{B_{s\varrho,(1-s)\sigma}}(z-k^*)^p d\eta}{\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{2}{3}\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right)^{\frac{2}{p}}} \leq \tag{11}$$

$$\leq \frac{\left((k^*)^2 + \frac{c}{(\varrho^2\sigma)^M(t-s)^{3M}}\int_{B_{t\varrho,(1-t)\sigma}}(z-k^*)^2 d\eta\right)^{\frac{2-p}{2}}(k^*)^p}{\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{2}{3}\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right)^{\frac{2}{p}}} +$$

$$+\frac{\left((k^*)^2 + \frac{c}{(\varrho^2\sigma)^M(t-s)^{3M}}\int_{B_{t\varrho,(1-t)\sigma}}(z-k^*)^2 d\eta\right)^{\frac{2-p}{2}}\frac{c}{(\varrho^2\sigma)^M}\int_{B_{s\varrho,(1-s)\sigma}}(z-k^*)^p d\eta}{\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{2}{3}\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right)^{\frac{2}{p}}} \leq$$

$$\leq \frac{\left((k^*)^2 + \frac{c}{(\varrho^2\sigma)^M(t-s)^{3M}}\int_{B_{t\varrho,(1-t)\sigma}}(z-k^*)^2 d\eta\right)^{\frac{2-p}{2}}}{\left((k^*)^p + \frac{c}{(\varrho^2\sigma)^M}\int_{B_{\frac{3}{2}\varrho,\frac{\sigma}{3}}}(z-k^*)^p d\eta\right)^{-1+\frac{2}{p}}} \leq \left(\frac{1}{(t-s)^{3M}}J(t)\right)^{\frac{2-p}{2}}.$$

We thus have

$$\log J(s) \leq \frac{2-p}{p}\left(-3M\log(t-s) + \log J(t)\right)$$

for every t and s. ¿From this inequality we can conclude, as in [5], that $J(\frac{2}{3}) \leq$ constant independent of ϱ and σ.

Consider now the linearized problem

$$\begin{cases} u_t = div(gDu) + f & in\ \Omega \times [\bar{t}, +\infty[\\ u = 0 & on\ \partial\Omega \times [\bar{t}, +\infty[\\ u(x,\bar{t}) = \widetilde{u}_0 & in\ \Omega. \end{cases} \tag{12}$$

Lemma 2.3. *Let ϕ_1 be the first eigenfunction of the operator $L_g u = div(gDu)$ in Ω with Dirichlet data and let u_0, f be orthogonal to ϕ_1 in L^2 norm. Suppose also that*

$$||f(\cdot,t)||_{L^2(\Omega)} \leq c\exp(-2\beta t),\ for\ t \geq \bar{t}$$

for some constant $\beta \neq \frac{\lambda_2}{2}$, where λ_2 is the second eigenvalue of the operator $L_g u = div(gDu)$. If u is a smooth solution of (12) then for every $t \geq \bar{t}$ we have

$$||u(\cdot,t)||^2_{L^2(\Omega)} \leq c\exp(-2\gamma t), \tag{13}$$

$$\int_t^{t+1}\int_\Omega |Du(x,s)|^2 g(x)dxds \le c\exp(-2\gamma t), \tag{14}$$

where $\gamma = \min\{2\beta, \lambda_2\}$.

Proof Let us first note that for any $t \ge \bar t$ the function $u(\cdot, t)$ is orthogonal to ϕ_1, indeed

$$\partial_t \int_\Omega u\phi_1 dx = \int_\Omega u_t\phi_1 dx = \int_\Omega \phi_1 div(gDu)dx + \int_\Omega f\phi_1 dx =$$

$$= -\int_\Omega gD\phi_1 Du dx = \int_\Omega div(gD\phi_1)udx = \lambda_1 \int_\Omega u\phi_1 dx.$$

