The multi-species mean-field spin-glass in the Nishimori line

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Abstract

In this paper we study a multi-species disordered model in the Nishimori line. The typical properties of this line, a set of identities and inequalities, allow us to prove the replica symmetry i.e. the concentration of the order parameter. When the interaction structure is elliptic we rigorously compute the exact solution of the model in terms of a finite-dimensional variational principle.

Keywords Multi-species spin glass, Nishimori line, replica symmetry.

1 Introduction

In this paper we investigate the properties of the elliptic multi-species Sherrington-Kirkpatrick model along the Nishimori line i.e. the sub-manifold of the phase space in which mean and variance of the random parameters, interactions and magnetic fields, coincide. The multi-species version of a mean field model is simply obtained by relaxing the full invariance under the symmetric group into the weaker one of the product of the symmetric groups on a given partition of the system. The ratios of the partition with respect to the whole, the form factors, are kept fixed in the thermodynamic limit. The ellipticity condition provides the positivity and monotonicity properties that allow to study the system with interpolation methods [9, 10, 16] and obtain a Parisi like solution [4, 15]. While we have some information on the properties of such solution for the multi-specie case many properties are still under investigation like, for instance, the precise location of phase separation among the replica symmetric and non-symmetric regions. The choice to study the model on the Nishimori line [14] (see also [5] for a case with a ferromagnetic mean of the interactions) reflects the importance of this sub-manifold of the phase space due to its ubiquitous appearence in inference problems and, especially, on the statistical physics approach to machine learning [1, 3, 11]. The main results of the paper, i.e. the variational expression for the pressure per particle in the thermodynamic limit and the proof that the magnetization per particle doesn't fluctuate, are indeed obtained by merging methods whose origins belong both to statistical mechanics and inference [2, 6, 7, 8, 9, 12, 13].

The paper is organized as follows. In Section 2 we give the definition of the model together with its main properties, such as the self-averaging of the pressure and the Nishimori identities. In Section 3 we extend to our multi-dimensional model the adaptive interpolation method due to Barbier and Macris [2] and we use it to compute the exact solution in Section 4 by writing the pressure in the thermodynamic limit in terms of a finite-dimensional variational principle. Finally we study the main properties of the extremizers of our variational expression. In Appendix A the reader can find the details of the proof of the concentration of the magnetization in the thermodynamic limit, which ultimately leads to replica symmetry. For completeness the properties of the mono-species case (SK) on the Nishimori line are studied in Appendix B.

2 Definitions and basic properties

Consider a set Λ of indices with cardinality $|\Lambda| = N$. Let us partition Λ in K disjoint subsets:

$$\Lambda = \bigcup_{r=1}^{K} \Lambda_r, \quad \Lambda_r \cap \Lambda_s = \emptyset \; \forall r \neq s, \quad |\Lambda_r| =: N_r, \quad \alpha_r := \frac{N_r}{N} \in (0,1) \tag{1}$$

Each subset will be called *species* from now on. The model is defined by the following Gaussian Hamiltonian:

$$H_N(\sigma) := -\sum_{r,s=1}^{K} \sum_{(i,j)\in\Lambda_r\times\Lambda_s} \tilde{J}_{ij}^{rs}\sigma_i\sigma_j - \sum_{r=1}^{K}\sum_{i\in\Lambda_r} \tilde{h}_i^r \sigma_i, \tag{2}$$

$$\tilde{J}_{ij}^{rs} \overset{\text{iid}}{\sim} \mathcal{N}\left(\frac{\mu_{rs}}{2N}, \frac{\mu_{rs}}{2N}\right), \qquad \tilde{h}_i^r \overset{\text{iid}}{\sim} \mathcal{N}(h_r, h_r) \tag{3}$$

where μ_{rs} and h_r are positive real numbers, and the $K\times K$ matrix μ_{rs} can be assumed to be symmetric without loss of generality. We explicitly notice that the gaussian variables are considered on a special line known as *Nishimori line* in the context of statistical mechanics, where mean and variance are tied to be the same.

It is also convenient to rewrite the Hamiltonian (2) in terms of centered Gaussians. To do that we introduce the following notation for species magnetizations and overlaps that will be used throughout:

$$m_r(\sigma) := \frac{1}{N_r} \sum_{i \in \Lambda_r} \sigma_i, \quad q_r(\sigma, \tau) := \frac{1}{N_r} \sum_{i \in \Lambda_r} \sigma_i \tau_i \tag{4}$$

$$\mathbf{m}(\sigma) := (m_r(\sigma))_{r=1,\dots,K}, \quad \mathbf{q}(\sigma, \tau) := (q_r(\sigma, \tau))_{r=1,\dots,K} \tag{5}$$

with $\sigma, \tau \in \Sigma_N := \{-1, 1\}^N$. We also set:

$$\Delta := (\alpha_r \mu_{rs} \alpha_s)_{r,s=1,\dots,K}, \quad \hat{\alpha} := \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_K), \quad \mathbf{h} := (h_r)_{r=1,\dots,K}. \tag{6}$$

We will call Δ the *effective interaction matrix* because it encodes the interactions and relative sizes of the species in our model. See Figure1 for a scheme.

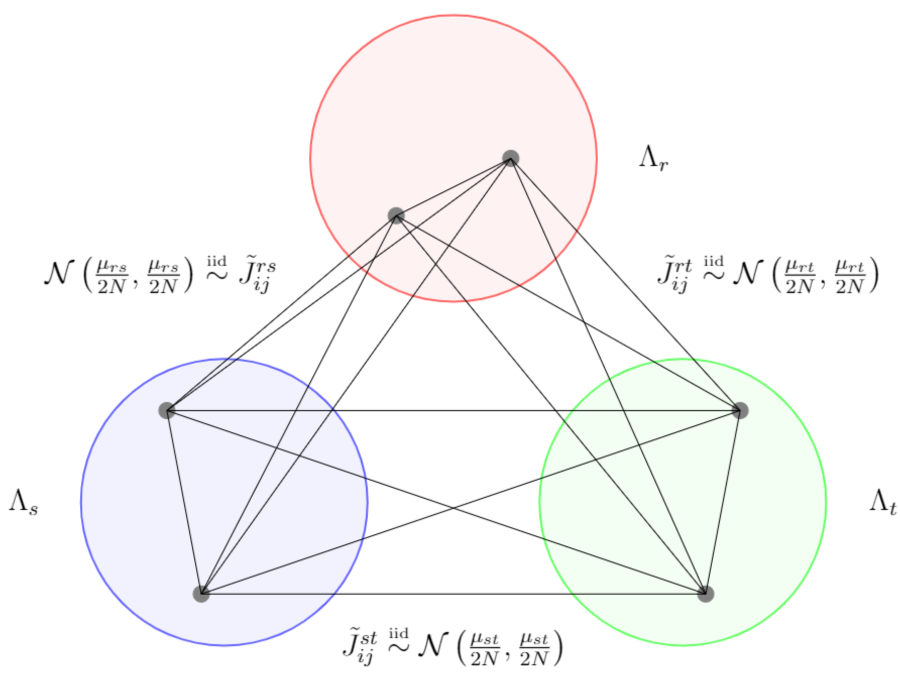

Figure 1: Scheme of the structure of the interactions.

With these notations and using the properties of Gaussian random variables the

Hamiltonian (2) is equivalent (in distribution) to the following:

$$H_N(\sigma) = -\frac{1}{\sqrt{2N}}\sum_{r,s=1}^{K}\sum_{(i,j)\in\Lambda_r\times\Lambda_s} J_{ij}^{rs}\sigma_i\sigma_j - \sum_{r=1}^{K}\sum_{i\in\Lambda_r} h_i^r\sigma_i - \frac{N}{2}(\mathbf{m},\Delta\mathbf{m}) - N(\hat{\alpha}\mathbf{h},\mathbf{m}) \tag{7}$$

$$J_{ij}^{rs} \overset{\text{iid}}{\sim} \mathcal{N}(0,\mu_{rs})\,, \qquad h_i^r \overset{\text{iid}}{\sim} \mathcal{N}(0,h_r)\ . \tag{8}$$

The last expression allows us to identify the model with a multi-species Sherrington-Kirkpatrick model (SK) with the addition of a ferromagnetic interaction and a positive external field whose intensity coincide with the variances of the random terms.

