A mean-field monomer-dimer model with random monomer activities. Exact solution and rigorous results.

Diego Alberici, Pierluigi Contucci, Emanuele Mingione

September 23, 2014

Abstract

Independent random monomer activities are considered on a mean-field monomer-dimer model. Under very general conditions on the randomness the model is shown to have a self-averaging pressure density that obeys a solvable variational principle. The dimer density is exactly computed in the thermodynamic limit and shown to be a smooth function.

1 Introduction

In this paper we study a mean field monomer-dimer model with randomness on the monomer activities. The model describes, in the mean field approximation, the equilibrium properties of systems of diatomic molecules (see e.g. [12, 5]) depositing on lattices with random impurities. The problem of depositing monoatomic molecules on lattices with random impurities has been approached, within the mean-field approximation, by the study of the Curie-Weiss random field model (see e.g. [15, 1, 14]) where a non-trivial feature is given by the presence of the ferromagnetic interaction among spins.

The properties of monomer-dimer systems emerge from the hard-core interactions among the particles, representing the repulsivity of the van der Waals potential at short distance. Our approach builds on the fundamental work by Heilmann and Lieb [8, 9] and their rigorous proof of the absence of phase transitions under very general conditions. In a previous work [3] we have shown, as it was expected on experimental observations and heuristic bases (see e.g. [5]), that the presence of an attractive interaction among particles, beyond the equilibrium distance of the van der Waals potential, induces a ferromagnetic phase transition with coexisting phases. The exact solution of the mean field theory that we provided in [3, 4] have opened the way to the investigation of more realistic situations like those in presence of impurities.

The present work is a first and necessary result to study the thermodynamic properties of monomer-dimer systems with a fully realistic interaction that includes both the hard-core and attractive components and considers also the presence of impurities. Our main result is the exact solution of the model with *i.i.d.* randomness on the local monomer activities x_i's. Precisely we prove that the pressure density exists, under very general conditions on the probability distribution, and is given by a variational principle of elementary nature, i.e. the maximisation of a function Φ on the positive real line, where

$$\Phi(\xi) \;=\; -\frac{\xi^2}{2w} + \mathbb{E}_x[\log(\xi+x)] \,, \quad \xi \geq 0 \,.$$

The solution turns out to be a smooth function of the dimer activity w. The dimer density $d = d(w)$ is therefore a smooth function too and it is described by the formula

$$d \;=\; \frac{(\xi^*)^2}{2w}$$

where ξ^* is the unique positive solution of the fixed point equation

$$\xi^* \;=\; w\,\mathbb{E}_x\left[\frac{1}{\xi^*+x}\right]\,.$$

The problem, otherwise expected to be difficult due the hard-core algebraic constraints, became accessible with the use of a Gaussian representation for the partition function. Then a careful application of the uniform law of large numbers and the Laplace method lead to the solution. Under the perspective of a Gaussian representation for the partition function the celebrated Heilmann-Lieb recursion relation, at the base of the exact solvability of some models [8, 2] and of the proof of the absence of phase transitions [8], admits the interpretation of a Gaussian integration by parts. In the present paper the Heilmann-Lieb recursion is one of the main tools, together with technical methods for martingales (like Azuma's inequality), used to prove the self-averaging of the pressure density.

The work is organised as follows. In the section 2 we describe the Gaussian representation for the partition function of a general monomer-dimer model and we deduce the Heilmann-Lieb recursion. In the section 3 we solve the monomer-dimer model on the complete graph with *i.i.d.* random monomer activities; in particular we compute the pressure density in the theorem 3.2 and the dimer density in the corollary 3.4. In the section 4 we show, under suitable assumptions, that the free energy density of a monomer-dimer model with independent random activities is self-averaging. The appendix collects the main technical results used in this paper, in order to facilitate the reader.

2 Gaussian representation for monomer-dimer models

In this section we recall the definition of a monomer-dimer model with pure hard-core interaction and we show how to write its partition function as a Gaussian expectation. This representation, which will be extensively used in this work, is not new [17] and it is an easy consequence of the Wick-Isserlis formula for Gaussian moments. As a first application we show in this section that the well-known Heilmann-Lieb recursion formula [8] for monomer-dimer models corresponds in fact to a Gaussian integration by parts.

Definition 2.1. Let $G=(V,E)$ be a finite simple graph. A *dimer configuration* (or *matching*) on G is a set D of pairwise non-incident edges (called *dimers*). The associated set of dimer-free vertices (called *monomers*) is denoted by $M_G(D)$. In other terms a dimer configuration D on G is a partition of a certain set $A\subseteq V$ into pairs belonging to E:

$$\begin{aligned}&D=\big\{\{i_1,i_2\},\dots,\{i_{|A|-1},i_{|A|}\}\big\}\\ &\text{with}\;\;\{i_1,i_2,\dots,i_{|A|}\}=A\;\;\text{and}\;\;\{i_s,i_{s+1}\}\in E\;;\end{aligned} \tag{1}$$

and the associated monomer set is $M_G(D)=V\smallsetminus A\,$.

Denote by $\mathscr{D}_G$ the space of all possible dimer configurations on the graph G. A *monomer-dimer model (with pure hard-core interaction)* on G is obtained by assigning a monomer weight $x_i > 0$ to each vertex $i \in V$, a dimer weight $w_{ij} \geq 0$ to each edge $ij \equiv \{i,j\} \in E$ and introducing the following Gibbs probability measure on $\mathscr{D}_G$:

$$\mu_G(D) := \frac{1}{Z_G} \prod_{ij \in D} w_{ij} \prod_{i \in M_G(D)} x_i \quad \forall\, D \in \mathscr{D}_G\,, \tag{2}$$

where $Z_G := \sum_{D \in \mathscr{D}_G} \prod_{ij \in D} w_{ij} \prod_{i \in M_G(D)} x_i$ is the normalizing factor, called *partition function.*

The following remark shows that, when the weights are kept so general, it is sufficient (and convenient) to work on a complete graph.

Remark 2.2. Consider the complete graph K_N, with vertex set $\{1,\dots,N\}$ and edge set made of all possible pairs of vertices. Because of the lack of geometric structure the space of dimer configurations $\mathscr{D}_N \equiv \mathscr{D}_{K_N}$ simplifies; precisely $D \in \mathscr{D}_N$ if and only if

$$D = \big\{\{i_1,i_2\},\dots,\{i_{|A|-1},i_{|A|}\}\big\} \;\text{ with }\; \{i_1,i_2,\dots,i_{|A|}\} = A \tag{3}$$

for a certain set of vertices $A \subseteq \{1,\dots,N\}$, and the monomer set associated to D is $M_N(D) \equiv M_{K_N}(D) = \{1,\dots,N\} \smallsetminus A$.
On the other hand any monomer-dimer model on a graph $G = (V,E)$ with N vertices can be thought as a monomer-dimer model on the complete graph K_N. Indeed the measure μ_G is equivalent to a measure $\mu_N \equiv \mu_{K_N}$ by setting $w_{ij} := 0$ for all pairs $ij \notin E$. Precisely introducing these zero dimer weights it holds $Z_N \equiv Z_{K_N} = Z_G$ and

$$\mu_N(D) = \begin{cases} \mu_G(D) & \text{if } D \in \mathscr{D}_G \\ 0 & \text{if } D \in \mathscr{D}_N \smallsetminus \mathscr{D}_G \end{cases}.$$

The next proposition describes the Gaussian representation for the monomer-dimer model. Without loss of generality we work with the partition function Z_N on the complete graph.

Proposition 2.3 (Gaussian representation)**.** *The partition function of any monomer-dimer model over N vertices can be written as*

$$Z_N = \mathbb{E}_{\boldsymbol{\xi}}\left[\prod_{i=1}^N (\xi_i + x_i)\right], \tag{4}$$

where $\boldsymbol{\xi} = (\xi_1,\dots,\xi_N)$ is a Gaussian random vector with mean 0 and covariance matrix $W = (w_{ij})_{i,j=1,\dots,N}$. Here the diagonal entries w_{ii} are arbitrary numbers, chosen in such a way that W is a positive semi-definite matrix.