Moreover,

$$\int_\Omega u\phi_1 dx|_{t=\bar t} = \int_\Omega \widetilde{u}_0\phi_1 dx = 0$$

so that

$$\int_\Omega u(x,t)\phi_1(x)dx = 0$$

for every $t \ge \bar t$. By definition of λ_2, it follows that

$$\lambda_2 ||u(\cdot,t)||^2_{L^2(\Omega)} \le \int_\Omega |Du(x,t)|^2 g(x)\,dx \ for\ any\ t \ge \bar t.$$

Put $\psi(t) = ||u(\cdot,t)||^2_{L^2(\Omega)}$. Multiplying equation in (12) by u, integrating over Ω and using the hypothesis on f, we obtain

$$\int_\Omega u_t\,udx = -\int_\Omega gDuDudx + \int_\Omega fudx \le -\lambda_2\psi^2(t) + c\exp(-2\beta t)\psi(t) \tag{15}$$

so that

$$\frac{1}{2}\partial_t\psi^2(t) \le -\lambda_2\psi^2(t) + c\exp(-2\beta t)\psi(t).$$

We immediately deduce:

$$\partial_t\psi(t) + \lambda_2\psi(t) \le c\exp(-2\beta t)\ for\ t \ge \bar t.$$

Then for any $t \ge \bar t$

$$\psi(t) \le \exp(-\lambda_2 t)\left(\exp(\lambda_2\bar t)\psi(\bar t) - \frac{c}{\lambda_2 - 2\beta}\exp(\lambda_2 - 2\beta)\bar t\right) +$$

$$+\frac{c}{\lambda_2 - 2\beta} \exp(-2\beta t).$$

This implies estimate (13). Assertion in (14) is obtained by integrating (15) from t to $t+1$ and using (13).

LEMMA 2.4. *Suppose that conditions of Lemma 2.3 are satisfied except for the L^2 estimate of f which is replaced by*

$$\sup_{\Omega} |f(\cdot,t)| \leq c \exp(-2\beta t) \text{ for } t \geq \bar{t}.$$

Then, for every $t \geq \bar{t} + \frac{1}{2}$

$$\sup_{\Omega} |u(\cdot,t)| \leq c \exp(-\gamma t)$$

where $\gamma = \min\{2\beta, \lambda_2\}$.

Proof The proof is a easy modification of the argument found in Lemma 5.3 in [22] and follows by Lemma 2.1 and Lemma 2.2.

3. THE NONLINEAR EQUATION

3.1. L^∞ gradient estimate and existence theorem

The object of this subsection is to prove the classical solvability of problem (1).

THEOREM 3.1. *There is a unique solution $u \in C^\infty(\Omega \times [0,\infty[)$ of the problem (1).*

Proof It is well known that the solvability of the problem (1) reduces to the apriori-estimates of the gradient. Let us show that the classical structure conditions stated for example in [18] are satisfied.
Let u be a $C^{2,1}$ solution of

$$-u_t + g \sum_{ij=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2} \right) u_{ij} + \langle Dg, Du \rangle = 0$$

in Ω. (Here and in the following we indicate by u_i and u_{ij} the partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i x_j}$, respectively).
Let ϕ be a strictly increasing function in $C^3([0,1],[m,M])$ to be chosen later, and $\omega = \frac{\phi''(\bar{u})}{(\phi'(\bar{u}))^2}$. Then the function $\bar{u} = \phi^{-1}(u)$, is a solution of

$$-\bar{u}_t + g \sum_{ij=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2} \right) \bar{u}_{ij} + \frac{1}{\phi'} \left(\langle Dg, Du \rangle + \omega\, E \right) = 0, \quad (16)$$

where E is the Bernstein function, defined by

$$E = \sum_{ij=1}^{n} g\left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right) u_i u_j = \frac{g|Du|^2}{1+|Du|^2}.$$

It is proved, for example in [18], that the function $\bar{v} = |D\bar{u}|^2$, is a solution of

$$-\bar{v}_t + \sum_{ij=1}^{n} a_{ij}\bar{v}_{ij} + \sum_{i=1}^{n} b_i \bar{v}_i + c_0\, E\, \bar{v} \geq 0, \tag{17}$$

where $a_{ij} = g\left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right)$, for $i, j = 1, \dots, n$, b_i are suitable regular functions and

$$c_0 = \frac{\omega'}{\phi'} + A\omega^2 + B,$$

$$A = -1 + \frac{2}{1+|Du|^2}, \quad B = \frac{1+|Du|^2}{|Du^4|} \sum_{ij=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{g_i}{g}\right) u_i u_j.$$