Now we define the main quantity under investigation, the random and average quenched pressure densities:

$$p_N := \frac{1}{N}\log\sum_{\sigma\in\Sigma_N}\exp\left(-H_N(\sigma)\right) \tag{9}$$

$$\bar{p}_N(\mu,h) := \mathbb{E}p_N \tag{10}$$

$$\tag{11}$$

where we emphasize the dependence of the quenched pressure on the mean parameters μ_{rs}, h and the symbol $\mathbb{E}$ stands for the Gaussian expectation with respect to the disorder. We also introduce the Gibbs expectation:

$$\langle\cdot\rangle_N := \frac{\sum_{\sigma\in\Sigma_N} e^{-H_N(\sigma)}(\cdot)}{Z_N}, \qquad Z_N := \sum_{\sigma\in\Sigma_N} e^{-H_N(\sigma)} \tag{12}$$

We will denote the dependence of the Gibbs measure on further parameters with subscripts or superscripts, for example $\langle\cdot\rangle_{N,t\dots}^{(\epsilon)}$. Notice that in this context the Gibbs measure is random.

2.1 Self-averaging of the pressure

We have an important concentration property of the pressure:

$$\mathbb{E}[(p_N - \bar{p}_N(\mu,h))^2] \le \frac{S}{N} \tag{13}$$

for a suitable constant S. This follows from Theorem 1.2 and equation (1.42) in [16]. In fact if we denote:

$$G_N(\sigma) := \exp\left(-\frac{N}{2}(\mathbf{m},\Delta\mathbf{m}) - N(\hat{\alpha}\mathbf{h},\mathbf{m})\right),$$

$$\hat{H}_N(\sigma) := -\frac{1}{\sqrt{2N}}\sum_{r,s=1}^{K}\sum_{(i,j)\in\Lambda_r\times\Lambda_s} J_{ij}^{rs}\sigma_i\sigma_j - \sum_{r=1}^{K}\sum_{i\in\Lambda_r} h_i^r\sigma_i$$

we can easily verify that they fulfill the hypothesis of the theorem for any given N:

$$\sum_{\sigma\in\Sigma_N} G_N(\sigma) \leq \exp\left[N\left(\log 2 + \frac{K}{2}\|\Delta\| + \|\hat{\alpha}\mathbf{h}\|\sqrt{K}\right)\right]$$
$$\mathbb{E}[\hat{H}_N^2(\sigma)] = N\left(\frac{(\mathbf{1},\Delta\mathbf{1})}{2} + (\hat{\alpha}\mathbf{h},\mathbf{1})\right) =: NC, \qquad \mathbf{1} = (1,1,\dots,1)$$

the latter follows from independence of the Gaussian variables. Now by Theorem 1.2 in [16] we infer that:

$$\mathbb{P}\left(|p_N - \bar{p}_N(\mu,h)| > x\right) = \mathbb{P}\left(N\,|p_N - \bar{p}_N(\mu,h)| > Nx\right) \leq$$
$$\leq 2\exp\left(-\frac{N^2x^2}{4NC}\right) = 2\exp\left(-\frac{Nx^2}{4C}\right), \quad \forall x > 0 \quad (14)$$

After a tail integration, and renaming constants ($S = 8C$), one easily gets the concentration (13). This property will be a crucial tool to prove replica symmetry when combined with the Nishimori identities introduced in the next section.

2.2 Nishimori identities and correlation inequalities

Here we will list some identities and inequalities proved in [12] and [13] that are valid for models on the Nishimori line, such as (2). The most useful for are purposes are the following:

$$\mathbb{E}[\langle\sigma_i\rangle_N^2] = \mathbb{E}[\langle\sigma_i\rangle_N] \quad (15)$$
$$\mathbb{E}[\langle\sigma_i\sigma_j\rangle_N^2] = \mathbb{E}[\langle\sigma_i\sigma_j\rangle_N] \quad (16)$$

for all $i,j\in\Lambda$. In particular they imply that:

$$\mathbb{E}[\langle q_s\rangle_N] = \sum_{i\in\Lambda_s}\frac{1}{N_s}\mathbb{E}[\langle\sigma_i\rangle_N\langle\tau_i\rangle_N] = \sum_{i\in\Lambda_s}\frac{1}{N_s}\mathbb{E}[\langle\sigma_i\rangle_N^2] = \sum_{i\in\Lambda_s}\frac{1}{N_s}\mathbb{E}[\langle\sigma_i\rangle_N] = \mathbb{E}[\langle m_s\rangle_N] \quad (17)$$

$$\mathbb{E}[\langle q_rq_s\rangle_N] = \sum_{(i,j)\in\Lambda_r\times\Lambda_s}\frac{\mathbb{E}[\langle\sigma_i\sigma_j\rangle_N\langle\tau_i\tau_j\rangle_N]}{N_rN_s} = \sum_{(i,j)\in\Lambda_r\times\Lambda_s}\frac{\mathbb{E}[\langle\sigma_i\sigma_j\rangle_N^2]}{N_rN_s} = \mathbb{E}[\langle m_rm_s\rangle_N] \quad (18)$$

and finally:

$$\mathbb{E}\Big\langle(\mathbf{q},\Delta\mathbf{q})\Big\rangle_N = \mathbb{E}\Big\langle(\mathbf{m},\Delta\mathbf{m})\Big\rangle_N \quad (19)$$

The previous identities show that the model has a unique order parameter, that can be regarded as a magnetization or equivalently an overlap. We will choose the first point of view. This intuitive statement will acquire a precise meaning when we will write down the sum rule for the quenched pressure.

Let us now move to correlation inequalities. By a simple computation and using the results proved in [12, 13, 6] we see that:

$$\frac{\partial \bar{p}_N}{\partial h_r} = \frac{1}{2N} \sum_{i \in \Lambda_r} \mathbb{E}[1 + \langle \sigma_i \rangle_N] = \frac{\alpha_r}{2}[1 + \mathbb{E}\langle m_r \rangle_N] \geq 0 \tag{20}$$

$$\frac{\partial^2 \bar{p}_N}{\partial h_r \partial h_s} = \frac{1}{2N} \sum_{(i,j) \in \Lambda_r \times \Lambda_s} \mathbb{E}[(\langle \sigma_i \sigma_j \rangle_N - \langle \sigma_i \rangle_N \langle \sigma_j \rangle_N)^2] \geq 0 \tag{21}$$

Analogous identities and inequalities hold for the first and second derivatives w.r.t. μ_{rs}. The generating function (pressure) and the first moments are monotonically increasing with respect to the Nishimori parameters μ_{rs}, h_r. Inequalities (20) and (21) are called respectively correlation inequalities of the I and II type in the Nishimori line [6]. In particular the magnetization is always increasing w.r.t. the external field mean:

$$\frac{\partial \mathbb{E}\langle m_r \rangle_N}{\partial h_s} \geq 0 \tag{22}$$

This monotonicity will be a key ingredient to prove replica symmetry.

3 Adaptive interpolation and sum rule

In this section we build up an interpolating model with some specific features. The method here employed is an extension of the standard Guerra-Toninelli interpolation [10], also called *adaptive interpolation technique*, developed in [2] by J. Barbier and N. Macris.

Definition 1 (Interpolating model)**.** Let $t \in [0, 1]$. The hamiltonian of the interpolating model is:

$$\begin{aligned} H_\sigma(t) := -\frac{\sqrt{1-t}}{\sqrt{2N}} \sum_{r,s=1}^{K} \sum_{(i,j) \in \Lambda_r \times \Lambda_s} J_{ij}^{rs} \sigma_i \sigma_j - (1-t)\frac{N}{2}(\mathbf{m}, \Delta \mathbf{m}) + \\ - \sum_{r=1}^{K} \sum_{i \in \Lambda_r} \left(\sqrt{Q_{\epsilon,r}(t)} J_i^r + Q_{\epsilon,r}(t) \right) \sigma_i - \sum_{r=1}^{K} \sum_{i \in \Lambda_r} h_i^r \sigma_i - N(\hat{\alpha}\mathbf{h}, \mathbf{m}) \end{aligned} \tag{23}$$

with $J_i^r \stackrel{\text{iid}}{\sim} \mathcal{N}(0,1)$ independent of all the other Gaussian random variables, and

$$\mathbf{Q}_\epsilon(t) := \boldsymbol{\epsilon} + \hat{\alpha}^{-1}\Delta \int_0^t \mathbf{q}_\epsilon(s)\, ds, \qquad \epsilon_r \in [s_N, 2s_N],\ s_N \propto N^{-\frac{1}{16K}} \ .$$

Here $\mathbf{Q}_{\epsilon,r} =: (Q_{\epsilon,r})_{r=1,\dots,K}$, while $\mathbf{q}_\epsilon := (q_{\epsilon,r})_{r=1,\dots,K}$ denotes a vector of K non-negative functions that will be suitably chosen in the following.