Proof. As already noticed the dimer configurations on the complete graph are the partitions into pairs of all possible $A \subseteq \{1,\dots,N\}$, hence

$$Z_N = \sum_{D \in \mathscr{D}_N} \prod_{ij \in D} w_{ij} \prod_{i \in M_N(D)} x_i = \sum_{A \subseteq \{1,\dots,N\}} \sum_{\substack{P \text{ partition} \\ \text{of } A \text{ into pairs}}} \prod_{ij \in P} w_{ij} \prod_{i \in A^c} x_i\,. \tag{5}$$

Now choose w_{ii} for $i = 1, \ldots, N$ such that the matrix $W = (w_{ij})_{i,j=1,\ldots,N}$ is positive semi-definite[1]. Then there exists an (eventually degenerate) Gaussian vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ with mean 0 and covariance matrix W. And by the Wick-Isserlis theorem (identity (A2) in the theorem A1)

$$\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{i\in A}\xi_i\Big] \;=\; \sum_{\substack{P \text{ partition} \\ \text{of } A \text{ into pairs}}} \;\prod_{ij\in P} w_{ij}\ . \tag{6}$$

Substituting (6) into (5) one obtains

$$Z_N \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\sum_{A\subseteq\{1,\ldots,N\}}\;\prod_{i\in A}\xi_i\,\prod_{i\in A^c}x_i\Big] \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{i=1}^{N}(\xi_i+x_i)\Big]\ . \tag{7}$$

□

As an application of the Gaussian representation we show that the well-know Heilmann-Lieb recursion [8] for the partition function of monomer-dimer models can be proved by means of a Gaussian integration by parts.

Proposition 2.4 (Heilmann-Lieb recursion)**.** *Let $G = (V, E)$ be a finite simple graph and consider a monomer-dimer model on G. Fix $i \in V$ and look at its adjacent vertices $j \sim i$, then it holds*

$$Z_G \;=\; x_i\,Z_{G-i} \;+\; \sum_{j\sim i} w_{ij}\,Z_{G-i-j}\ . \tag{8}$$

Here $G - i$ denotes the graph obtained from G deleting the vertex i and all its incident edges.

Proof using Gaussian integration by parts. Set $N := |V|$. Introduce zero dimer weights $w_{hk} = 0$ for the pairs $hk \notin E$, so that $Z_G = Z_N$ (see remark 2.2). Following proposition 2.3, introduce an N-dimensional Gaussian vector $\boldsymbol{\xi}$ with mean 0 and covariance matrix W. Then write the identity (4) isolating the vertex i:

$$Z_G \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k=1}^{N}(\xi_k+x_k)\Big] \;=\; x_i\,\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big] \;+\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\xi_i\,\prod_{k\neq i}(\xi_k+x_k)\Big]\ . \tag{9}$$

Now apply the Gaussian integration by parts (identity (A1) in the theorem A1) to the second term on the r.h.s. of (9):

$$\mathbb{E}_{\boldsymbol{\xi}}\Big[\xi_i\,\prod_{k\neq i}(\xi_k+x_k)\Big] \;=\; \sum_{j=1}^{N}\,\mathbb{E}_{\boldsymbol{\xi}}[\xi_i\xi_j]\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\frac{\partial}{\partial\xi_j}\prod_{k\neq i}(\xi_k+x_k)\Big] \;=\; \sum_{j\neq i} w_{ij}\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\ . \tag{10}$$

Notice that summing over $j \neq i$ in the r.h.s. of (10) is equivalent to sum over $j \sim i$, since by definition $w_{ij} = 0$ if $ij \notin E$. Substitute (10) in (9):

$$Z_G \;=\; x_i\,\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big] \;+\; \sum_{j\sim i} w_{ij}\;\mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\ . \tag{11}$$

[1] For example one can choose $w_{ii} \geq \sum_{j\neq i} w_{ij}$ for every $i = 1, \ldots, N$. W can be diagonalized and has non-negative eigenvalues by the Gershgorin circle theorem, hence it is positive semi-definite.

To conclude observe that $(\xi_k)_{k\neq i}$ is an $(N-1)$-dimensional Gaussian vector with mean 0 and covariance $(w_{hk})_{h,k\neq i}$. Hence by proposition 2.3

$$Z_{G-i} \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i}(\xi_k+x_k)\Big]\;. \tag{12}$$

And similarly

$$Z_{G-i-j} \;=\; \mathbb{E}_{\boldsymbol{\xi}}\Big[\prod_{k\neq i,j}(\xi_k+x_k)\Big]\;. \tag{13}$$

Substitute the identities (12), (13) into (11) to obtain the identity (8). $\square$

3 Monomer-dimer model with random monomer weights

In this section we fix a uniform dimer weight on the complete graph, while we choose i.i.d. random monomer weights. Under quite general integrability hypothesis, we show that this model is exactly solvable and it does not present a phase transition (in agreement with the general results by Heilmann and Lieb [8, 9]).

Let $w>0$. Let $x_i>0$, $i\in\mathbb{N}$, be *independent identically distributed* random variables. In order to keep the logarithm of the partition function of order N, a normalization of the dimer weight as w/N is needed. Therefore during all this section we will denote

$$Z_N \;=\; \sum_{D\in\mathscr{D}_N}\Big(\frac{w}{N}\Big)^{|D|}\prod_{i\in M_N(D)}x_i\;. \tag{14}$$

μ_N will denote the corresponding Gibbs measure and $\langle\,\cdot\,\rangle_N$ will be the expected value with respect to μ_N. Notice that now the partition function is a random variable and the Gibbs measure is a random measure.

Remark 3.1. Since the dimer weight is uniform, the Gaussian representation of (14) simplifies:

$$Z_N \;=\; \mathbb{E}_\xi\Big[\prod_{i=1}^N(\xi+x_i)\Big]\;, \tag{15}$$

where ξ is a 1-dimensional Gaussian random variable with mean 0 and variance w/N. Indeed by proposition 2.3, $Z_N=\mathbb{E}_{\boldsymbol{\xi}}\big[\prod_{i=1}^N(\xi_i+x_i)\big]$ where $\boldsymbol{\xi}=(\xi_1,\dots,\xi_N)$ is an N-dimensional Gaussian random vector with mean 0 and constant covariance matrix[2] $(w/N)_{i,j=1,\dots,N}$. It is easy to check that $\boldsymbol{\xi}$ has the same joint distribution of the constant random vector $(\xi,\dots,\xi)$. Therefore the identity (15) follows.

Z_N can be expressed as an expectation in the Gaussian variable ξ; but on the other hand Z_N is a random variable dependent on the monomer weights x_i's. To avoid confusion we rewrite (15) as an explicit integral in dξ:

$$Z_N \;=\; \frac{\sqrt{N}}{\sqrt{2\pi w}}\int_{\mathbb{R}} e^{-\frac{N}{2w}\xi^2}\prod_{i=1}^N(\xi+x_i)\,\mathrm{d}\xi\;. \tag{16}$$

[2] It is important to notice that setting also the diagonal entries to w/N, the resulting matrix is positive semi-definite: $\sum_{i=1}^N\sum_{j=1}^N(w/N)\,\alpha_i\alpha_j=(w/N)\big(\sum_{i=1}^N\alpha_i\big)^2\geq 0$ for every $\alpha\in\mathbb{R}^N$.

Theorem 3.2. *Let $w > 0$. Let $x_i > 0$, $i \in \mathbb{N}$ be i.i.d. random variables. Denote by x a random variable distributed like x_i; suppose that $\mathbb{E}_x[x] < \infty$ and $\mathbb{E}_x[(\log x)^2] < \infty$. Then:*

$$\exists \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_{\boldsymbol{x}}[\,\log Z_N] \;=\; \sup_{\xi\geq 0} \Phi(\xi) \tag{17}$$

where

$$\Phi(\xi) \;:=\; -\frac{\xi^2}{2w} + \mathbb{E}_x[\,\log(\xi+x)] \quad \forall\, \xi \geq 0\;. \tag{18}$$

Furthermore the function Φ attains its maximum at a unique point ξ^*. ξ^* is the only solution in $[0,\infty[$ of the fixed point equation

$$\xi^* \;=\; \mathbb{E}_x\left[\frac{w}{\xi^*+x}\right]. \tag{19}$$

Thus the following bounds hold:

$$\frac{-\mathbb{E}_x[x]+\sqrt{\mathbb{E}_x[x]^2+4w}}{2} \vee \sup_{t>0} \frac{-t+\sqrt{t^2+4w\,\mathbb{P}_x(x\leq t)}}{2} \;\leq\; \xi^* \;\leq\; \sqrt{w}\wedge \mathbb{E}_x\left[\frac{w}{x}\right]. \tag{20}$$

In consequence of the theorem 3.2 it is not hard to prove that the system does not present a phase transition in the parameter $w > 0$. It is also easy to compute the main macroscopic quantity of physical interest, that is the *dimer density*, in terms of the positive solution ξ^* of the fixed point equation (19). Therefore we state the following two corollaries before starting to prove the theorem.

