A subriemannian model of visual perception

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joint work with A. Sarti

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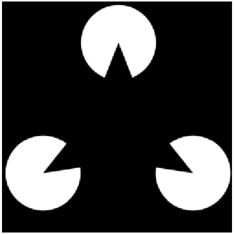
Modal completion

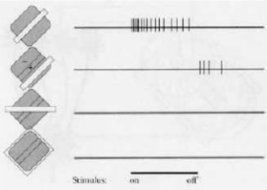


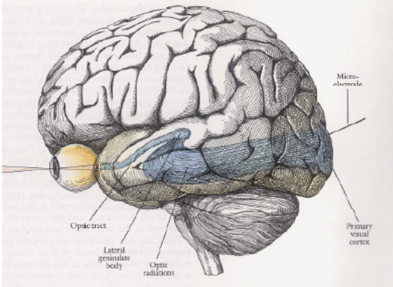
- contrast perception
- completion of boundaries
- pop up of the triangle

Contrast perception - retinex model

[Land, '71, '77],
[J. M. McCann et al. 2002],
[R. Kimmel, M. Elad, D. Shaked, R. Keshet, I. Sobel, 2003] ,
[J-M Morel, A. Beln Petro, C. Sbert, 2010],

Models of the visual cortex with differential instruments





[Hubel Wiesel] [W.C. Hoffmann '89]
[Petitot and Tondut '99], [Petitot '03]
[S. Zucker ' 05]

[C.- Sarti] [Sarti, C- Petitot '08]

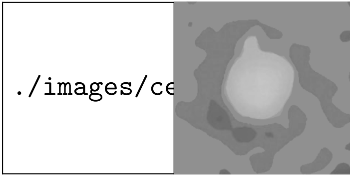
[Hladky and Pauls '08].

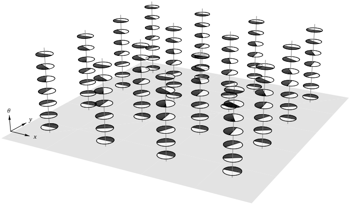
[Duits, van Almsick, Franken, ter Haar Romeny '05, '08]

Receptive profiles

The receptive field of a cortical neuron is the portion of the retina which the neuron reacts to, retinal and receptive profile ψ models the activation.
Over each retinal point there is a whole fiber of
In presence of a visual stimulus I, output of the cells is an integral

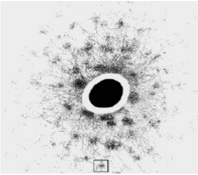
$$O(x,y)_{LGN} = \int \phi_{xy}(\xi,\eta) \log I(\xi,\eta) d\xi d\eta \approx \Delta \log I(x,y).$$





The input is propagated via the lateral connectivity.

$$A = \Gamma * O(x, y)_{LGN}$$



Example

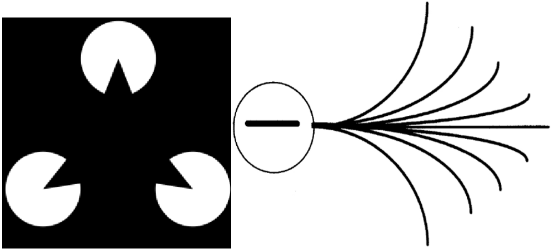
The output of LGN cells is propagated via The total activity contribution will be modelled as

$$u = \Gamma * \Delta \log(I)$$

We do not recover the initial image, but $u - \log(I)$ is harmonic. Euler Lagrange equation of the functional

$$F_1(u) = \int |\nabla u(\xi,\eta) - \nabla \log(I(\xi,\eta))|^2 d\xi d\eta. \tag{1}$$

Modal completion



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Simple cells of the visual cortex V1

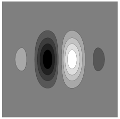
Simple cells of V1 are strongly oriented cells. They can be represented as a first derivative of a Gaussian bell.

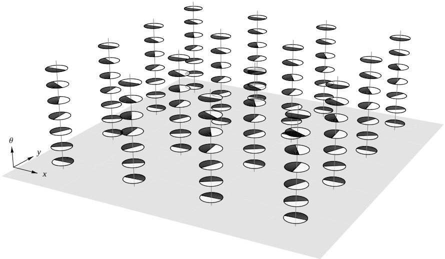
$$\psi_0(\xi,\eta) = \partial_\eta G(\xi,\eta)$$

Over each point (x, y) there is a whole fiber of cells each one sensible to a direction θ. Each cell is identified by 3 parameters

The set of cells is obtained by rotation and translation from a fixed one

$$\psi_{(x,y,\theta)}(\xi,\eta) = R_\theta T_{x,y}\psi_0(\xi,\eta) = \psi_\theta(x-\xi, y-\eta)$$





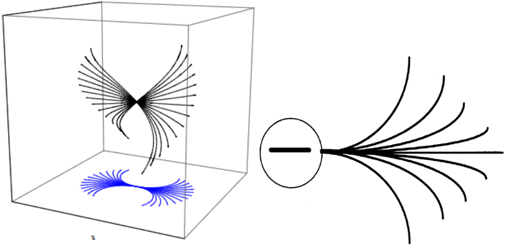
The action of simple cells and non maxima suppression