Since A is negative if $|Du| > 1$, if ω is a negative constant then c_0 is negative. Hence, with this choice of function ϕ, by the maximum principle it follows

$$\sup_{\Omega\times[0,+\infty[} |Du|^2 \leq \max\left\{\frac{\max\phi'}{\min\phi'} \sup_{\partial_p\Omega} |Du|^2, 1\right\}.$$

In order to estimate $\sup_{\partial_p\Omega} |Du|^2$, it is sufficient to observe that Ω is a convex set and a barrier function can be chosen in the form

$$v(x) = f(\langle \nu, x_0 - x\rangle),$$

where ν is the outer normal and f a suitable function.

In the follow subsections we will prove asymptotic estimate of a normalized solution of (1) stated in Theorem 1.1.
First, we prove uniform convergence to zero of the solutions as $t \to +\infty$ and a priori C^0- asymptotic estimates. Next, we establish a boundary and a globally gradient asymptotic estimate and finally we give the proof of Theorem 1.1.

3.2. Asymptotic estimate of the solutions

PROPOSITION 3.1. *If u is the solution of problem (1) in $\Omega \times [0, \infty[$ then u uniformly converges to zero in Ω as $t \to +\infty$.*

Proof Write equation in (1) as

$$\frac{u_t}{\sqrt{1+|Du|^2}} = div\left(\frac{gDu}{\sqrt{1+|Du|^2}}\right).$$

Multiplying by u_t, integrating the result over $\Omega \times [t_1, t_2]$, $0 < t_1 < t_2 < +\infty$ and taking account vanishing of u_t on $\partial\Omega \times]0, \infty[$, we get

$$\int_{t_1}^{t_2}\int_\Omega \frac{u_t^2}{\sqrt{1+|Du|^2}}dxdt = \int_{t_1}^{t_2}\int_\Omega u_t div\left(\frac{gDu}{\sqrt{1+|Du|^2}}\right)dxdt =$$

$$= -\int_{t_1}^{t_2}\int_\Omega \frac{gDu_tDu}{\sqrt{1+|Du|^2}}dxdt = -\int_{t_1}^{t_2}\int_\Omega \frac{d}{dt}g\sqrt{1+|Du|^2}dxdt =$$

$$= -\int_\Omega g\sqrt{1+|Du|^2}dx\ |_{t_1}^{t_2} \le c,$$

where the constant c depends on g and on $\sup_{\Omega\times[0,\infty[}|Du|$. Thus

$$\int_{t_1}^{t_2}\int_\Omega \frac{u_t^2}{\sqrt{1+|Du|^2}}dxdt \le c \tag{18}$$

and then, there is a sequence $(t_k)_k$ such that $t_k \to +\infty$ and

$$r(t_k) := \int_\Omega \frac{u_t^2}{\sqrt{1+|Du|^2}}dx\ |_{t=t_k} \to 0 \ \ as\ k \to +\infty.$$

We note that $r(t) \to 0 \ \ as\ t \to +\infty$. Indeed, because of the boundness of u_t, u_{tt} and Du_t, also the function $\frac{dr}{dt}$ is bounded. Consequently if there exists a sequence $t_k \to +\infty$ such that $r(t_k) \ge c$ for every k, then $r(t) \ge \frac{c}{2}$ in a neighborhood of fixed length of each t_k, contradicting the integrability of r in $\mathbb{R}$.
We consider any sequence $(t_k)_k$, $t_k \to +\infty$ and the sequence $(u_k(\cdot))_k \equiv (u(\cdot, t_k))_k$. Now $r(t_k) \to 0 \ \ as\ k \to +\infty$, by Hölder inequality we get

$$\int_\Omega \frac{\partial_t u_k}{\sqrt{1+|Du_k|^2}}dx \le (r(t_k)|\Omega|)^{\frac{1}{2}} \to 0 \ \ as\ k \to +\infty,$$

where $|\cdot|$ indicate the Lebesgue measure on $\mathbb{R}^n$.
Consequently, by equation (1) and the boundness of u_k, we have

$$-\int_\Omega g\frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}}dx = \int_\Omega \frac{u_k\partial_t u_k}{\sqrt{1+|Du_k|^2}}dx \to 0 \ as\ k \to +\infty.$$