Corollary 3.3. *In the hypothesis of the theorem 3.2, consider the limiting pressure density function $p(w) := \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_{\boldsymbol{x}}\big[\log Z_N(w)\big]$ for all $w > 0$. Then $p \in C^\infty\big(]0,\infty[\big)$.*

Proof. By the theorem 3.2 $p(w) = \Phi(w,\xi^*)$, where $\Phi(w,\xi) = -\xi^2/(2w) + \mathbb{E}_x[\,\log(\xi+x)]$ and $\xi^* = \xi^*(w)$ is *the only* positive solution of the equation $F(w,\xi) = 0$ with $F := \frac{\partial\Phi}{\partial\xi}$.
F is a smooth function on $]0,\infty[\,\times\,]0,\infty[$, because Φ is smooth as it will be proven in the lemma 3.5. In addition $\frac{\partial F}{\partial \xi}(w,\xi^*) \neq 0$ for all $w > 0$, by the lemma 3.5 equation (23).
As a consequence, by the implicit function theorem (see e.g. [13]), ξ^* is a smooth function of $w \in]0,\infty[$. Hence, by composition, also $p(w) = \Phi\big(w,\xi^*(w)\big)$ is a smooth function of $w \in]0,\infty[$. □

Corollary 3.4. *In the hypothesis of the theorem 3.2, the limiting dimer density*

$$d := \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}_{\boldsymbol{x}}\big[\langle\, |D|\,\rangle_N\,\big]$$

can be computed as

$$d \;=\; w\,\frac{\mathrm{d}\,p}{\mathrm{d}w} \;=\; \frac{(\xi^*)^2}{2w}\;. \tag{21}$$

Proof. Set $p_N := \frac{1}{N}\log Z_N$ and perform the change of parameter $w =: e^h$. Clearly $\frac{\mathrm{d}}{\mathrm{d}h} = w\,\frac{\mathrm{d}}{\mathrm{d}w}$ and it is easy to check that

$$\frac{\mathrm{d}\,\mathbb{E}_{\boldsymbol{x}}[p_N]}{\mathrm{d}h} \;=\; \mathbb{E}_{\boldsymbol{x}}\big[\langle\, |D|\,\rangle_N\,\big]\;.$$

By the theorem 3.2 and its corollary 3.3, $\mathbb{E}_{\boldsymbol{x}}[p_N]$ converges pointwise to a smooth function p as $N \to \infty$ for all values of $h \in \mathbb{R}$. A standard computation shows that $\mathbb{E}_{\boldsymbol{x}}[p_N]$ is a convex function of h. Therefore

$$\frac{\mathrm{d}\,\mathbb{E}_{\boldsymbol{x}}[p_N]}{\mathrm{d}h} \xrightarrow[N\to\infty]{} \frac{\mathrm{d}\,p}{\mathrm{d}h}\,.$$

Since $p(h) = \Phi\big(h, \xi^*(h)\big)$, where ξ^* is the critical point of Φ and is a smooth function of h, it is easy to compute

$$\frac{\mathrm{d}\,p}{\mathrm{d}h}(h) \;=\; \frac{\partial\Phi}{\partial h}(h,\xi^*) \,+\, \underbrace{\frac{\partial\Phi}{\partial\xi}(h,\xi^*)}_{=\,0}\,\frac{\mathrm{d}\xi^*}{\mathrm{d}h}(h) \;=\; \frac{(\xi^*)^2}{2\,e^h}\,. \qquad \square$$

Now let us start to prove the theorem 3.2. The logic structure of the proof is divided in three main parts. First we study the basic properties of the function Φ. Then we use the uniform law of large numbers and other observations to show that for large N the integrated function in (16) can be well approximated by $e^{N\Phi}$. Finally we will be able to exploit the Laplace's method in order to compute a lower and an upper bound for $\frac{1}{N}\,\mathbb{E}_{\boldsymbol{x}}[\,\log Z_N\,]$.

Lemma 3.5. *Φ is continuous on $[0,\infty[$, it is smooth on $]0,\infty[$ and the derivatives can be taken inside the expectation. In particular for all $\xi > 0$ it holds*

$$\Phi'(\xi) \;=\; -\frac{\xi}{w} + \mathbb{E}_x\left[\frac{1}{\xi + x}\right]; \tag{22}$$

$$\Phi''(\xi) \;=\; -\frac{1}{w} - \mathbb{E}_x\left[\frac{1}{(\xi + x)^2}\right] < 0\,. \tag{23}$$

As a consequence Φ has exactly one critical point ξ^* in $]0,\infty[$, that is the equation (19) *has exactly one solution in $]0,\infty[$. ξ^* is* the only global maximum point *of Φ on $[0,\infty[$.*

Proof. **I.** First of all $\Phi(\xi)$ is well-defined for all $\xi \geq 0$. Indeed for $\xi > 0$

$$\log(\xi + x) \begin{cases} \leq\, \xi + x - 1\, \in L^1(\mathbb{P}_x) \\ \geq\, 1 - \frac{1}{\xi+x} \,\geq\, 1 - \frac{1}{\xi}\, \in L^1(\mathbb{P}_x) \end{cases} ;$$

while for $\xi = 0$, $\mathbb{E}_x[|\log x|] \leq \mathbb{E}_x[(\log x)^2]^{1/2} < \infty$ by the Hölder inequality.
Φ is continuous at $\xi = 0$ by monotone convergence: $\log(\xi + x)$ decreases to $\log x$ as $\xi \searrow 0$ and $\mathbb{E}_x[\,\log(\xi + x)] < \infty$.
Let now $\xi > 0$ and let $\delta > 0$ such that $\xi - \delta > 0$. The first derivative of Φ at ξ can be computed inside the expectation, obtaining (22), since the difference quotient of $\xi \mapsto \log(\xi + x)$ satisfies the dominated convergence hypothesis. Indeed for all $\xi' \in]\xi - \delta, \xi + \delta[$

$$\left|\frac{\log(\xi' + x) - \log(\xi + x)}{\xi' - \xi}\right| \leq \sup_{\widetilde{\xi}\in[\xi,\xi']} \frac{1}{\widetilde{\xi} + x} \leq \sup_{\widetilde{\xi}\in[\xi,\xi']} \frac{1}{\widetilde{\xi}} \leq \frac{1}{\xi - \delta} \,\in L^1(\mathbb{P}_x)\,.$$

Now the second derivative of Φ at ξ can be computed inside the expectation, obtaining (23), since the difference quotient of $\xi \mapsto \frac{1}{\xi+x}$ satisfies the dominated convergence hypothesis. Indeed for all $\xi' \in]\xi - \delta, \xi + \delta[$

$$\left|\frac{\frac{1}{\xi'+x} - \frac{1}{\xi+x}}{\xi' - \xi}\right| \leq \sup_{\widetilde{\xi}\in[\xi,\xi']} \frac{1}{(\widetilde{\xi} + x)^2} \leq \sup_{\widetilde{\xi}\in[\xi,\xi']} \frac{1}{\big(\widetilde{\xi}\big)^2} \leq \frac{1}{(\xi - \delta)^2} \,\in L^1(\mathbb{P}_x)\,.$$

This reasoning can be iterated up to the derivative of any order, since $1/\big(\widetilde{\xi}+x\big)^k \le 1/\big(\widetilde{\xi}\,\big)^k \le 1/(\xi-\delta)^k \in L^1(\mathbb{P}_x)$ for all $\widetilde{\xi} \in]\xi-\delta, \xi+\delta[$ and all $k \ge 1$.