In presence of visual signal, the whole ipercolumn fires, the cell sensible to the perceived feature has the highest response.

$$O_{V1}(x, y, \theta) = \int \psi_{x,y,\theta}(\xi, \eta) f(\xi, \eta) d\xi d\eta.$$

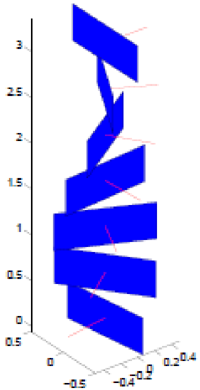
lateral connectivity induced

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y \quad X_2 = \partial_\theta$$



Curves on the LGN plane are lifted to 3D curves. The tangent vector to any lifted curve lies on the plane generated by

Definition of distance



$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad X_2 = \partial_\theta$$

admissible curves: $\gamma' = \sum_i^2 \alpha_i X_i(\gamma)$,
metric $||a_1X_1 + a_2X_2|| = \sqrt{a_1^2 + a_2^2}$
length $l(\gamma) = \int ||\gamma'(t)||dt$,
$d(\xi, \eta) = inf\{l(\gamma) : \gamma \text{ connects } \xi, \eta\}$.

$X_3 = [X_1, \partial_\theta] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$,
[Bony], [A.Nagel, E.M.Stein, S.Wainger '85]

Theorem: Chow

$X_1, \cdots X_m$ are called bracket generating if the generated Lie algebra has dimension n at every point. In this case each couple of points can be connected with integral curves of the vector fields.

Connectivity kernel and sub-Laplacian operator

A 2D gradient can be defined $\nabla_{V1} f = (X_1, X_2) f$
$f \in C^1_{V1}$ if $\nabla_{V1} = (X_1, X_2)$ is continuous
$f \in C^2_{V1}$ if $X_j X_i$ exists and are continous

A Laplace operator is well defined

$$\Delta_{V1} = X_1^2 + X_2^2$$

It has a fundamental solution Γ_{V1}

If u is a solution of $X_1^2 u + X_2^2 u = f \in C^\infty(R^2 \times S^1)$ then $u \in C^\infty$

(SanchezCalle '84, Folland-Stein, Rothschild-Stein '87)

Note that if u is a solution of $\partial_1^2 u + \partial_2^2 u = 0 \in C^\infty(R^3)$ u is not regular along x_3

Induced gradient on a surface

We are interested in is the geometry induced on the retinal plane by the geometry of V1 in presence of a visual stimulus.
The stimulus induces two functions $\rho(x, y)$ and $\theta : \mathbb{R}^2 \to S^1$

Theorem: [Ambrosio Serracassano Vittone], [C- Manfredini]

If the function $f(x, y, s) = s - \theta(x, y)$ is of class C^1_{V1} then θ is C^1 with respect to the vector field $X_{1,\theta(x,y)}$.

This defines a differential structure on R^2. Associated differential equation in more general setting: [Capogna, Citti, Manfredini] [Serracassano Vittone]

Induced Dirichlet functional

The two output functions define a vector

$$\vec{A}(x, y) = \rho(x, y)(\cos(\theta(x, y)), \sin(\theta(x, y))$$

The intrinsic gradient is

$$X_{1,\theta}\vec{A}(x, y)$$

The associated propagation of the $\vec{A}$ along the connectivity is

$$F_2 = \int |X_{1,\theta(x,y)}A(\vec{x, y})|^2 dxdy. \quad (2)$$

the joint action of LGN cells and cells in V1

The first term is

$$\mathcal{L}_1 = \int |\nabla u(x,y) - \nabla Log(I(x,y))|^2 dxdy \tag{3}$$

and is directly inspired by the Retinex model.
The field term is

$$\tilde{\mathcal{L}}_2 = \int |X_{1,\theta(x,y)} A(\vec{x,} y)|^2 dxdy.$$

The last term describes the interaction between the particle $u(x,y)$ and the field $A(x,y)$.

$$\mathcal{L}_3 = \int |\nabla u(x,y) - A(x,y)|^2 dxdy. \tag{4}$$

The gauge functional

$$\mathcal{L} = \int |\nabla u - \nabla log(I)|^2 dxdy + \int |\nabla u - A|^2 dxdy + \int |X_{1\theta}\vec{A}|^2 dxdy \quad (5)$$

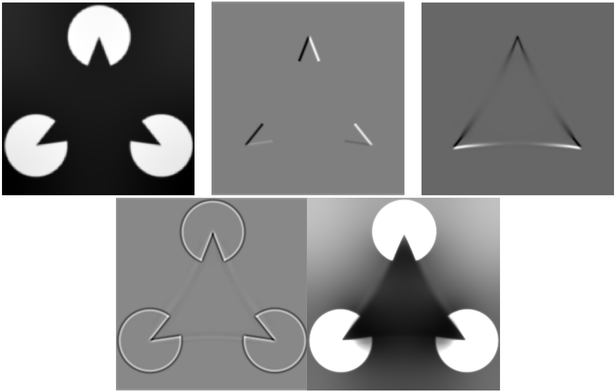
Theorem

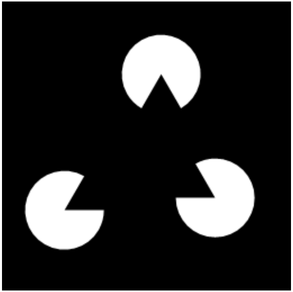
The functional $\mathcal{L}(u, A)$, is weakly lower semi-continuous in suitable Sobolev spaces. Hence it has a minimum.