Remark 1. We notice that the interpolating model is on the Nishimori line for any $t \in [0,1]$. This can be seen directly from (23) by observing that the square of the multiplicative factor introduced for each centered r.v., for both the two and one body terms, coincides with the factor introduced for the corresponding deterministic term. Therefore the Nishimori identities (15), (16) and (19) can be use by replacing $\langle\cdot\rangle$ with the Gibbs measure induced by the interpolating hamiltonian (23), that is $\langle\cdot\rangle_{N,t}^{(\epsilon)}$. Notice also that the role played by the functions $\mathbf{Q}_\epsilon(t)$ is that of an external magnetic field.

The corresponding interpolating pressure will be denoted as

$$\bar{p}_{N,\epsilon}(t) := \frac{1}{N}\mathbb{E}\log\sum_\sigma e^{-H_\sigma(t)} \tag{24}$$

The following lemma will lead to the sum rule of the model.

Lemma 1 (Interpolating pressure at $t = 0, 1$)**.** *Setting*

$$\psi(Q) := \mathbb{E}_z \log 2\cosh\left[z\sqrt{Q} + Q\right], \qquad z \sim \mathcal{N}(0,1) \tag{25}$$

we have the following:

$$\bar{p}_{N,\epsilon}(1) \ = \ \sum_{r=1}^K \alpha_r\, \psi(Q_{\epsilon,r}(1) + h_r) \ = \ \mathcal{O}(s_N) + \sum_{r=1}^K \alpha_r \psi\left(\left(\hat{\alpha}^{-1}\Delta\int_0^1 \mathbf{q}_\epsilon(t)\, dt + \mathbf{h}\right)_r\right) \tag{26}$$

$$\bar{p}_{N,\epsilon}(0) \ = \ \mathcal{O}(s_N) + \bar{p}_N(\mu, h) \tag{27}$$

Proof. Each ϵ_r can be regarded as the mean (or variance) of a small magnetic field.

At $t = 1$ the system is *free*, non interacting. Its pressure can be explicitly computed. Take $z_i^r \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. Then:

$$\begin{aligned}\bar{p}_{N,\epsilon}(1) = \frac{1}{N}\mathbb{E}\log\prod_{r=1}^{K}\sum_{\sigma\in\Sigma_{N_r}}\exp\Bigg(\sum_{r=1}^{K}\sum_{i\in\Lambda_r}\left(\sqrt{Q_{\epsilon,r}(1)}J_i^r + Q_{\epsilon,r}(1)\right)\sigma_i + \\ + \sum_{r=1}^{K}\sum_{i\in\Lambda_r}(\sqrt{h_r}z_i^r + h_r)\sigma_i\Bigg) = \\ = \sum_{r=1}^{K}\frac{\alpha_r}{N_r}\mathbb{E}\log\sum_{\sigma\in\Sigma_{N_r}}\exp\left(\sum_{r=1}^{K}\sum_{i\in\Lambda_r}\left(J_i^r\sqrt{Q_{\epsilon,r}(1)+h_r} + Q_{\epsilon,r}(1)+h_r\right)\sigma_i\right)\end{aligned}$$

where we have used the rule for the sum of independent gaussian at the exponent. Finally:

$$\bar{p}_{N,\epsilon}(1) = \sum_{r=1}^{K}\alpha_r\mathbb{E}_z\log 2\cosh\left[z\sqrt{Q_{\epsilon,r}(1)+h_r} + Q_{\epsilon,r}(1)+h_r\right], \quad z\sim\mathcal{N}(0,1)$$

Since the derivatives of the pressure w.r.t. magnetic fields are bounded by α_r, we can get rid of the explicit dependence on ϵ_r at the expense of a term $\mathcal{O}(s_N)$, thus getting (26).

By setting $t = 0$, the interpolating Hamiltonian simply reduces to the original one (7) except for the ϵ_r's that can be neglected again at the expense of terms $\mathcal{O}(s_N)$ for the same reason. □

Proposition 2 (Sum rule)**.** *The quenched pressure of the model* (10) *obeys to the following sum rule:*

$$\begin{aligned}\bar{p}_N(\mu,h) = \mathcal{O}(s_N) + \sum_{r=1}^{K}\alpha_r\psi(Q_{\epsilon,r}(1)+h_r) + \\ + \int_0^1 dt\left[\frac{(\mathbf{1}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{1}-\mathbf{q}_\epsilon(t)))}{4} - \frac{(\mathbf{q}_\epsilon(t),\Delta\mathbf{q}_\epsilon(t))}{2}\right] + \frac{1}{4}\int_0^1 dt\,R_\epsilon(t,\mu,h)\end{aligned} \quad (28)$$

where the remainder is:

$$R_\epsilon(t,\mu,h) = \mathbb{E}\Big\langle(\mathbf{m}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{m}-\mathbf{q}_\epsilon(t)))\Big\rangle_{N,t}^{(\epsilon)} \quad (29)$$

Proof. The proof consists in computing the first derivative by using Gaussian integration by parts for the terms containing the disorder.

$$\dot{\bar{p}}_{N,\epsilon}(t) = -\frac{1}{4}\mathbb{E}\Big\langle(\mathbf{1},\Delta\mathbf{1})-(\mathbf{q},\Delta\mathbf{q})\Big\rangle_{N,t}^{(\epsilon)} - \frac{1}{2}\mathbb{E}\Big\langle(\mathbf{m}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{m}-\mathbf{q}_\epsilon(t)))\Big\rangle_{N,t}^{(\epsilon)}+$$
$$+\frac{1}{2}(\mathbf{q}_\epsilon(t),\Delta\mathbf{q}_\epsilon(t))+\frac{1}{2}\mathbb{E}\Big\langle(\mathbf{1},\Delta\mathbf{q}_\epsilon(t))-(\mathbf{q}_\epsilon(t),\Delta\mathbf{q})\Big\rangle_{N,t}^{(\epsilon)} = -\frac{1}{4}(\mathbf{1}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{1}-\mathbf{q}_\epsilon(t)))+$$
$$+\frac{1}{2}(\mathbf{q}_\epsilon(t),\Delta\mathbf{q}_\epsilon(t))+\frac{1}{4}\mathbb{E}\Big\langle(\mathbf{q}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{q}-\mathbf{q}_\epsilon(t)))\Big\rangle_{N,t}^{(\epsilon)}-\frac{1}{2}\mathbb{E}\Big\langle(\mathbf{m}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{m}-\mathbf{q}_\epsilon(t)))\Big\rangle_{N,t}^{(\epsilon)}$$

Using the Nishimori identities (15) and (16) we can sum the last two terms together:

$$\dot{\bar{p}}_{N,\epsilon}(t) = -\frac{1}{4}(\mathbf{1}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{1}-\mathbf{q}_\epsilon(t))) + \frac{1}{2}(\mathbf{q}_\epsilon(t),\Delta\mathbf{q}_\epsilon(t))+ \\ -\frac{1}{4}\underbrace{\mathbb{E}\Big\langle(\mathbf{m}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{m}-\mathbf{q}_\epsilon(t)))\Big\rangle_{N,t}^{(\epsilon)}}_{R_\epsilon(t,\mu,h)} \tag{30}$$

The sum rule then follows from a simple application of the Fundamental Theorem of Calculus and the previous Lemma:

$$\bar{p}_{N,\epsilon}(0) = \mathcal{O}(s_N) + \bar{p}_N(\mu,h) = \bar{p}_{N,\epsilon}(1) - \int_0^1 dt\,\dot{\bar{p}}_{N,\epsilon}(t) \tag{31}$$

□

4 Solution of the model

In this section we present the main result of the paper, namely the thermodynamic limit of the model under the hypothesis of a positive semi-definite effective interaction matrix: $\Delta \geq 0$. First, we need a couple of lemmas listed below.