II. In virtue of (23) Φ is a strictly convex function on $]0,\infty[$. At the boundaries of this domain $\lim_{\xi\to 0+} \Phi'(\xi) = \mathbb{E}_x[x^{-1}] > 0$ and $\lim_{\xi\to\infty} \Phi'(\xi) = -\infty < 0$ by (22) and monotone converge. Therefore Φ has exactly one critical point ξ^* in $]0,\infty[$ and it is the only global maximum point of Φ. $\square$

Remark 3.6. Since ξ^* satisfies the fixed point equation (19), it is easy to obtain the bounds (20) for ξ^*. Since $\xi^* > 0$ and $x > 0$,

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^*+x}\right] \le \frac{1}{\xi^*} \;\Rightarrow\; \xi^* \le \sqrt{w}\,; \quad \xi^* = \mathbb{E}_x\left[\frac{w}{\xi^*+x}\right] \le \mathbb{E}_x\left[\frac{w}{x}\right].$$

Using the Jensen inequality,

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^*+x}\right] \ge \frac{w}{\xi^*+\mathbb{E}_x[x]} \;\Rightarrow\; (\xi^*)^2+\xi^*\,\mathbb{E}_x[x]-w \ge 0 \;\Rightarrow\; \xi^* \ge \frac{-\mathbb{E}_x[x]+\sqrt{\mathbb{E}_x[x]^2+4w}}{2}.$$

Finally, since $\xi^* + x > 0$, it holds for all $t > 0$

$$\xi^* = \mathbb{E}_x\left[\frac{w}{\xi^*+x}\right] \ge \frac{w}{\xi^*+t}\,\mathbb{P}_x(x\le t) \;\Rightarrow\; (\xi^*)^2+\xi^*\,t-w\,\mathbb{P}_x(x\le t) \ge 0 \;\Rightarrow$$
$$\Rightarrow\; \xi^* \ge \frac{-t+\sqrt{t^2+4w\,\mathbb{P}_x(x\le t)}}{2}.$$

Lemma 3.7. *Define the random function*

$$\Phi_N(\xi) \;:=\; -\frac{\xi^2}{2w} + \frac{1}{N}\sum_{i=1}^N \log|\xi+x_i| \quad \forall\,\xi\in\mathbb{R}\,. \tag{24}$$

This function is defined also for negative values of ξ and it takes the value $-\infty$ at the random points $-x_1,\dots,-x_N$. It is important to observe that

$$\Phi_N(-\xi) \;<\; \Phi_N(\xi) \quad \forall\,\xi>0\,. \tag{25}$$

i. *Let $0<M<\infty$. Then for all $\varepsilon>0$*

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[0,M] \;\; |\Phi_N(\xi)-\Phi(\xi)|<\varepsilon\Big) \xrightarrow[N\to\infty]{} 1\,. \tag{26}$$

ii. *Let $0<m<M<\infty$. Then there exists $\lambda_{m,M}>0$ such that*

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[m,M] \;\; \Phi_N(-\xi)<\Phi_N(\xi)-\lambda_{m,M}\Big) \xrightarrow[N\to\infty]{} 1\,. \tag{27}$$

iii. *Let $C\in\mathbb{R}$. Then there exists $M_C>0$ such that*

$$\mathbb{P}_{\boldsymbol{x}}\Big(\,\forall\,\xi\in[M_C,\infty[\;\; \Phi_N(\xi)<C \;\text{ and }\; \Phi_N(\xi)<\varphi(\xi)\Big) \xrightarrow[N\to\infty]{} 1\,; \tag{28}$$

where φ is the following deterministic function

$$\varphi(\xi) \;:=\; -\frac{\xi^2}{2w} + \log\xi + \frac{1}{\xi}\,(\mathbb{E}_x[x]+1) \quad \forall\,\xi>0\,. \tag{29}$$

Notice that $\Phi_N(\xi)-\Phi(\xi)=\frac{1}{N}\sum_{i=1}^N\log(\xi+x_i)-\mathbb{E}_x[\log(\xi+x)]$ for all $\xi>0$. Since the x_i, $i\in\mathbb{N}$ are i.i.d., the basic idea behind the lemma 3.7 is to approximate Φ_N with Φ by the law of large numbers. But this approximation is needed to hold at every ξ at the same time, hence a *uniform* law of large numbers is required.
To prove the theorem 3.2 it will be important to have found a good uniform approximation near the global maximum point ξ^* of Φ. Far from ξ^* instead such a uniform approximation cannot hold: for example Φ_N diverges to $-\infty$ at certain negative points, while, if the distribution of x is absolutely continuous and satisfies some integrability hypothesis, it is possible to show that $\Phi(\xi)=-\frac{\xi^2}{2w}+\mathbb{E}_x[\log|\xi+x|]$ is continuous on $\mathbb{R}$. But fortunately, far from ξ^*, it will be sufficient for our purposes to bound suitably Φ_N from above.

Proof. **i.** For every $x>0$ the function $\xi\mapsto\log(\xi+x)$ is continuous on $[0,M]$ compact. Moreover there is domination:

$$\log(\xi+x)\begin{cases}\leq\ \log(M+x)\ \in L^1(\mathbb{P}_x)\\ \geq\ \log x\ \in L^1(\mathbb{P}_x)\end{cases}\qquad\forall\,\xi\in[0,M]\ .$$

Therefore (26) holds by the uniform weak law of large numbers (theorem A3).
ii. Clearly $\log(\xi+x)>\log|-\xi+x|$ for all $\xi,x>0$. Furthermore an elementary computation shows that for all $\xi,x,\tau>0$

$$\log(\xi+x)-\log|-\xi+x|\ \geq\ \tau\quad\Leftrightarrow\quad\frac{e^\tau-1}{e^\tau+1}\,\xi\ \leq\ x\ \leq\ \frac{e^\tau+1}{e^\tau-1}\,\xi\ .$$

Therefore for all $\xi\in[m,M]$ and all $\tau>0$,

$$\begin{aligned}\Phi_N(\xi)-\Phi_N(-\xi)\ &=\ \frac{1}{N}\sum_{i=1}^N\big(\log(\xi+x_i)-\log|-\xi+x_i|\big)\ \geq\\ &\geq\ \frac{1}{N}\sum_{i=1}^N\tau\,\mathbb{1}\bigg(\frac{e^\tau-1}{e^\tau+1}\,\xi\ \leq\ x_i\ \leq\ \frac{e^\tau+1}{e^\tau-1}\,\xi\bigg)\ \geq\\ &\geq\ \tau\,\frac{1}{N}\sum_{i=1}^N\mathbb{1}\bigg(\frac{e^\tau-1}{e^\tau+1}\,M\ \leq\ x_i\ \leq\ \frac{e^\tau+1}{e^\tau-1}\,m\bigg)\ .\end{aligned}\tag{30}$$

Set $I^\tau_{m,M}:=\big[\,\frac{e^\tau-1}{e^\tau+1}\,M\,,\,\frac{e^\tau+1}{e^\tau-1}\,m\,\big]$. Now by the weak law of large numbers, for all $\varepsilon>0$

$$\mathbb{P}_{\boldsymbol{x}}\bigg(\frac{1}{N}\sum_{i=1}^N\mathbb{1}\big(x_i\in I^\tau_{m,M}\big)\ >\ \mathbb{P}_x\big(x\in I^\tau_{m,M}\big)-\varepsilon\bigg)\ \xrightarrow[N\to\infty]{}\ 1\ .\tag{31}$$

Hence, using (30) and (31), for all $\tau,\varepsilon>0$

$$\mathbb{P}_{\boldsymbol{x}}\bigg(\,\Phi_N(\xi)-\Phi_N(-\xi)\ >\ \tau\,\big(\mathbb{P}_x(x\in I^\tau_{m,M})-\varepsilon\big)\bigg)\ \xrightarrow[N\to\infty]{}\ 1\ .\tag{32}$$

To conclude observe that $I^\tau_{m,M}\nearrow\,]0,\infty[$ (which is the support of the distribution of x) as $\tau\searrow0$. Hence there exists $\tau_0>0$ such that $\mathbb{P}_x(x\in I^{\tau_0}_{m,M})>0$. Choose $0<\varepsilon_0<\mathbb{P}_x(x\in I^{\tau_0}_{m,M})$ and set

$$\lambda_{m,M}\ :=\ \tau_0\,\big(\mathbb{P}_x(x\in I^{\tau_0}_{m,M})-\varepsilon_0\big)\ >0\ .$$