Then, u_k weakly converges to zero in $W^{1,2}(\Omega)$. But, it is uniformly bounded and uniformly continuous, then by Ascoli-Arzelá theorem it is uniformly convergent to zero. Thanks to the arbitrariness of $(t_k)_k$ the function $u(\cdot,t)$ uniformly converges to zero in Ω as $t \to +\infty$.

Proposition 3.2. *If u is a solution of the problem (1) and λ_1 is the first eigenvalue of the linear operator $L_g u = div(gDu)$ in Ω with Dirichlet boundary condition, then there exists $\bar{t} > 0$ such that for every $\lambda < \lambda_1$ we have*

$$\sup_\Omega |u(\cdot,t)| \le c(\lambda)\exp(-\lambda t) \quad for\ every\ t \ge \bar{t}.$$

Proof Let Ω_s be a tubular neighborhood of the domain Ω at a distance $s > 0$ small enough so that $\partial\Omega_s$ is still smooth. Let γ_s, ϕ_s be, correspondingly, the first eigenvalue and the first eigenfunction of the operator L_g in Ω_s. Since $\overline{\Omega} \subset \Omega_s$, the function ϕ_s is positive in $\bar{\Omega}$, then we may assume it to be normalized so that $\inf_\Omega \phi_s = 1$ in $\overline{\Omega}$. Fix any $\lambda \in]0,\gamma_s[$ and consider the function

$$\omega_s(x,t) = A_s\phi_s(x)\exp\left(-\lambda(t-t_s)\right),$$

where $A_s = \sup_\Omega u(\cdot,t_s)$ and t_s is a positive constant to be chosen later. If $t > t_s$ then

$$L\omega_s := \frac{\partial\omega_s}{\partial t} - \sqrt{1+|D\omega_s|^2}\frac{d}{dx_i}\left(\frac{g\partial_i\omega_s}{\sqrt{1+|D\omega_s|^2}}\right) =$$

$$\left\{[-\lambda+\gamma_s]\phi_s + \frac{A_s^2\phi_i^s\phi_j^s\phi_{ij}^s\exp\left(-2\lambda(t-t_s)\right)}{1+A_s^2|D\phi^s|^2\exp\left(-2\lambda(t-t_s)\right)}\right\}A_s\exp\left(-\lambda(t-t_s)\right) \ge$$

$$\ge (-\lambda+\gamma_s - A_s^2|\phi_i^s\phi_j^s\phi_{ij}^s|)A_s\exp\left(-\lambda(t-t_s)\right).$$

Because of Proposition 3.1, the function A_s converges to 0 as $t \to +\infty$ and choosing t_s sufficiently large we obtain

$$L\omega_s \ge 0\ in\ \Omega \times [t_s,+\infty[.$$

Since $\omega_s > 0$ on $\partial\Omega$ and $\omega_s \geq u$ in $\Omega \times \{t_s\}$, applying the maximum principle we get

$$|u(x,t)| \leq \omega_s(x,t) \leq c \exp(-\lambda t) \; in \; \overline{\Omega} \times [t_s, +\infty[.$$

3.3. Asymptotic estimate of the gradient of the solutions

PROPOSITION 3.3. *Under the same assumption of Proposition 3.2 there exists $\bar{t} > 0$ such that for every $\lambda < \lambda_1$ we have*

$$\sup_{\partial\Omega} |Du(\cdot,t)| \leq c(\lambda) \exp(-\lambda t) \quad for\, every\, t \geq \bar{t}.$$