Lemma 3 (Liouville's formula)**.** *Consider two matrices whose elements depend on a real parameter:* $\Phi(t)$, $A(t)$. *Suppose that* Φ *satisfies:*

$$\dot{\Phi}(t) = A(t)\Phi(t) \tag{32}$$
$$\Phi(0) = \Phi_0 \tag{33}$$

Then:

$$\det(\Phi(t)) = \det(\Phi_0)\exp\left\{\int_0^t ds\,\mathrm{Tr}(A(s))\right\} \tag{34}$$

Definition 2 (Regularity of $\boldsymbol{\epsilon} \longmapsto \mathbf{Q}_\epsilon(\cdot)$)**.** We will say that the map $\boldsymbol{\epsilon} \longmapsto \mathbf{Q}_\epsilon(\cdot)$ is *regular* if

$$\det\left(\frac{\partial \mathbf{Q}_\epsilon(t)}{\partial \boldsymbol{\epsilon}}\right) \geq 1 \quad \forall t \in [0,1] \tag{35}$$

Remark 2. Choosing $\mathbf{Q}_\epsilon$ as the solution of the following ODE

$$\dot{\mathbf{Q}}_\epsilon(t) = \hat{\alpha}^{-1}\Delta\,\mathbb{E}\langle \mathbf{m}\rangle^{(\epsilon)}_{N,t}, \qquad \mathbf{Q}_\epsilon(0) = \boldsymbol{\epsilon} \tag{36}$$

the map $\epsilon \longmapsto \mathbf{Q}_\epsilon(\cdot)$ turns out to be regular. Indeed

$$\frac{d}{dt}\frac{\partial \mathbf{Q}_\epsilon(t)}{\partial \boldsymbol{\epsilon}} = \underbrace{\frac{\partial}{\partial \mathbf{Q}_\epsilon(t)}\hat{\alpha}^{-1}\Delta\mathbb{E}\langle \mathbf{m}\rangle^{(\epsilon)}_{N,t}}_{=:A(t)}\frac{\partial \mathbf{Q}_\epsilon(t)}{\partial \boldsymbol{\epsilon}}\ ; \tag{37}$$

since $Q_{\epsilon,r}(t)$ play the role of magnetic fields and the entries of $\hat{\alpha}^{-1}$ and Δ are non-negative we have

$$\mathrm{Tr}A(t) \geq 0\,, \tag{38}$$

by the correlation inequalities of type II (21), (22). Finally using Liouville's formula we get:

$$\det\left(\frac{\partial \mathbf{Q}_\epsilon(t)}{\partial \boldsymbol{\epsilon}}\right) = \underbrace{\det\left(\frac{\partial \mathbf{Q}_\epsilon(0)}{\partial \boldsymbol{\epsilon}}\right)}_{=\mathbb{1}}\exp\left\{\int_0^t ds\,\mathrm{Tr}(A(s))\right\} \geq 1 \tag{39}$$

We stress that the sign of Δ plays no role yet, since we have used only the positivity of its entries so far.

Lemma 4 (Concentration)**.** *Suppose $\boldsymbol{\epsilon} \longmapsto \mathbf{Q}_\epsilon(\cdot)$ is a regular map. Consider the quantity:*

$$\mathcal{L}_r := \frac{1}{N_r}\sum_{i\in\Lambda_r}\left(\sigma_i + \frac{J_i^r\sigma_i}{2\sqrt{Q_{\epsilon,r}(t)}}\right), \qquad J_i^r \overset{\text{iid}}{\sim} \mathcal{N}(0,1) \tag{40}$$

and introduce the $\boldsymbol{\epsilon}$-average:

$$\mathbb{E}_{\boldsymbol{\epsilon}}[\cdot] = \prod_{r=1}^{K}\left(\frac{1}{s_N}\int_{s_N}^{2s_N} d\epsilon_r\right)(\cdot)\ . \tag{41}$$

We have:

$$\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}\Big\langle \Big(\mathcal{L}_r - \mathbb{E}\langle\mathcal{L}_r\rangle^{(\epsilon)}_{N,t}\Big)^2 \Big\rangle^{(\epsilon)}_{N,t} \longrightarrow 0\,, \quad \text{when } N\to\infty \tag{42}$$

and

$$\mathbb{E}\Big\langle \Big(\mathcal{L}_r - \mathbb{E}\langle\mathcal{L}_r\rangle^{(\epsilon)}_{N,t}\Big)^2 \Big\rangle^{(\epsilon)}_{N,t} \geq \frac{1}{4}\mathbb{E}\Big\langle \Big(m_r - \mathbb{E}\langle m_r\rangle^{(\epsilon)}_{N,t}\Big)^2 \Big\rangle^{(\epsilon)}_{N,t} \tag{43}$$

therefore the magnetization (or the overlap) concentrates in $\boldsymbol{\epsilon}$-average.

The proof, simple but lengthy (see the Appendix), is based on controlling the thermal and disorder-related fluctuations of $\mathcal{L}_r$. This implies the control of the fluctuations of the magnetization thus ensuring the replica symmetry of the model, which is independent of the sign of Δ and depends only on the positivity of its elements.

We have laid the ground for our main result: the computation of the quenched pressure in the thermodynamic limit in form of a finite dimensional (due to the concentration lemma) variational principle.

Theorem 5 (Thermodynamic limit)**.** *In the Nishimori line, when $\Delta \geq 0$, the thermodynamic limit of the pressure $\bar{p}(\mu,h) := \lim_{N\to\infty}\bar{p}_N(\mu,h)$ exists and:*

$$\bar{p}(\mu,h) = \sup_{\mathbf{x}\in\mathbb{R}^K_{\geq 0}} \bar{p}(\mu,h;\mathbf{x}) \tag{44}$$

where

$$\bar{p}(\mu,h;\mathbf{x}) := \frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r) \tag{45}$$

with the following stationary condition:

$$\mathbf{x} - \mathbb{E}_z \tanh\Big(z\sqrt{\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}} + \hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}\Big) \in Ker\Delta\,, \quad z\sim\mathcal{N}(0,1) \tag{46}$$

Proof of Theorem 5. Let us divide the proof in two steps.

Lower Bound: We initially fix $\mathbf{q}_\epsilon(t) = \mathbf{x} \in \mathbb{R}^K_{\geq 0}$ in (28). Up to orders $\mathcal{O}(s_N)$ we find:

$$\begin{aligned}\bar{p}_N(\mu,h) &= \mathcal{O}(s_N) + \frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \\ &+ \sum_{r=1}^{K}\alpha_r\mathbb{E}_z\log 2\cosh\Big(z\sqrt{(\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r} + (\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r\Big) + \frac{1}{4}\int_0^1 dt\, R_\epsilon(t,\mu,h)\end{aligned} \tag{47}$$

We have exploited the result in Lemma 1. Being Δ positive semi-definite, the rest has a positive sign, for it is a quadratic form exactly with matrix Δ.

Hence:

$$\bar{p}_N(\mu,h) \geq \mathcal{O}(s_N) + \frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r)$$

Then, taking the $\liminf_{N\to\infty}$ on both sides and optimizing with $\sup_{\mathbf{x}}$ we get the first bound:

$$\liminf_{N\to\infty}\bar{p}_N(\mu,h) \geq \sup_{\mathbf{x}}\left\{\frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r)\right\} \tag{48}$$

Upper Bound: We start with a key observation: $\psi(\cdot)$ is a convex function. It can be seen as a consequence of the correlation inequality (20) on the Nishimori line. In fact, $\psi(Q)$ can be recast in the following way:

$$\psi(Q) = \mathbb{E}_z \log\sum_{\sigma=\pm1} e^{\sigma(z\sqrt{Q}+Q)} = \mathbb{E}_{z(Q)}\log\sum_{\sigma=\pm1} e^{\sigma z(Q)},\ z(Q)\sim\mathcal{N}(Q,Q)$$

This is a simple 1-particle, free system on the Nishimori line. For this model we have:

$$\frac{\partial\psi}{\partial Q} = \frac{1}{2}\mathbb{E}_z[1+\langle\sigma\rangle]\,,\quad \frac{\partial^2\psi}{\partial Q^2} = \frac{1}{2}\mathbb{E}[(1-\langle\sigma\rangle^2)^2]\,,\quad \langle\sigma\rangle = \frac{\sum_{\sigma=\pm1}e^{\sigma z(Q)}\sigma}{\sum_{\sigma=\pm1}e^{\sigma z(Q)}} \tag{49}$$

This allows us to use Jensen's inequality to extract the integral in $Q_{\epsilon,r}(1)$ from the "free term" of the pressure (28). Thanks to Lemma 1, we have that up to $\mathcal{O}(s_N)$:

$$\begin{aligned}\bar{p}_N(\mu,h) \leq{}& \mathcal{O}(s_N) + \int_0^1 dt\,\Bigg[\frac{(\mathbf{1}-\mathbf{q}_\epsilon(t),\Delta(\mathbf{1}-\mathbf{q}_\epsilon(t)))}{4} - \frac{(\mathbf{q}_\epsilon(t),\Delta\mathbf{q}_\epsilon(t))}{2} + \\ &+ \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{q}_\epsilon(t)+\mathbf{h})_r)\Bigg] + \frac{1}{4}\int_0^1 dt\,R_\epsilon(t,\mu,h) \leq \\ \leq{}& \mathcal{O}(s_N) + \sup_{\mathbf{x}}\left\{\frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r)\right\} + \\ &+ \frac{1}{4}\int_0^1 dt\,R_\epsilon(t,\mu,h)\end{aligned} \tag{50}$$