Then (27) follows from (32).

iii. For all $\xi > 0$ the following bound holds:

$$\begin{aligned}\Phi_N(\xi) \;&=\; -\frac{\xi^2}{2w} + \frac{1}{N}\sum_{i=1}^{N}\log(\xi + x_i) \;=\; -\frac{\xi^2}{2w} + \log\xi + \frac{1}{N}\sum_{i=1}^{N}\log\left(1 + \frac{x_i}{\xi}\right) \;\leq \\ &\leq\; -\frac{\xi^2}{2w} \,+\, \log\xi \,+\, \frac{1}{\xi}\,\frac{1}{N}\sum_{i=1}^{N} x_i\;.\end{aligned} \tag{33}$$

Now by the weak law of large numbers (no uniformity in ξ is needed here), for all $\varepsilon > 0$

$$\mathbb{P}_{\boldsymbol{x}}\left(\frac{1}{N}\sum_{i=1}^{N} x_i \;<\; \mathbb{E}_x[x] + \varepsilon\right) \;\xrightarrow[N\to\infty]{}\; 1\;. \tag{34}$$

Hence, using (33) and (34), for all $0 < \varepsilon < 1$

$$\mathbb{P}_{\boldsymbol{x}}\Big(\forall\,\xi\!>\!0\;\;\Phi_N(\xi) < \varphi(\xi)\Big) \;\xrightarrow[N\to\infty]{}\; 1\;. \tag{35}$$

Furthermore it holds $\varphi(\xi) \to -\infty$ as $\xi \to \infty$. Hence for all $C \in \mathbb{R}$ there exists $M_C > 0$ such that

$$\varphi(\xi) < C \quad \forall\,\xi > M_C\;. \tag{36}$$

In conclusion (28) follows from (35) and (36). $\square$

Lemma 3.8. *There exists a constant $C_0 < \infty$ such that*

$$\mathbb{E}_{\boldsymbol{x}}\left[\left(\frac{\log Z_N}{N}\right)^2\right] \;\leq\; C_0 \quad \forall\,N \in \mathbb{N}\;. \tag{37}$$

Proof. Since $x \mapsto (\log x)^2$ is concave for $x \geq e$, the Jensen inequality can be used as follows:

$$\begin{aligned}\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\;\mathbb{1}(Z_N \geq e)\big] \;&=\; \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\,\big|\,Z_N \geq e\big]\;\mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;\leq \\ &\leq\; \big(\log\mathbb{E}_{\boldsymbol{x}}\big[Z_N\,\big|\,Z_N \geq e\big]\,\big)^2\;\mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;= \\ &=\; \left(\log\frac{\mathbb{E}_{\boldsymbol{x}}\big[Z_N\;\mathbb{1}(Z_N \geq e)\big]}{\mathbb{P}_{\boldsymbol{x}}(Z_N \geq e)}\right)^2\mathbb{P}_{\boldsymbol{x}}(Z_N \geq e) \;\leq \\ &\leq\; 2\,\big(\log\mathbb{E}_{\boldsymbol{x}}\big[Z_N\big]\big)^2 \,+\, 2\,\max_{p\in[0,1]}(\log p)^2\,p\;.\end{aligned} \tag{38}$$

Since the x_i, $i \in \mathbb{N}$ are i.i.d. $\mathbb{E}_{\boldsymbol{x}}[Z_N]$ equals a deterministic partition function with uniform weights. Hence it is easy to bound it as follows:

$$\begin{aligned}\mathbb{E}_{\boldsymbol{x}}\big[Z_N\big] \;&=\; \sum_{D\in\mathscr{D}_N}\left(\frac{w}{N}\right)^{|D|}\mathbb{E}_x[x]^{|M(D)|} \;\leq\; \sum_{d=0}^{|E_N|}\binom{|E_N|}{d}\left(\frac{w}{N}\right)^{d}\mathbb{E}_x[x]^{N-2d} \;= \\ &=\; \mathbb{E}_x[x]^N\left(1 + \frac{w}{N}\,\mathbb{E}_x[x]^{-2}\right)^{|E_N|} \;\leq\; \mathbb{E}_x[x]^N\,\exp\left(\frac{N-1}{2}\;\frac{w}{\mathbb{E}_x[x]^2}\right)\end{aligned} \tag{39}$$

(here $|E_N| = \frac{N(N-1)}{2}$ denotes the number of edges in the complete graph over N vertices). Therefore, substituting (39) into (38),

$$\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\ \mathbb{1}(Z_N \geq e)\big] \ \leq\ 2\,N^2 \left(\log \mathbb{E}_x[x] + \frac{w}{2\,\mathbb{E}_x[x]^2}\right)^2 + 2 \max_{p\in[0,1]} (\log p)^2\, p\ . \tag{40}$$

It remains to deal with the case $Z_N < e$. When $1 < Z_N < e$, it holds $0 < \log Z_N < 1$ hence trivially

$$\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\ \mathbb{1}(1 < Z_N < e)\big] \ \leq\ \mathbb{E}_{\boldsymbol{x}}\big[(\log e)^2\ \mathbb{1}(1 < Z_N < e)\big] \ \leq\ 1\ . \tag{41}$$

When instead $Z_N \leq 1$, it holds $\log Z_N \leq 0$ hence we need a lower bound for Z_N. For example, considering only the configuration with no dimers, $Z_N \geq \prod_{i=1}^N x_i$. Therefore:

$$\begin{aligned}\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\ \mathbb{1}(Z_N \leq 1)\big] \ &\leq\ \mathbb{E}_{\boldsymbol{x}}\left[\left(\log \prod_{i=1}^N x_i\right)^2 \mathbb{1}(Z_N \leq 1)\right] \ \leq\ \mathbb{E}_{\boldsymbol{x}}\left[\left(\sum_{i=1}^N \log x_i\right)^2\right] \ \leq \\ &\leq\ N^2\,\mathbb{E}_x\big[\log x\big]^2 \ +\ N\,\mathbb{E}_x\big[(\log x)^2\big]\ .\end{aligned} \tag{42}$$

In conclusion the lemma is proved splitting $\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\big]$ as $\mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\,\mathbb{1}(Z_N \geq e)\big] + \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\,\mathbb{1}(1 < Z_N < e)\big] + \mathbb{E}_{\boldsymbol{x}}\big[(\log Z_N)^2\,\mathbb{1}(Z_N \leq 1)\big]$ and applying the bounds (40), (41), (42). ☐

Proof of the theorem 3.2. It remains to prove only the convergence (17). Fix $C < \Phi(\xi^*)$. Fix $0 < m < M_C =: M < \infty$ such that (28) holds and $m < \xi^* < M$: it is possible to make such a choice thanks to the bounds (20) for ξ^* proven in the remark 3.6. Fix $\lambda_{m,M} =: \lambda > 0$ such that (27) holds. Let $\varepsilon > 0$. Then consider the following random events depending on $x_1, \dots, x_N$

$$E^1_{N,\varepsilon} := \{\,\forall\,\xi\in[0,M]\ \ |\Phi_N(\xi) - \Phi(\xi)| < \varepsilon\,\}$$

$$E^2_N := \{\,\forall\,\xi\in[m,M]\ \ \Phi_N(-\xi) < \Phi_N(\xi) - \lambda\,\}$$

$$E^3_N := \{\,\forall\,\xi\in[M,\infty[\ \ \Phi_N(\xi) < C\,,\ \ \Phi_N(\xi) < \varphi(\xi)\,\}$$

and set $E_{N,\varepsilon} := E^1_{N,\varepsilon}\cap E^2_N \cap E^3_N$. It is convenient to split the expectation of $\log Z_N$ as follows:

$$\mathbb{E}_{\boldsymbol{x}}\left[\frac{1}{N}\log Z_N\right] \ =\ \mathbb{E}_{\boldsymbol{x}}\left[\frac{1}{N}\log Z_N\ \mathbb{1}\big(E_{N,\varepsilon}\big)\right] \ +\ \mathbb{E}_{\boldsymbol{x}}\left[\frac{1}{N}\log Z_N\ \mathbb{1}\big((E_{N,\varepsilon})^c\big)\right]. \tag{43}$$

In the following we are going to see that in the limit $N \to \infty$ the second term on the r.h.s. of (43) is negligible, while the first term can be computed using the Laplace's method.