Proof Arguing as in [22], we define a function ω on $\Omega_\delta \times [T-1, \infty[$, $\Omega_\delta = \{x \in \Omega \,:\, d(x, \partial\Omega) < \delta\}$, such that
(i) $\omega \geq 0$ on $\partial\Omega \times [0, \infty[$, $\omega(x,t) \geq |u(x,t)|$ for every $x \in \Omega$ such that $d(x, \partial\Omega) = \delta$;
(ii) $L\omega \geq 0$ in $\Omega_\delta \times]T-1, \infty[$.
We can choose the barrier function ω in such a form

$$\omega(x,t) = c_1 \left[f(d(x)) + p(t)\right] \exp(-\lambda t)$$

where d is the distance from $\partial\Omega$, $f \in C^2(\mathbb{R})$ such that $f(0) = 0$, $0 < f' \leq 1$, $f'' < 0$ and $p(t) = [(T-t)^+]^2$. Moreover, δ is small enough so that $d \in C^2(\bar{\Omega}_\delta)$.
In the following, to compute $L\omega$, we use the fact that $g d_i^2 = 1$. Now,

$$L\omega := \omega_t - \sqrt{1 + |D\omega|^2} \frac{d}{dx_i} \left(\frac{h^{\frac{n}{2}} \omega_i}{\sqrt{1 + |D\omega|^2}} \right) =$$

$$= \omega_t - h^{\frac{n}{2}+1} \Delta_h \omega - h^{\frac{n}{2}-1} \langle Dh, D\omega \rangle + \frac{1}{2} h^{\frac{n}{2}} \frac{\omega_i}{1 + |D\omega|^2} \frac{d}{dx_i} \left(|D\omega|^2 \right) =$$

$$= c_1 e^{-\lambda t} \Bigg[-\lambda(f+p) + p_t - h^{\frac{n}{2}+1} f' \Delta_h d - h^{\frac{n}{2}-1} \frac{f''}{1 + c_1^2 exp(-2\lambda t)(f')^2 h^{-1}} +$$

$$+ h^{\frac{n}{2}} \frac{(f')^3 \langle D(h^{-1}), Dd \rangle}{1 + c_1^2 exp(-2\lambda t)(f')^2 h^{-1}} - h^{\frac{n}{2}-1} f' \langle Dh, Dd \rangle \Bigg].$$

Then, with the same arguments in [22], for a suitable choice of δ, T and f, which depends also on the metric g and on its gradient, we obtain (i) and (ii), so that from maximum principle

$$|u(x,t)| \leq c_1(\lambda) d(x) \exp(-\lambda t).$$

Since, $u = 0$ on $\partial\Omega$, we get the desired estimate.

PROPOSITION 3.4. *Under the same assumption of Proposition 3.2 there exists $\bar{t} > 0$ such that for every $\lambda < \lambda_1$, we have*

$$\sup_\Omega |Du(\cdot, t)| \le c(\lambda)\exp(-\lambda t) \quad for\, every\, t \ge \bar{t}.$$

Proof First, we prove that, for some positive constant β sufficiently small, we have

$$\sup_\Omega |Du(\cdot, t)| \le c\, \exp\left(-\beta(t - \bar{t})\right) for\ t \ge \bar{t}. \tag{19}$$

As in the proof of Theorem 3.1, the function $\bar{v} = |D\bar{u}|^2$ satisfies inequality (17)

$$-\bar{v}_t + \sum_{i,j=1}^{n} a_{ij}\bar{v}_{ij} + \sum_{i=1}^{n} b_i\bar{v}_i + c_0 E\bar{v} \ge 0.$$

Thanks to Proposition 3.2 the oscillation of u is small, for t sufficiently large, and the number c_0 is negative (see Theorem 1.1 in [18]), so that

$$\bar{L}\bar{v} \equiv \bar{v}_t - \sum_{i,j=1}^{n} a_{ij}\bar{v}_{ij} - \sum_{i=1}^{n} b_i\bar{v}_i \le 0\ for\, any\ t \ge \bar{t}. \tag{20}$$