If we finally take the expectation $\mathbb{E}_{\boldsymbol{\epsilon}}$ on both sides of the previous inequality we get:

$$\begin{aligned}\bar{p}_N(\mu,h) \leq \mathcal{O}(s_N) + \sup_{\mathbf{x}}\Bigg\{ \frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{2} - (\mathbf{x},\Delta\mathbf{x}) + \\ + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r)\Bigg\} + \frac{1}{2}\mathbb{E}_{\boldsymbol{\epsilon}}\int_0^1 dt\, R_\epsilon(t,\mu,h) \quad (51)\end{aligned}$$

This time we choose $\mathbf{q}_\epsilon(t)$ according to a different criterion. We would like to have: $\Delta\mathbf{q}_\epsilon(t) = \Delta\mathbb{E}\langle\mathbf{m}\rangle^{(\epsilon)}_{N,t}$. In this way we could use the concentration Lemma 4. This can be achieved through the following ODE:

$$\dot{\mathbf{Q}}_\epsilon(t) = \hat{\alpha}^{-1}\Delta\mathbb{E}\langle\mathbf{m}\rangle^{(\epsilon)}_{N,t} =: \mathbf{F}(t,\mathbf{Q}_\epsilon(t)), \quad \mathbf{Q}_\epsilon(0) = \boldsymbol{\epsilon} \quad (52)$$

As seen in (21), the derivatives of $\mathbf{F}$ are positive and bounded for any fixed N. This guarantees the existence of a unique solution over $[0,1]$.

Then, exchanging the two integrals by Fubini's theorem in (51), and applying the concentration result we get:

$$\limsup_{N\to\infty}\bar{p}_N(\mu,h) \leq \sup_{\mathbf{x}}\left\{\frac{(\mathbf{1}-\mathbf{x},\Delta(\mathbf{1}-\mathbf{x}))}{4} - \frac{(\mathbf{x},\Delta\mathbf{x})}{2} + \sum_{r=1}^{K}\alpha_r\psi((\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h})_r)\right\}$$

The two bounds match and this proves (44). Moreover, using the properties (49) the gradient of (44) is:

$$\begin{aligned}\nabla_{\mathbf{x}}\bar{p}(\mu,h;\mathbf{x}) &= -\frac{\Delta}{2}(\mathbf{1}-\mathbf{x}) - \Delta\mathbf{x} + \frac{\Delta}{2}\mathbf{1} + \\ &+ \frac{\Delta}{2}\mathbb{E}_z\tanh\left(z\sqrt{\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}} + \hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}\right) = \\ &= \frac{\Delta}{2}\left[-\mathbf{x} + \mathbb{E}_z\tanh\left(z\sqrt{\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}} + \hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}\right)\right] \quad (53)\end{aligned}$$

and it vanishes exactly when (46) holds. □

Proposition 6. *Let Δ be strictly positive definite in* (45). *Denote by $\rho(A)$ the spectral radius of a matrix A and by $\mathscr{H}_{\mathbf{x}}$ the Hessian matrix operator. The following implication holds:*

$$\rho(\hat{\alpha}^{-1}\Delta) < 1 \quad \Rightarrow \quad \mathscr{H}_{\mathbf{x}}\bar{p}(\mu,h;\mathbf{x}) < 0,\ \forall\mathbf{x}\in\mathbb{R}^K_{\geq 0} \quad (54)$$

or equivalently $\bar{p}(\mu,h;\mathbf{x})$ is strictly concave w.r.t. $\mathbf{x}$.

Proof. The Hessian matrix can be computed starting from the gradient (53) and using properties (49):

$$\mathscr{H}_{\mathbf{x}}\bar{p}(\mu, h; \mathbf{x}) = -\frac{\Delta}{2} + \frac{1}{2}\Delta\mathcal{D}(\mathbf{x},\mathbf{h})\hat{\alpha}^{-1}\Delta = \frac{1}{2}\Delta^{1/2}\left[-\mathbb{1} + \Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right]\Delta^{1/2} \tag{55}$$

$$\mathcal{D}(\mathbf{x},\mathbf{h}) := \operatorname{diag}\left\{\mathbb{E}_z\left[\left(1-\tanh^2\left(z\sqrt{\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}}+\hat{\alpha}^{-1}\Delta\mathbf{x}+\mathbf{h}\right)_r\right)^2\right]\right\}_{r=1,\dots,K} \tag{56}$$

By similarity we have:

$$\begin{aligned}\rho\left(\Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right) &= \rho(\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta) = \\ &= \rho\left(\mathcal{D}^{1/2}(\mathbf{x},\mathbf{h})\hat{\alpha}^{-1/2}\Delta\hat{\alpha}^{-1/2}\mathcal{D}^{1/2}(\mathbf{x},\mathbf{h})\right)\end{aligned} \tag{57}$$

Now we use the fact that spectral radius and matrix 2-norm are equal for symmetric matrices:

$$\begin{aligned}&\rho\left(\Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right) = \|\mathcal{D}^{1/2}(\mathbf{x},\mathbf{h})\hat{\alpha}^{-1/2}\Delta\hat{\alpha}^{-1/2}\mathcal{D}^{1/2}(\mathbf{x},\mathbf{h})\| \leq \\ &\leq \|\mathcal{D}(\mathbf{x},\mathbf{h})\|\|\hat{\alpha}^{-1/2}\Delta\hat{\alpha}^{-1/2}\| = \|\mathcal{D}(\mathbf{x},\mathbf{h})\|\rho\left(\hat{\alpha}^{-1/2}\Delta\hat{\alpha}^{-1/2}\right) \leq \rho\left(\hat{\alpha}^{-1/2}\Delta\hat{\alpha}^{-1/2}\right)\end{aligned} \tag{58}$$

Finally, again by similarity:

$$\rho\left(\Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right) \leq \rho\left(\hat{\alpha}^{-1}\Delta\right) < 1 \tag{59}$$

by hypothesis. The previous one implies that:

$$-\mathbb{1} + \Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2} < 0 \tag{60}$$

whence, for any test vector $\mathbf{v}$:

$$\begin{aligned}&\left(\mathbf{v}, \Delta^{1/2}\left[-\mathbb{1} + \Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right]\Delta^{1/2}\mathbf{v}\right) = \\ &\qquad = \left(\Delta^{1/2}\mathbf{v}, \left[-\mathbb{1} + \Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x},\mathbf{h})\Delta^{1/2}\right](\Delta^{1/2}\mathbf{v})\right) < 0\,.\end{aligned} \tag{61}$$

□

Remark 3. The previous proposition further implies that, whenever Δ is invertible, $\mathbf{h} = 0$ and $\rho(\hat{\alpha}^{-1}\Delta) < 1$, the point $\mathbf{x} = 0$ is the unique maximizer of (45). On the contrary, when $\rho(\hat{\alpha}^{-1}\Delta) > 1$ we have

$$\mathscr{H}_{\mathbf{x}}\bar{p}(\mu, 0; 0) = \frac{1}{2}\Delta^{1/2}\left[-\mathbb{1} + \Delta^{1/2}\hat{\alpha}^{-1}\Delta^{1/2}\right]\Delta^{1/2}$$

and the matrix in square brackets has at least one positive eigenvalue, therefore $\mathbf{x} = 0$ becomes an unstable saddle point for the variational pressure. Notice that this instability can be generated both varying the parameters Δ_{rs} and the form factors α_r.