By the lemma 3.7, using the Hölder inequality and the lemma 3.8,

$$\left|\,\mathbb{E}_{\boldsymbol{x}}\left[\frac{1}{N}\log Z_N\ \mathbb{1}\big((E_{N,\varepsilon})^c\big)\right]\right| \ \leq\ \mathbb{E}_{\boldsymbol{x}}\left[\left(\frac{1}{N}\log Z_N\right)^2\right]^{1/2}\ \mathbb{P}_{\boldsymbol{x}}\big((E_{N,\varepsilon})^c\big)^{1/2} \ \xrightarrow[N\to\infty]{}\ 0\ . \tag{44}$$

[*Upper bound*] Using the Gaussian representation (16), a simple upper bound for Z_N is

$$Z_N \ \leq\ \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{-\frac{N}{2w}\xi^2} \prod_{i=1}^N |\xi + x_i|\ \mathrm{d}\xi \ =\ \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{N\,\Phi_N(\xi)}\ \mathrm{d}\xi\ . \tag{45}$$

If the event $E_{N,\varepsilon}$ holds true, remembering also the inequality (25), then the following upper bound holds:

$$
\begin{aligned}
&\int_{\mathbb{R}} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \,\leq \\
&\leq\, 2\int_0^m e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \,+\, \int_m^M e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi_N(\xi)-\lambda)}\,\mathrm{d}\xi \,+\, 2\int_M^\infty e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \,\leq \\
&\leq\, 2\int_0^m e^{N\,(\Phi(\xi)+\varepsilon)}\,\mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi(\xi)+\varepsilon)}\,\mathrm{d}\xi \,+\, \int_m^M e^{N\,(\Phi(\xi)+\varepsilon-\lambda)}\,\mathrm{d}\xi \,+\, 2\,e^{(N-1)\,C}\int_M^\infty e^{\varphi(\xi)}\,\mathrm{d}\xi \,= \\
&\underset{N\to\infty}{=}\, O\big(e^{N\,(\max_{[0,m]}\Phi+\varepsilon)}\big) \,+\, e^{N\,(\Phi(\xi^*)+\varepsilon)}\,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}} \,+\, O\big(e^{N\,(\Phi(\xi^*)+\varepsilon-\lambda)}\big) \,+\, O\big(e^{N\,C}\big)\,;
\end{aligned}
\tag{46}
$$

the last step is obtained by applying the Laplace's method (theorem A2) to the function Φ, which by lemma 3.5 satisfies all the necessary hypothesis. Now since $\max_{[0,m]}\Phi$, $\Phi(\xi^*)-\lambda$ and C are strictly smaller than $\Phi(\xi^*)$, it holds

$$
\text{r.h.s. of (46)} \underset{N\to\infty}{\sim} e^{N\,(\Phi(\xi^*)+\varepsilon)}\,\frac{\sqrt{2\pi}}{\sqrt{-N\,\Phi''(\xi^*)}}\,. \tag{47}
$$

As a consequence of (45), (46), (47),

$$
\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon}) \,\leq\, \Phi(\xi^*)+\varepsilon+O\bigg(\frac{\log N}{N}\bigg)\,,
$$

where the $O(\frac{\log N}{N})$ is deterministic. Therefore for all $\varepsilon>0$

$$
\limsup_{N\to\infty}\mathbb{E}_{\boldsymbol{x}}\bigg[\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon})\bigg] \,\leq\, \Phi(\xi^*)+\varepsilon\,. \tag{48}
$$

[*Lower bound*] Observe that the product $\prod_{i=1}^N(\xi+x_i)$ is always positive for $\xi\geq 0$, while it is negative for some $\xi<0$. Hence using the Gaussian representation (16), a lower bound for Z_N is

$$
\begin{aligned}
Z_N \,&\geq\, \frac{\sqrt{N}}{\sqrt{2\pi w}}\bigg(\int_0^\infty e^{-\frac{N}{2w}\,\xi^2}\prod_{i=1}^N|\xi+x_i|\,\mathrm{d}\xi \,-\, \int_{-\infty}^0 e^{-\frac{N}{2w}\,\xi^2}\prod_{i=1}^N|\xi+x_i|\,\mathrm{d}\xi\bigg) \,= \\
&=\, \frac{\sqrt{N}}{\sqrt{2\pi w}}\bigg(\int_0^\infty e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \,-\, \int_{-\infty}^0 e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi\bigg)\,.
\end{aligned}
\tag{49}
$$

If the event $E_{N,\varepsilon}$ holds true, remembering also the inequality (25), then the following lower bound holds:

$$
\begin{aligned}
&\int_0^{\infty} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;-\; \int_{-\infty}^{0} e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;\geq \\
&\geq \int_m^M e^{N\,\Phi_N(\xi)}\,\mathrm{d}\xi \;-\; \int_m^M e^{N\,(\Phi_N(\xi)-\lambda)}\,\mathrm{d}\xi \;\geq \\
&\geq \int_m^M e^{N\,(\Phi(\xi)-\varepsilon)}\,\mathrm{d}\xi \;-\; \int_m^M e^{N\,(\Phi(\xi)+\varepsilon-\lambda)}\,\mathrm{d}\xi \;= \\
&\underset{N\to\infty}{=} \; e^{N\,(\Phi(\xi^*)-\varepsilon)}\,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}} \;-\; e^{N\,(\Phi(\xi^*)+\varepsilon-\lambda)}\,\frac{\sqrt{2\pi}\,(1+o(1))}{\sqrt{-N\,\Phi''(\xi^*)}}\;;
\end{aligned}
\tag{50}
$$

the last step is obtained by applying the Laplace's method (theorem A2) to the function Φ, which by lemma 3.5 satisfies all the necessary hypothesis. Now since $\Phi(\xi^*)+\varepsilon-\lambda<\Phi(\xi^*)-\varepsilon$ for all $0<\varepsilon<\frac{1}{2}\lambda$, for such a choice of ε it holds

$$
\text{r.h.s. of (50)} \underset{N\to\infty}{\sim} e^{N\,(\Phi(\xi^*)-\varepsilon)}\,\frac{\sqrt{2\pi}}{\sqrt{-N\,\Phi''(\xi^*)}}\,. \tag{51}
$$

As a consequence of (49), (50), (51), for all $0<\varepsilon<\frac{1}{2}\lambda$

$$
\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon}) \;\geq\; \left(\Phi(\xi^*)-\varepsilon+O\!\left(\frac{\log N}{N}\right)\right)\mathbb{1}(E_{N,\varepsilon})\,,
$$

where the $O(\frac{\log N}{N})$ is deterministic. Therefore, using also the lemma 3.7, for all $0<\varepsilon<\frac{1}{2}\lambda$

$$
\liminf_{N\to\infty}\mathbb{E}_{\boldsymbol{x}}\!\left[\frac{1}{N}\log Z_N\;\mathbb{1}(E_{N,\varepsilon})\right] \;\geq\; \liminf_{N\to\infty}\left(\Phi(\xi^*)-\varepsilon+O\!\left(\frac{\log N}{N}\right)\right)\mathbb{P}_{\boldsymbol{x}}(E_{N,\varepsilon}) \;=\; \Phi(\xi^*)-\varepsilon\,. \tag{52}
$$

In conclusion the convergence $\mathbb{E}_{\boldsymbol{x}}[\frac{1}{N}\log Z_N]\to\Phi(\xi^*)$ as $N\to\infty$ is proven by considering (43) for $0<\varepsilon<\frac{1}{2}\lambda$, then letting $N\to\infty$ exploiting (44), (48), (52), and finally letting $\varepsilon\to 0+$. □

Remark 3.9. In the deterministic case, namely when the distribution of the x_i's is a Dirac delta centred at a point x, the theorem 3.2 and its corollary 3.4 reproduce the results obtained in the Proposition 6 of [3] by a combinatorial computation. Indeed the fixed point equation (19) reduces to $\xi^*=\frac{w}{\xi^*+x}$, whose positive solution is

$$
\xi^* = \frac{-x+\sqrt{x^2+4w}}{2}\,.
$$

As a consequence, by (21) the limiting dimer and monomer density are respectively

$$
d=\frac{(\xi^*)^2}{2w}=\frac{x^2-x\sqrt{x^2+4w}+2w}{2w}\,,\quad m=1-2\,d=\frac{-x^2+x\sqrt{x^2+4w}}{2w}\,.
$$