Let $\bar{x} \in \mathbb{R}^n$ such that, for every $x \in \Omega$ the first component of $x - \bar{x}$ is nonnegative, and consider the function

$$w(x,t) = A\exp(-\beta t)\left(2 - \exp(-\mu(x - \bar{x})_1)\right)$$

where the constants A, β, and μ are to be chosen later. We have

$$A^{-1}\exp(\beta t + \mu(x - \bar{x})_1)\left(w_t - \sum_{i,j=1}^{n} a_{ij}w_{ij} - \sum_{i=1}^{n} b_i w_i\right) =$$

$$= [1 - 2\exp(-\mu(x - \bar{x})_1)]\beta + \left(1 - \frac{u_1^2}{v^2}\right) g\,\mu^2 + b_1\mu.$$

Choose μ so that

$$\left(1 - \frac{u_1^2}{v^2}\right) g\,\mu > |b_1|$$

and β small enough so that $\beta \le \lambda$ and

$$\bar{L}w > 0 \ge \bar{L}\bar{v}.$$

Put also $A = c\exp(\beta\bar{t})$, then

$$|\bar{v}(x,t)| \le c\exp(-\beta(t-\bar{t})) = A\exp(-\beta t) \le w \; on \; \partial\Omega \times [\bar{t}, +\infty[$$

and

$$|\bar{v}(x,\bar{t})| \le A\exp(-\beta\bar{t}) \le w(x,\bar{t}) \; in \, \Omega.$$

It follows from the maximum principle that

$$|\bar{v}(x,t)| \le w(x,t) \le 2c\exp(-\beta(t-\bar{t})) \; in \; \Omega \times [\bar{t}, +\infty[.$$

Estimate (19) is proved.
Choose $t_0 > \bar{t}$, and $T = t_0 + 1$ then, by definition of $\bar{v}$ and from the second part of Lemma 2.2, with $p = 1$, we get

$$|Du(x,t_0)|^2 \le c|\bar{v}(x,t_0)|^2 \le c\max\{\sup_{\partial\Omega} \bar{v}, ||\bar{v}||_{L^1(\bar{\Omega}\times[t_0+\frac{1}{2},T])}\} \le$$

$$\le c\max\{\sup_{\partial\Omega} |Du|^2, ||Du||^2_{L^2(\bar{\Omega}\times[t_0+\frac{1}{2},T])}\} \le$$

from Proposition 3.3 and Lemma 2.3

$$\le c\exp(-2\lambda t).$$

Then the proposition is proved.

3.4. Proof of main theorem

We are now ready to give the
Proof of Theorem 1.1 Write u as a sum of the $L^2(\Omega)$-projections on ϕ_1 and on its orthogonal complement H_1, that is

$$u = \bar{u} + \widetilde{u}, \quad \bar{u} = (u, \phi_1)\phi_1,$$

($(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$), then $\widetilde{u}$ is solution of the problem

$$\begin{cases} \widetilde{u}_t = div(gD\widetilde{u}) + \widetilde{f} & in \; \Omega \times [\bar{t}, +\infty[\\ \widetilde{u} = 0 & on \; \partial\Omega \times [\bar{t}, +\infty[\\ \widetilde{u} = u - \bar{u} & in \; \Omega \times \{\bar{t}\}, \end{cases} \tag{21}$$

where

$$\widetilde{f} = f - (f,\phi_1)\phi_1, \quad f = -\frac{u_i u_j u_{ij}}{1+|Du|^2} g.$$

Indeed, multiplying equation $u_t = div(gDu) + f$ by ϕ_1 and integrating over Ω we get

$$\int_\Omega u_t \phi_1 dx = -\int_\Omega gD\phi_1 Du dx + \int_\Omega f\phi_1 dx =$$

$$= \lambda_1 \int_\Omega u\phi_1 dx + \int_\Omega f\phi_1 dx.$$

Then

$$\phi_1 \partial_t \int_\Omega u\phi_1 dx = \phi_1 \lambda_1 \int_\Omega u\phi_1 dx + \phi_1 \int_\Omega f\phi_1 dx = \tag{22}$$

$$= \left(\int_\Omega u\phi_1 dx \right) div(gD\phi_1) + \phi_1 \int_\Omega f\phi_1 dx.$$