Remark 4. If Δ is non singular, our variational pressure (45) goes to $-\infty$ as $\|\mathbf{x}\| \to \infty$, because the concave quadratic form always dominates the sum of the gas terms containing ψ, which is Lispchitz with $\mathrm{Lip}(\psi) \leq 1$ (again by (49)). This, together with the regularity of $\bar{p}$ ensures that there is a global maximum satisfying the fixed point equation:

$$\mathbf{x} = \mathbb{E}_z \tanh\left(z\sqrt{\hat{\alpha}^{-1}\Delta\mathbf{x} + \mathbf{h}} + \hat{\alpha}^{-1}\Delta\mathbf{x} + \mathbf{h}\right) =: \mathbf{T}(\mathbf{x}; \mathbf{h}) \,. \tag{62}$$

The Jacobian matrix of $\mathbf{T}(\cdot; \mathbf{h})$ is:

$$\mathrm{D}\mathbf{T}(\mathbf{x}; \mathbf{h}) = \mathcal{D}(\mathbf{x}, \mathbf{h})\hat{\alpha}^{-1}\Delta \tag{63}$$

and satisfies:

$$\rho(\mathrm{D}\mathbf{T}(\mathbf{x}; \mathbf{h})) = \rho(\mathcal{D}(\mathbf{x}, \mathbf{h})\hat{\alpha}^{-1}\Delta) \leq \rho(\hat{\alpha}^{-1}\Delta) \tag{64}$$

as proved in Proposition 6. Equality holds at $\mathbf{h} = 0$ and $\mathbf{x} = 0$. Hence when $\rho(\hat{\alpha}^{-1}\Delta) < 1$ the iteration of $\mathbf{T}(\cdot; \mathbf{h})$ converges to a fixed point. If this does not hold, we still have that at one local maximum point, say $\mathbf{x}^*$:

$$\mathscr{H}_{\mathbf{x}}\bar{p}(\mu, h; \mathbf{x}^*) < 0 \quad \text{or equivalently} \quad \rho(\Delta^{1/2}\hat{\alpha}^{-1}\mathcal{D}(\mathbf{x}^*, \mathbf{h})\Delta^{1/2}) = \rho(\mathrm{D}\mathbf{T}(\mathbf{x}^*; \mathbf{h})) < 1 \,. \tag{65}$$

The latter implies that the iteration $\mathbf{x}_{n+1} = \mathbf{T}(\mathbf{x}_n; \mathbf{h})$ converges to $\mathbf{x}^*$ (locally) provided that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ with δ sufficiently small.

Remark 5. Our parameters lie in $\mathbb{R}^K_{\geq 0}$, thus the vanishing gradient condition *a priori* allows us only to find maximizers of (45) in the interior, namely when $x_r > 0\,\forall\, r = 1, \dots, K$. More rigorously, the necessary conditions for a point $\bar{\mathbf{x}} \in \mathbb{R}^K_{\geq 0}$ to be a maximizer are:

$$\begin{cases} \partial_{x_r}\bar{p}(\mu, h; \bar{\mathbf{x}}) = \frac{1}{2}\left[\Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}))\right]_r \leq 0 \\ \bar{x}_r\partial_{x_r}\bar{p}(\mu, h; \bar{\mathbf{x}}) = \frac{1}{2}x_r\left[\Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}))\right]_r = 0 \end{cases} \tag{66}$$

If we notice that $T_r(\mathbf{x}; \mathbf{h}) \geq 0$ these conditions imply:

$$\begin{cases} (\mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}), \Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}))) \leq 0 \\ (\bar{\mathbf{x}}, \Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}))) = 0 \end{cases} \quad \Rightarrow \quad (-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}), \Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}}; \mathbf{h}))) \leq 0 \,. \tag{67}$$

However, since $\Delta > 0$ we must necessarily have:

$$(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}};\mathbf{h}), \Delta(-\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}};\mathbf{h}))) = 0 \quad \Leftrightarrow \quad -\bar{\mathbf{x}} + \mathbf{T}(\bar{\mathbf{x}};\mathbf{h}) = 0 \,. \tag{68}$$

From the previous we can see that the consistency equation (62) is necessarily satisfied also by maximizers on the boundary.

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A Appendix: Proof of the Concentration Lemma

Proof of Lemma 4. Let us split the proof into three steps for the sake of clarity. As anticipated, it is convenient to split the total fluctuation of $\mathcal{L}_r$ into two parts, thus proving that:

$$\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}\Big\langle \Big(\mathcal{L}_r - \langle \mathcal{L}_r\rangle_{N,t}^{(\epsilon)}\Big)^2 \Big\rangle_{N,t}^{(\epsilon)} \longrightarrow 0 \quad \text{as } N \to \infty \tag{A.1}$$

$$\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}\left(\langle \mathcal{L}_r\rangle_{N,t}^{(\epsilon)} - \mathbb{E}\langle \mathcal{L}_r\rangle_{N,t}^{(\epsilon)}\right)^2 \longrightarrow 0 \quad \text{as } N \to \infty \,. \tag{A.2}$$

From this moment on, we neglect sub and superscripts in the Gibbs brackets as well as time dependencies. We start by proving the last inequality (43).

Proof of inequality (43)**:** To begin with, we compute:

$$\mathbb{E}[\langle \mathcal{L}_r\rangle] = \frac{1}{N_r}\sum_{i\in\Lambda_r}\mathbb{E}\Big\langle \sigma_i + \frac{J_i^r\sigma_i}{2\sqrt{Q_{\epsilon,r}}}\Big\rangle = \mathbb{E}\langle m_r\rangle + \frac{1}{2}\mathbb{E}[1-\langle m_r\rangle] = \frac{1}{2}\mathbb{E}[1+\langle m_r\rangle] \tag{A.3}$$

$$(\mathbb{E}\langle \mathcal{L}_r\rangle)^2 = \frac{1}{4} + \frac{1}{2}\mathbb{E}\langle m_r\rangle + \frac{1}{4}(\mathbb{E}\langle m_r\rangle)^2 \tag{A.4}$$

where integration by parts has been used.

Then, we proceed with:

$$\mathbb{E}\langle \mathcal{L}_r^2\rangle = \frac{1}{N_r^2}\sum_{i,j\in\Lambda_r}\mathbb{E}\Big\langle \sigma_i\sigma_j + \frac{J_i^r\sigma_i\sigma_j}{\sqrt{Q_{\epsilon,r}}} + \frac{J_i^rJ_j^r\sigma_i\sigma_j}{4Q_{\epsilon,r}}\Big\rangle = \underbrace{\mathbb{E}\langle m_r^2\rangle}_{R_1} + \underbrace{\frac{1}{N_r^2}\sum_{i,j\in\Lambda_r}\mathbb{E}\Big\langle\frac{J_i^r\sigma_i\sigma_j}{\sqrt{Q_{\epsilon,r}}}\Big\rangle}_{R_2} + \underbrace{\frac{1}{N_r^2}\sum_{i,j\in\Lambda_r}\mathbb{E}\Big\langle\frac{J_i^rJ_j^r\sigma_i\sigma_j}{4Q_{\epsilon,r}}\Big\rangle}_{R_3} \tag{A.5}$$

We treat the three pieces R_1, R_2 and R_3 separately with repeated integrations by parts.

$$R_2 = \frac{1}{N_r^2} \sum_{i,j\in\Lambda_r} \mathbb{E}\left[\langle\sigma_j\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_i\rangle\right] = \mathbb{E}\langle m_r\rangle - \mathbb{E}\langle m_r\rangle^2 \tag{A.6}$$

We have used the Nishimori identity: $\mathbb{E}[\langle\sigma_i\rangle\langle\sigma_i\sigma_j\rangle] = \mathbb{E}[\langle\sigma_i\rangle\langle\sigma_j\rangle]$

$$R_3 = \frac{1}{4N_r^2} \sum_{i,j\in\Lambda_r} \mathbb{E}\left[\frac{\delta_{ij}\overbrace{\langle\sigma_i\sigma_j\rangle}^{=1}}{Q_{\epsilon,r}} + J_j^r \frac{(\langle\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_i\sigma_j\rangle)}{\sqrt{Q_{\epsilon,r}}}\right] = \frac{1}{4N_rQ_{\epsilon,r}} +$$

$$+ \frac{1}{4N_r^2} \sum_{i,j\in\Lambda_r} \mathbb{E}\left[1 - \langle\sigma_j\rangle^2 - \langle\sigma_i\rangle(\langle\sigma_i\rangle - \langle\sigma_j\rangle\langle\sigma_i\sigma_j\rangle) - \langle\sigma_i\sigma_j\rangle(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle)\right] =$$

$$= \frac{1}{4N_rQ_{\epsilon,r}} + \frac{1}{4} - \frac{1}{2}\mathbb{E}\langle m_r\rangle + \frac{1}{2}\mathbb{E}\langle m_r\rangle^2 - \frac{1}{4}\mathbb{E}\langle m_r^2\rangle \tag{A.7}$$

Hence:

$$R_1 + R_2 + R_3 = \frac{1}{4} + \frac{1}{4N_rQ_{\epsilon,r}} + \frac{3}{4}\mathbb{E}\langle m_r^2\rangle + \frac{1}{2}\mathbb{E}\langle m_r\rangle - \frac{1}{2}\mathbb{E}\langle m_r\rangle^2 \tag{A.8}$$