Moreover by (17) and (21) the limiting pressure can be written as

$$
p=\Phi(\xi^*)=-\frac{(\xi^*)^2}{2w}+\log(\xi^*+x)=-d-\frac{1}{2}\,\log\frac{2\,d}{w}\,.
$$

4 Self-averaging for monomer-dimer models

In this section we prove that under quite general hypothesis a monomer-dimer model with independent random weights has self-averaging pressure density. In particular it will follows that the convergence (17) of the theorem 3.2 can be strengthen as

$$\mathbb{P}_{\boldsymbol{x}} \text{ - almost surely } \exists \lim_{N\to\infty} \frac{1}{N} \log Z_N \;=\; \sup_{\xi\geq 0} \Phi(\xi) \;, \tag{53}$$

when in the hypothesis of the theorem 3.2 one substitutes $\mathbb{E}_x[x] < \infty$, $\mathbb{E}_x[(\log x)^2] < \infty$ with the stronger $\mathbb{E}_x[x] < \infty$, $\mathbb{E}_x[x^{-1}] < \infty$.

In general let $w_{ij}^{(N)} \geq 0\,,\ 1 \leq i < j \leq N,\ N \in \mathbb{N}$, and $x_i > 0\,,\ i \in \mathbb{N}$, be *independent* random variables. Since the dimer weights may be allowed to take the value 0 (or to be identically 0), we do not really know on which kind of graph the model lives, on the contrary the framework is very general (for example the complete graph is included, but also finite-dimensional lattices or diluted random graphs are). This is why we allow a generic dependence of the dimer weights on N, in case a normalisation is needed. During all this section we will denote

$$Z_N \;:=\; \sum_{D\in\mathscr{D}_N} \prod_{ij\in D} w_{ij}^{(N)} \prod_{i\in M_N(D)} x_i \;. \tag{54}$$

Denote simply by $\mathbb{E}[\,\cdot\,]$ the expectation with respect to all the weights and assume that

$$\sup_N \sup_{1\leq i<j\leq N} \mathbb{E}[w_{ij}^{(N)}] =: C_1 < \infty\,, \quad \sup_{i\in\mathbb{N}} \mathbb{E}[x_i] =: C_2 < \infty\,, \quad \sup_{i\in\mathbb{N}} \mathbb{E}[x_i^{-1}] =: C_3 < \infty\,. \tag{55}$$

Clearly the pressure $p_N := \frac{1}{N}\log Z_N$ is a random variable and it has finite expectation, indeed

$$N\,p_N \begin{cases} \geq \log \prod_{i=1}^N x_i \;=\; \sum_{i=1}^N \log x_i \;\geq\; \sum_{i=1}^N (1 + x_i^{-1}) \;\in L^1(\mathbb{P}) \\ \leq\; Z_N - 1 \;\in L^1(\mathbb{P}) \end{cases} .$$

The following theorem shows that in the limit $N \to \infty$ the pressure p_N concentrates around its expectation, or in other terms it tends to become a deterministic quantity.

Theorem 4.1. *Let* $w_{ij}^{(N)} \geq 0\,,\ 1 \leq i < j \leq N,\ N \in \mathbb{N}$, *and* $x_i > 0\,,\ i \in \mathbb{N}$, *be independent random variables that satisfy* (55). *Then for all* $t > 0$, $N \in \mathbb{N}$, $q \geq 1$

$$\mathbb{P}\big(\,|p_N - \mathbb{E}[p_N]| \geq t\big) \;\leq\; 2\,\exp\left(-\,\frac{t^2\,N}{4\,q^2\,\log^2 N}\right) \;+\; (a + b\,N)\,N^{1-q}\;, \tag{56}$$

where $a := 4 + 2C_2C_3\,,\ b := 2C_1C_3^2\,.$ *As a consequence, choosing* $q > 3$,

$$|p_N - \mathbb{E}[p_N]| \xrightarrow[N\to\infty]{} 0 \;\; \mathbb{P}\text{-almost surely}\,. \tag{57}$$

If the random variables $w_{ij}^{(N)}$, x_i, x_i^{-1} are bounded, then one could obtain an exponential rate of convergence instead of (56), but here we prefer to obtain the result (57) with minimal assumptions.

Proof. Fix $N \in \mathbb{N}$. Set $w_i := \big(w^{(N)}_{i(i+1)}, \dots, w^{(N)}_{iN}\big)$ for all $i = 1, \dots, N-1$. We consider the filtration of length $2N-1$ such that in the first N steps the monomer weights x_i are exposed, while in the last $N-1$ steps the vectors w_i of dimer weights are exposed. Since p_N is a function of $x_1, \dots, x_N,\ w_1, \dots, w_{N-1}$ and $\mathbb{E}[|p_N|] < \infty$, we may define the Doob martingale of p_N with respect to this filtration

$$M_i := \mathbb{E}\big[\, p_N \,\big|\, x_1, \dots, x_i \big] \quad \forall\, i = 0, \dots, N \;,$$
$$M_{N+i} := \mathbb{E}\big[\, p_N \,\big|\, x_1, \dots, x_N, w_1, \dots, w_i \big] \quad \forall\, i = 1, \dots, N-1 \;;$$

in particular it holds $M_0 = \mathbb{E}[\,p_N]$ and $M_{2N-1} = p_N$.
Now we want to bound the increments $|M_i - M_{i-1}|$ for every $i = 1, \dots, 2N-1$, in order to apply the Azuma's inequality. By hypothesis $x_1, \dots, x_N,\ w_1, \dots, w_{N-1}$ are stochastically independent, hence the conditional expectations are simply $M_i = \mathbb{E}_{\boldsymbol{x}^{i+1},\boldsymbol{w}}[p_N]$ for $i = 0, \dots, N$ and $M_{N+i} = \mathbb{E}_{\boldsymbol{w}^{i+1}}[p_N]$ for $i = 1, \dots, N-1$. As a consequence it is easy to check that for $i = 1, \dots, N$ it holds

$$|M_i - M_{i-1}| \leq \sup_{\tilde{\boldsymbol{x}}_{i-1},\, \tilde{\boldsymbol{x}}^{i+1},\, \tilde{\boldsymbol{w}}} \big|\, p_N\big(\tilde{\boldsymbol{x}}_{i-1},\, x_i,\, \tilde{\boldsymbol{x}}^{i+1},\, \tilde{\boldsymbol{w}}\big) - \mathbb{E}_{x_i}\big[\, p_N\big(\tilde{\boldsymbol{x}}_{i-1},\, x_i,\, \tilde{\boldsymbol{x}}^{i+1},\, \tilde{\boldsymbol{w}}\big)\big]\big| \tag{58}$$

and for $i = 1, \dots, N-1$ it holds

$$|M_{N+i} - M_{N+i-1}| \leq \sup_{\tilde{\boldsymbol{w}}_{i-1},\, \tilde{\boldsymbol{w}}^{i+1}} \big|\, p_N\big(\boldsymbol{x},\, \tilde{\boldsymbol{w}}_{i-1},\, w_i,\, \tilde{\boldsymbol{w}}^{i+1}\big) - \mathbb{E}_{w_i}\big[\, p_N\big(\boldsymbol{x},\, \tilde{\boldsymbol{w}}_{i-1},\, w_i,\, \tilde{\boldsymbol{w}}^{i+1}\big)\big]\big| \,. \tag{59}$$

Here we have adopted the following notation $\boldsymbol{x} := (x_1, \dots, x_N)$, $\boldsymbol{x}_k := (x_1, \dots, x_k)$, $\boldsymbol{x}^k := (x_k, \dots, x_N)$ and similarly $\boldsymbol{w} := (w_1, \dots, w_{N-1})$, $\boldsymbol{w}_k := (w_1, \dots, w_k)$, $\boldsymbol{w}^k := (w_k, \dots, w_N)$; the symbols with a tilde denote a deterministic value taken by the corresponding random quantity.