So that $\bar{u}_t = div(gD\bar{u}) + \bar{f}$. Analogously for $\widetilde{u}$.
¿From Proposition 3.4, the function $\widetilde{f}$ satisfies

$$\sup_\Omega |\widetilde{f}(\cdot, t)| \leq c \exp(-2\beta t), \ for\, every\ t \geq \bar{t}$$

with $\beta = \lambda < \lambda_1$, and, by Lemma 2.4 we obtain that

$$\sup_\Omega |\widetilde{u}(\cdot, t)| \leq c \exp\left(-\gamma t\right), \ for\, every\ t \geq \bar{t}, \tag{23}$$

where $\gamma = \min\{2\lambda, \lambda_2\} > \lambda_1$. On the other hand, from (22), we have

$$\bar{u}(x,t) = e^{-\lambda_1(t-\bar{t})} \left((u(\cdot,\bar{t}), \phi_1) + \int_{\bar{t}}^{\infty} e^{-\lambda_1(\bar{t}-\tau)} f_1(\tau) d\tau \right) \phi_1(x) + r(x,t), \tag{24}$$

where

$$f_1(t) = (f(\cdot,t), \phi_1),$$

$$r(x,t) = -\phi_1(x) \int_t^\infty e^{-\lambda_1(t-\tau)} f_1(\tau) d\tau.$$

By Proposition 3.4, the function f_1 is once again bounded by the function $c\exp(-2\lambda t)$ and since, $2\lambda > \lambda_1$, we get

$$|r(x,t)| \leq \frac{c}{2\lambda - \lambda_1} \exp(-2\lambda t).$$

Finally, from this inequality, and (23), (24), we obtain

$$u(x,t) = \bar{u}(x,t) + \widetilde{u}(x,t) = ce^{-\lambda_1 t}\phi_1(x) + r(x,t) + \widetilde{u}(x,t) =$$

$$= ce^{-\lambda_1 t}\phi_1(x) + O(e^{-2\lambda}) \ \ as \ \ t \ \to \infty.$$

This gives the conclusion of the theorem with

$$c = e^{-\lambda_1 \bar{t}}\left((u(\cdot,\bar{t}),\phi_1) + \int_{\bar{t}}^{\infty} e^{-\lambda_1(\bar{t}-\tau)} f_1(\tau)d\tau\right)$$

and

$$\mu = \min\{2\lambda - \lambda_1, \lambda_2 - \lambda_1\}.$$

Theorem 1.1 implies that the model which motivates this study exhibits a non-linear behavior for short period of time and it's able to reconstruct correctly the image in this period.

REFERENCES

1. G. Bellettini, M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry* , Hokkaido Math. Jour., 25, (1996), 537-566.
2. V.Caselles, F.Catte', T.Coll, F.Dibos, *A Geometric model for Active Contours*, Numerische Mathematik,Vol.66, (1993), 1-31.
3. V. Caselles, R. Kimmel, G. Sapiro, C. Sbert, *Minimal surfaces: A geometric three dimensional segmentation approach*, Numer. Math. 77, No.4, (1997), 423-451.
4. Y.G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom., 33, (1991), 749-786.
5. F. Chiarenza, E. Fabes, N. Garofalo, *Harnack's inequality for Schrödinger operators and continuity of solutions*, Proc. Amer. Math. Soc., 98, (1986), 415-425.
6. K. Ecker, G. Huisken, *Mean curvature evolution of entire graphs*, Ann. Math., II. Ser. 130, No.3, (1989), 453-471.
7. L.C. Evans, J. Spruck, *Motion of level sets by mean curvature*, J. Diff. Geom, 33, (1991), 635-681.
8. L.C. Evans, J. Spruck, *Motion of level sets by mean curvature. II*, Trans. Am. Math. Soc. 330, No.1, (1992), 321-332.
9. L.C. Evans, J. Spruck, *Motion of level sets by mean curvature. III*, J. Geom. Anal. 2, No.2, (1992), 121-150.
10. L.C. Evans, J. Spruck, *Motion of level sets by mean curvature. IV*, J. Geom. Anal. 5, No.1, (1995), 79-116.
11. Y. Giga, S. Goto, *Motion of hypersurfaces and geometric equations*, J. Math. Soc. Jap., 44, No. 1, (1992), 99-112.
12. G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differ. Geom. 20, (1984), 237-266.