Summing up all the contributions:

$$\mathbb{E}\langle\mathcal{L}_r^2\rangle - (\mathbb{E}\langle\mathcal{L}_r\rangle)^2 = \frac{1}{4N_rQ_{\epsilon,r}} + \frac{3}{4}\mathbb{E}\langle m_r^2\rangle - \frac{1}{2}\mathbb{E}\langle m_r\rangle^2 - \frac{1}{4}(\mathbb{E}\langle m_r\rangle)^2 =$$

$$= \frac{1}{4N_rQ_{\epsilon,r}} + \frac{1}{4}(\mathbb{E}\langle m_r^2\rangle - (\mathbb{E}\langle m_r\rangle)^2) + \frac{1}{2}(\mathbb{E}\langle m_r^2\rangle - \mathbb{E}\langle m_r\rangle^2) \geq \frac{1}{4}\mathbb{E}\Big\langle(m_r - \mathbb{E}\langle m_r\rangle)^2\Big\rangle \tag{A.9}$$

Proof of (A.1)**:** Notice that:

$$\frac{\partial\bar{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}} = \frac{1}{N}\mathbb{E}\Big\langle \sum_{i\in\Lambda_r}\left(\sigma_i + \frac{J_i^r\sigma_i}{2\sqrt{Q_{\epsilon,r}}}\right)\Big\rangle = \alpha_r\mathbb{E}\langle\mathcal{L}_r\rangle = \frac{\alpha_r}{2}\mathbb{E}[1 + \langle m_r\rangle], \qquad J_i^r \overset{\text{iid}}{\sim} \mathcal{N}(0,1) \tag{A.10}$$

$$\frac{\partial^2\bar{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}^2} = \alpha_r N_r\mathbb{E}\langle(\mathcal{L}_r - \langle\mathcal{L}_r\rangle)^2\rangle - \frac{1}{4NQ_{\epsilon,r}^{3/2}}\sum_{i\in\Lambda_r}\mathbb{E}\langle J_i^r\sigma_i\rangle \tag{A.11}$$

From the last one, after an integration by parts and using the regularity of the map $\epsilon \longmapsto \mathbf{Q}_\epsilon(\cdot)$ and Liouville's formula we get:

$$\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}\langle(\mathcal{L}_r - \langle\mathcal{L}_r\rangle)^2\rangle \leq \frac{1}{N_r\alpha_r s_N^K}\prod_{s=1}^{K}\int_{Q_{s_N,s}}^{Q_{2s_N,s}} dQ_{\epsilon,s}\frac{\partial^2\bar{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}^2} + \mathbb{E}_{\boldsymbol{\epsilon}}\frac{1}{4N_r\epsilon_r}\mathbb{E}[1-\langle m_r\rangle] \leq$$

$$\leq \frac{1}{N_r\alpha_r s_N^K}\prod_{s\neq r,1}^{K}\int_{Q_{s_N,s}}^{Q_{2s_N,s}} dQ_{\epsilon,s}\left[\frac{\partial\bar{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}}\bigg|_{Q_{2s_N,r}} - \frac{\partial\bar{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}}\bigg|_{Q_{s_N,r}}\right] + \frac{\log 2}{4N_r s_N} \leq$$

$$\leq \frac{2K_r(\Delta)}{N_r s_N^K} + \frac{\log 2}{4N_r s_N} = \mathcal{O}\left(\frac{1}{N_r s_N^K}\right) \longrightarrow 0 \quad \text{(A.12)}$$

where:

$$\prod_{s\neq r,1}^{K}(Q_{2s_N,s} - Q_{s_N,s}) \leq K_r(\Delta) \quad \text{(A.13)}$$

Proof of (A.2)**:** Let $p_{N,\epsilon}$ be the random interpolating pressure, such that $\mathbb{E}p_{N,\epsilon} = \bar{p}_{N,\epsilon}$. Define:

$$\hat{p}_{N,\epsilon} = p_{N,\epsilon} - \alpha_r\sqrt{Q_{\epsilon,r}}\sum_{i\in\Lambda_r}\frac{|J_i^r|}{N_r}, \qquad \hat{\bar{p}}_{N,\epsilon} = \mathbb{E}\hat{p}_{N,\epsilon} \quad \text{(A.14)}$$

$$\frac{\partial^2\hat{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}^2} = \alpha_r N_r\langle(\mathcal{L}_r - \langle\mathcal{L}_r\rangle)^2\rangle + \frac{\alpha_r}{4Q_{\epsilon,r}^{3/2}}\sum_{i\in\Lambda_r}\frac{|J_i^r| - J_i^r\langle\sigma_i\rangle}{N_r} \geq 0 \quad \text{(A.15)}$$

Let us evaluate:

$$\left|\frac{\partial\hat{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}} - \frac{\partial\hat{\bar{p}}_{N,\epsilon}}{\partial Q_{\epsilon,r}}\right| \geq \alpha_r\left|\langle\mathcal{L}_r\rangle - \mathbb{E}\langle\mathcal{L}_r\rangle\right| - \frac{\alpha_r|A_r|}{2\sqrt{Q_{\epsilon,r}}} \quad \text{(A.16)}$$

where:

$$A_r := \frac{1}{N_r}\sum_{i\in\Lambda_r}[|J_i^r| - \mathbb{E}|J_i^r|] \quad \text{(A.17)}$$

Thanks to the independence of the J_i^r it is immediate to verify that $\exists\, a \geq 0$ s.t.:

$$\mathbb{E}[A_r^2] \leq \frac{a}{N_r} \quad \text{(A.18)}$$

Using Lemma 3.2 in [16], with notations used in [2]:

$$\left|\frac{\partial \hat{p}_{N,\epsilon}}{\partial Q_{\epsilon,r}} - \frac{\partial \hat{\hat{p}}_{N,\epsilon}}{\partial Q_{\epsilon,r}}\right| \leq \frac{1}{\delta} \sum_{u\in\{Q_{\epsilon,r}+\delta, Q_{\epsilon,r}, Q_{\epsilon,r}-\delta\}} [|\hat{p}_{N,\epsilon} - \hat{\hat{p}}_{N,\epsilon}| + \alpha_r \sqrt{u}|A_r|] + \\ + C^{+}_{\delta}(Q_{\epsilon,r}) + C^{-}_{\delta}(Q_{\epsilon,r}) \quad (A.19)$$

with:

$$C^{\pm}_{\delta}(Q_{\epsilon,r}) = |\hat{\hat{p}}'_{N,\epsilon}(Q_{\epsilon,r} \pm \delta) - \hat{\hat{p}}'_{N,\epsilon}(Q_{\epsilon,r})| \quad (A.20)$$

$$|\hat{\hat{p}}'_{N,\epsilon}| = \left|\frac{\alpha_r}{2}\mathbb{E}[1 + \langle m_r\rangle] - \frac{\alpha_r \mathbb{E}|J^r_1|}{2\sqrt{Q_{\epsilon,r}}}\right| \leq \alpha_r \left(1 + \frac{C}{2\sqrt{s_N}}\right) \quad (A.21)$$

$$C^{\pm}_{\delta}(Q_{\epsilon,r}) \leq \alpha_r \left(2 + \frac{C}{\sqrt{s_N}}\right) \quad (A.22)$$

where for simplicity we have kept the dependence on $Q_{\epsilon,r}$ only, $\hat{\hat{p}}'_{N,\epsilon}$ is the derivative w.r.t. it and $\delta > 0$. Notice that δ will be chosen strictly smaller than s_N, so that $Q_{\epsilon,r} - \delta \geq \epsilon - \delta \geq s_N - \delta > 0$.