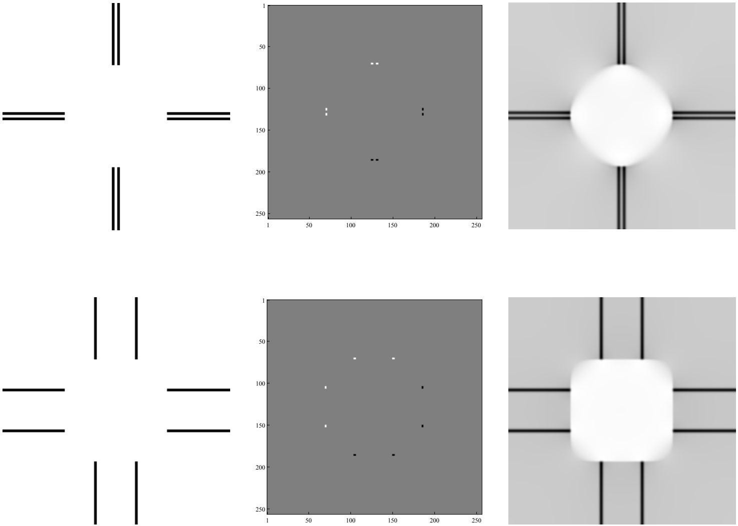
The Euler Lagrange Equations are

$$\begin{cases} \Delta u = \frac{1}{2}(\Delta log(I) + div_\theta(A)) \\ \Delta_\theta A = -\nabla u + A. \end{cases} \quad (6)$$

$$\begin{cases} \Delta_A A = -\nabla u \\ \Delta u = \frac{1}{2}(\Delta log(I) + div_\theta(A)) \end{cases} \quad (7)$$




1
50
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1 50 100 150 200 250
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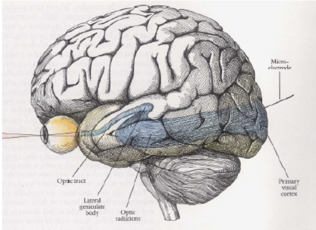
Multilayer extension of the model



- other features: scale, color, movement
- the invariance of images characterize the space of cells
- the receptive fields and connectivity kernels have the sane invariance properties

The GHAIA project

Marie Curie Rise Project, funded by EU Community.

Title: Geometric and Harmonic Analysis with Interdisciplinar Applications

Topics: Geometric, Harmonic analysis, Lie groups, PDE and variational methods, probabilistic methods, models of brain, vision, data analysis, industrial application.

Scope: mobility from some European Universities to WPI and other US Universities and viceversa

EU Partners: University of Bologna, Univ. Autonoma Madrid, CAMS - EHESS - Paris, University of Paris Sud, University of Granada

Dates: november 2017 - october 2021

Website: https://site.unibo.it/ghaia-eu-project/en

Thank you
for your attention

fundamental solution and grouping

Ermentraut-Cowan, Bressloff-Cowan field population model,

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\lambda a + \sigma\Big(\Gamma_{V1} * a - \chi_{O_{V1}>0}\Big) \tag{8}$$

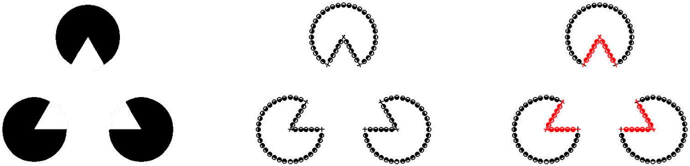
σ is the transfer function of the population with sigmoidal behavior.
Calling to the set Ω where O_{V1} does not vanish,

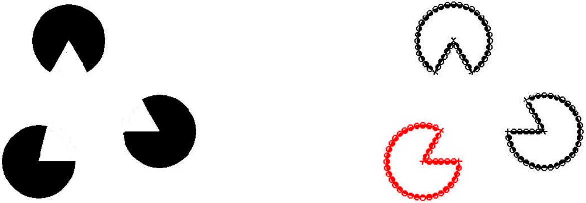
$$\frac{\mathrm{d}a(x,y,\theta,t)}{\mathrm{d}t} = -\lambda a(x,y,\theta,t) + \sigma(\Gamma * a - 1) \in \Omega \tag{9}$$

Linearizing the perturbation around a stationary solution:

$$-\lambda a_1(x,y,\theta,t) + \Gamma * a_1 = 0$$

Eigenvalues on the set Ω





Favali Citti Sarti