13. G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. 84, (1986), 463-480.

14. G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differ. Equations 77, No.2, (1989), 369-378.

15. T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. Math. J. 41, No.3, (1992), 671-705.

16. S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, A. jun Yezzi, *Conformal curvature flows: from phase transitions to active vision*, Arch. Ration. Mech. Anal. 134, No.3, (1996), 275-301.

17. A. Lichnewsky, R. Temam, *Pseudo solutions of the time-dependent minimal surface problem*, J. Differ. Equations 30, (1978), 340-364.

18. G.M. Lieberman, *Second order parabolic differential equations*, Singapore: World Scientific. xi, (1996).

19. R. Malladi, J. A. Sethian, B. Vemuri, *Topology independent shape modeling scheme*, SPIE Proceedings on Geometric Methods in Computer Vision II, San Diego, (1993), 246-258.

20. P.Marcellini, K. Miller, *Elliptic versus parabolic regularization for the equation of prescribed mean curvature*, J. Differ. Equations 137, No.1, (1997), 1-53.

21. V.I. Oliker, N.N. Ural'tseva, *Evolution of nonparametric surfaces with speed depending on curvature. III: Some remarks on mean curvature and anisotropic flows*, Ni, Wei-Ming (ed.) et al., Degenerate diffusions. Proceedings of the IMA workshop, held at the University of Minnesota, MN, USA, from May 13 to May 18, 1991. New York: Springer-Verlag. IMA Vol. Math. Appl. 47, (1993), 141-156.

22. V.I. Oliker, N.N. Ural'tseva, *Evolution of nonparametric surfaces with speed depending on curvature. II: The mean curvature case*, Commun. Pure Appl. Math. 46, No.1, (1993), 97-135.

23. V.I. Oliker, N.N. Ural'tseva, *Long time behavior of flows moving by mean curvature*, Uraltseva, N. N. (ed.), Nonlinear evolution equations. Providence, RI: American Mathematical Society. Transl., Ser. 2, Am. Math. Soc. 164 (22), (1995), 163-170.

24. V.I. Oliker, N.N. Ural'tseva, *Long time behavior of flows moving by mean curvature. II*, Topol. Methods Nonlinear Anal., 9, No.1, (1997), 17-28.

25. P. Olver, G. Sapiro, A. Tannenbaum, *Invariant geometric evolutions of surfaces and volumetric smoothing*, SIAM J. Appl. Math, 57, 1, (1997), 176-194.

26. S. Osher, J.A. Sethian, *Front Propagating with Curvature Dependent Speed: Algorithms Based on Hamilton Jacobi Formulation*, Journal of Computational Physics, Vol. 79,(1988), 12-49.

27. A. Sarti, R. Malladi, J.A. Sethian, *Subjective surfaces: A method for completing missing boundaries*, Proceedings of the National Academy of Science of the United States of America, Vol 12, n. 97, (2000), 6258-6263.

28. J.A. Sethian, *Curvature and the Evolution of Fronts*, Comm. Math. Phys., 101, (1985), 487-499.

29. J. Shah, *Riemanninian Drums Anisotropic curve evolution and Segmentation*, J.Visual Communication and Image Representation 11 (2000), 142-153.

30. H.M. Soner, *Motion of a set by curvature of its boundary*, J. Diff. Eqs, 101, 2, (1993), 313-372.

31. P. Sternberg, W.P. Ziemer, *Generalized motion by curvature with a Dirichlet condition*, Differ. Equations 114, No.2, (1994), 580-600.