Then, using the the previous ones, and thanks to the fact that $(\sum_{i=1}^{p} \nu_i)^2 \leq p\sum_{i=1}^{p} \nu_i^2$, we get:

$$\frac{\alpha_r^2}{9}\left|\langle \mathcal{L}_r\rangle - \mathbb{E}\langle \mathcal{L}_r\rangle\right|^2 \leq \frac{1}{\delta^2} \sum_{u\in\{Q_{\epsilon,r}+\delta, Q_{\epsilon,r}, Q_{\epsilon,r}-\delta\}} [|\hat{p}_{N,\epsilon} - \hat{\hat{p}}_{N,\epsilon}|^2 + \alpha_r^2 u|A_r|^2] + \\ + C^{+}_{\delta}(Q_{\epsilon,r})^2 + C^{-}_{\delta}(Q_{\epsilon,r})^2 + \frac{\alpha_r^2 A_r^2}{4\epsilon_r} \quad (A.23)$$

We first evaluate the two terms containing $C^{\pm}_{\delta}$:

$$\mathbb{E}_{\boldsymbol{\epsilon}}[C^{+}_{\delta}(Q_{\epsilon,r})^2 + C^{-}_{\delta}(Q_{\epsilon,r})^2] \leq 2\alpha_r \left(2 + \frac{C}{\sqrt{s_N}}\right) \mathbb{E}_{\boldsymbol{\epsilon}}[C^{+}_{\delta}(Q_{\epsilon,r}) + C^{-}_{\delta}(Q_{\epsilon,r})] \leq \\ \leq \frac{2\alpha_r}{s_N^K}\left(2 + \frac{C}{\sqrt{s_N}}\right) \prod_{s=1}^{K} \int_{Q_{s_N,s}}^{Q_{2s_N,s}} dQ_{\epsilon,s} [\hat{\hat{p}}'_{N,\epsilon}(Q_{\epsilon,r} + \delta) - \hat{\hat{p}}'_{N,\epsilon}(Q_{\epsilon,r} - \delta)] = \\ = \frac{2\alpha_r}{s_N^K}\left(2 + \frac{C}{\sqrt{s_N}}\right) \prod_{s\neq r,1}^{K} \int_{Q_{s_N,s}}^{Q_{2s_N,s}} dQ_{\epsilon,s} [\hat{\hat{p}}_{N,\epsilon}(Q_{2s_N,r} + \delta) - \hat{\hat{p}}_{N,\epsilon}(Q_{2s_N,r} - \delta) + \\ - \hat{\hat{p}}_{N,\epsilon}(Q_{s_N,r} + \delta) + \hat{\hat{p}}_{N,\epsilon}(Q_{s_N,r} - \delta)] \leq \frac{8\alpha_r^2 K_r(\Delta)}{s_N^K}\delta \left(2 + \frac{C}{\sqrt{s_N}}\right)^2 \quad (A.24)$$

Taking the expectation $\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}$ in (A.23), and defining W_r s.t. $Q_{\epsilon,r} \leq W_r$, we get:

$$\frac{\alpha_r^2}{9}\mathbb{E}_{\boldsymbol{\epsilon}}\mathbb{E}\left|\langle\mathcal{L}_r\rangle - \mathbb{E}\langle\mathcal{L}_r\rangle\right|^2 \leq \frac{3}{\delta^2}\left[\frac{S}{N}+\frac{\alpha_r^2 W_r a}{N_r}\right]+ \\ +\frac{8\alpha_r^2 K_r(\Delta)}{s_N^K}\delta\left(2+\frac{C}{\sqrt{s_N}}\right)^2+\frac{\alpha_r^2 a\log 2}{4N_r s_N} \quad \text{(A.25)}$$

We can make the r.h.s. vanish by choosing for example: $\delta = s_N^{2K/3}N^{-1/3}$. The choice $s_N \propto N^{-1/16K}$ makes the r.h.s. (A.25) behave like $\mathcal{O}(N^{-1/4})$. □

B Appendix: the SK case

In the case $K = 1$ the equation (44) reduces to:

$$\lim_{N\to\infty}\bar{p}_N(\mu,h) = \sup_{x\in\mathbb{R}_{\geq 0}}\left\{\mu\frac{(1-x)^2}{4}-\frac{\mu x^2}{2}+\psi(\mu x+h)\right\} \quad \text{(B.1)}$$

while (46) simply becomes:

$$x = \mathbb{E}_z \tanh\left(z\sqrt{\mu x+h}+\mu x+h\right) := T(x;\mu,h)\ . \quad \text{(B.2)}$$

We collect the main results on this model in the following proposition.

Proposition 7. *Define:*

$$\bar{p}_{var}(x;\mu,h) = \mu\frac{(1-x)^2}{4}-\frac{\mu x^2}{2}+\psi(\mu x+h)\ . \quad \text{(B.3)}$$

The following hold:

1. *if $\mu < 1$ then $\bar{p}_{var}$ is concave in x. Equivalently if $\mu < 1$ then $T(x;\mu,h)$ is a contraction, and if further $h = 0$ then $x = 0$ is its fixed point;*

2. *the stable solution of the consistency equation* (B.2) *is continuous at $(\mu,h) = (1,0)$:*

$$\lim_{(\mu,h)\to(1,0)}\bar{x}(\mu,h) = 0 = \bar{x}(1,0)\ ; \quad \text{(B.4)}$$

3. *for fixed $h = 0$, the magnetization goes to 0 linearly with $\mu - 1$ as $\mu \to 1_+$, more precisely:*

$$\bar{x} = (1+o(1))\frac{\mu-1}{\mu^2} \quad \text{(B.5)}$$

where $o(1)$ goes to 0 when $\mu \to 1_+$. Therefore the critical exponent β (in the Landau classification) is 1, which means that the derivative of the magnetization w.r.t. μ does not diverge at the critical point, it only jumps from 0 to 1 and then decreases;

4. *Along the line $(\mu, \lambda(\mu-1))$, $\lambda > 0$ in the plane (μ, h) the magnetization goes to 0 as follows:*

$$\bar{x} = \sqrt{\frac{\lambda(\mu-1)}{\mu^2}}(1+o(1)) \tag{B.6}$$

when $\mu \to 1_+$, therefore with a critical exponent $1/2$;

5. *For fixed $\mu = 1$ and $h \to 0_+$ the magnetization behaves as:*

$$\bar{x}^2 = h(1+o(1)) \tag{B.7}$$

therefore we have a critical exponent $\delta = 2$ (according to Landau's classification).

Proof. 1. The fist assertion follows immediately from (54), since $\hat{\alpha} \equiv 1$ and $\Delta \equiv \mu$. Then, by (49):

$$\frac{dT}{dx}(x;\mu,h) = \mu \mathbb{E}_z \left[\left(1 - \tanh^2\left(z\sqrt{\mu x + h} + \mu x + h\right)\right)^2 \right] \leq \mu < 1$$

that implies T is a contraction. It is easy to see that if $h = 0$ then $x = 0$ is a solution of the fixed point equation which must be unique by Banach's fixed point theorem.

2. Using continuity and monotonicity of T (see (49)):

$$\limsup_{(\mu,h)\to(1,0)} \bar{x}(\mu,h) = T(\limsup_{(\mu,h)\to(1,0)} \bar{x}(\mu,h); 1, 0)$$
$$\liminf_{(\mu,h)\to(1,0)} \bar{x}(\mu,h) = T(\liminf_{(\mu,h)\to(1,0)} \bar{x}(\mu,h); 1, 0)$$

hence both $\limsup_{(\mu,h)\to(1,0)} \bar{x}(\mu,h)$ and $\liminf_{(\mu,h)\to(1,0)} \bar{x}(\mu,h)$ satisfy the consistency equation:

$$m = \mathbb{E}_z \tanh(z\sqrt{m} + m)$$

whose solution $m = 0$ is unique, since the derivative of $T(m;1,0)$ is ≤ 1 and equality holds only at $m = 0$. We conclude that there exists

$$\lim_{(\mu,h)\to(1,0)} \bar{x}(\mu,h) = 0 = \bar{x}(1,0) \ . \tag{B.8}$$

3. By computing the first and second derivatives of the map $T(x;\mu,0)$ and using the Nishimori identities we get:

$$T'(0;\mu,0)=\mu\,,\quad T''(0;\mu,0)=-2\mu^2$$

$$\bar{x}=\mu\bar{x}-\mu^2\bar{x}^2(1+o(1))\quad\Rightarrow\quad\bar{x}=(1+o(1))\left(\frac{\mu-1}{\mu^2}\right)\,,$$

which implies that, in proximity of $\mu=1$, the magnetization goes to 0 with a critical exponent $\beta=1$ and with slope 1.

4. An analogous expansion of T yields:

$$\bar{x}=T(\bar{x};\mu,\lambda(\mu-1))=\mu\bar{x}+\lambda(\mu-1)-(\mu^2\bar{x}^2+o(\mu-1))(1+o(1))$$

which in turn entails:

$$\bar{x}^2=\frac{\lambda(\mu-1)}{\mu^2}(1+o(1))\,.$$

*5.*As in the previous steps:

$$\bar{x}=T(\bar{x};1,h)=\bar{x}+h-(\bar{x}^2+o(h))(1+o(1))\,,$$

then we get:

$$\bar{x}^2=h(1+o(1))\quad\Rightarrow\quad\delta=2\,.$$

□

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