First fix $i = 1, \dots, N$, fix the deterministic vectors $\tilde{\boldsymbol{x}}_{i-1}$, $\tilde{\boldsymbol{x}}^{i+1}$, $\tilde{\boldsymbol{w}}$ and let x'_i, x''_i be two independent random variables distributed as x_i. Set

$$p'_N := p_N\big(\tilde{\boldsymbol{x}}_{i-1}, x'_i,\, \tilde{\boldsymbol{x}}^{i+1},\, \tilde{\boldsymbol{w}}\big) \,, \quad p''_N := p_N\big(\tilde{\boldsymbol{x}}_{i-1},\, x''_i,\, \tilde{\boldsymbol{x}}^{i+1},\, \tilde{\boldsymbol{w}}\big) \,.$$

To estimate the difference between p'_N, p''_N we use the Heilmann-Lieb recursion for the partition function of a monomer-dimer model (see [8] and the proposition 2.4):

$$\begin{aligned} p'_N - p''_N \;&=\; \frac{1}{N} \log \frac{Z'_N}{Z''_N} \;=\; \frac{1}{N} \log \frac{x'_i\, Z_{-i} \,+\, \sum_{j=1}^{i-1} \tilde{w}_{ji}\, Z_{-j-i} \,+\, \sum_{j=i+1}^{N} \tilde{w}_{ij}\, Z_{-i-j}}{x''_i\, Z_{-i} \,+\, \sum_{j=1}^{i-1} \tilde{w}_{ji}\, Z_{-j-i} \,+\, \sum_{j=i+1}^{N} \tilde{w}_{ij}\, Z_{-i-j}} \;\leq \\ &\leq\; \frac{1}{N} \log \left(\frac{x'_i}{x''_i} \,+\, 1 \right) \;; \end{aligned} \tag{60}$$

here we denote by Z_{-i}, Z_{-i-j} the partitions function of the model over the vertices $\{1, \dots, N\} \smallsetminus \{i\}$, $\{1, \dots, N\} \smallsetminus \{i, j\}$ respectively, with weights $\tilde{\boldsymbol{x}}_{i-1}$, $\tilde{\boldsymbol{x}}^{i+1}$, $\tilde{\boldsymbol{w}}_{i-1}$, $\tilde{\boldsymbol{w}}^{i+1}$. It is important (for the inequality in (60)) to notice that these partition functions do not depend on the weights x'_i, x''_i. In the same way one finds

$$p''_N - p'_N \;\leq\; \frac{1}{N} \log \left(\frac{x''_i}{x'_i} \,+\, 1 \right) \,. \tag{61}$$

Denote by $\mathbb{E}''$ the expectation with respect to the random variable x_i'' only. Then the inequalities (60), (61) provide respectively the following random bounds

$$p_N' - \mathbb{E}[p_N''] \;=\; \mathbb{E}''[p_N' - p_N''] \;\overset{(60)}{\leq}\; \mathbb{E}''\left[\frac{1}{N}\log\left(\frac{x_i'}{x_i''}+1\right)\right] \;\leq\; \frac{1}{N}\log\left(x_i'\,\mathbb{E}[x_i^{-1}]+1\right)\,; \tag{62}$$

$$\mathbb{E}[p_N''] - p_N' \;=\; \mathbb{E}''[p_N'' - p_N'] \;\overset{(61)}{\leq}\; \mathbb{E}''\left[\frac{1}{N}\log\left(\frac{x_i''}{x_i'}+1\right)\right] \;\leq\; \frac{1}{N}\log\left(\mathbb{E}[x_i]\,(x_i')^{-1}+1\right)\,. \tag{63}$$

Choose $q>0$ and the previous inequalities provide a bound for $|M_i - M_{i-1}|$ that holds true "with high probability":

$$\begin{aligned}
&\mathbb{P}\bigg(|M_i - M_{i-1}| > \frac{q}{N}\log N\bigg) \;\overset{(58)}{\leq}\; \mathbb{P}\bigg(\sup_{\tilde{\boldsymbol{x}}_{i-1},\,\tilde{\boldsymbol{x}}^{i+1},\,\tilde{\boldsymbol{w}}}\big|p_N' - \mathbb{E}[p_N'']\big| > \frac{q}{N}\log N\bigg) \;\leq \\
&\leq\; \mathbb{P}\bigg(\sup_{\tilde{\boldsymbol{x}}_{i-1},\,\tilde{\boldsymbol{x}}^{i+1},\,\tilde{\boldsymbol{w}}}\big(p_N' - \mathbb{E}[p_N'']\big) > \frac{q}{N}\log N\bigg) \;+\; \mathbb{P}\bigg(\sup_{\tilde{\boldsymbol{x}}_{i-1},\,\tilde{\boldsymbol{x}}^{i+1},\,\tilde{\boldsymbol{w}}}\big(\mathbb{E}[p_N''] - p_N'\big) > \frac{q}{N}\log N\bigg) \;\overset{(62),(63)}{\leq} \\
&\leq\; \mathbb{P}\bigg(\frac{1}{N}\log\left(x_i\,\mathbb{E}[x_i^{-1}]+1\right) > \frac{q}{N}\log N\bigg) \;+\; \mathbb{P}\bigg(\frac{1}{N}\log\left(\mathbb{E}[x_i]\,x_i^{-1}+1\right) > \frac{q}{N}\log N\bigg) \;= \\
&=\; \mathbb{P}\bigg(1 + x_i\,\mathbb{E}[x_i^{-1}] > N^q\bigg) \;+\; \mathbb{P}\bigg(1 + \mathbb{E}[x_i]\,x_i^{-1} > N^q\bigg) \;\leq \\
&\leq\; \mathbb{E}\bigg[1 + x_i\,\mathbb{E}[x_i^{-1}]\bigg]\,N^{-q} \;+\; \mathbb{E}\bigg[1 + \mathbb{E}[x_i]\,x_i^{-1}\bigg]\,N^{-q} \;\leq \\
&\leq\; 2\,(1 + C_2C_3)\,N^{-q}\;;
\end{aligned} \tag{64}$$

here at the penultimate step we have used the Markov inequality.

Now instead fix $i = 1,\dots,N-1$, fix the deterministic vectors $\tilde{\boldsymbol{w}}_{i-1}$, $\tilde{\boldsymbol{w}}^{i+1}$, let w_i', w_i'' be two independent random vectors distributed as w_i and leave the vector of monomer weights $\boldsymbol{x}$ random (choose w_i', w_i'' independent of $\boldsymbol{x}$ too). Reassign the notation previously used, setting now:

$$p_N' := p_N\big(\boldsymbol{x},\,\tilde{\boldsymbol{w}}_{i-1},\,w_i',\,\tilde{\boldsymbol{w}}^{i+1}\big)\,,\quad p_N'' := p_N\big(\boldsymbol{x},\,\tilde{\boldsymbol{w}}_{i-1},\,w_i'',\,\tilde{\boldsymbol{w}}^{i+1}\big)\,.$$

To estimate the difference between p_N', p_N'' we use again the Heilmann-Lieb recursion for the partition function (see [8] and the proposition 2.4):

$$\begin{aligned}
p_N' - p_N'' \;&=\; \frac{1}{N}\log\frac{Z_N'}{Z_N''} \;=\; \frac{1}{N}\log\frac{x_i\,Z_{-i} \,+\, \sum_{j=1}^{i-1}\tilde{w}_{ji}\,Z_{-j-i} \,+\, \sum_{j=i+1}^{N} w_{ij}'\,Z_{-i-j}}{x_i\,Z_{-i} \,+\, \sum_{j=1}^{i-1}\tilde{w}_{ji}\,Z_{-j-i} \,+\, \sum_{j=i+1}^{N} w_{ij}''\,Z_{-i-j}} \;\leq \\
&\leq\; \frac{1}{N}\log\left(1 \,+\, \frac{\sum_{j=i+1}^{N} w_{ij}'\,Z_{-i-j}}{x_i\,Z_{-i}}\right) \;=\; \frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N}\frac{w_{ij}'}{x_i\,x_j}\,\langle \mathbb{1}_{j\in M}\rangle_{-i}\right) \;\leq \\
&\leq\; \frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N}\frac{w_{ij}'}{x_i\,x_j}\right)\,;
\end{aligned} \tag{65}$$

we have denoted by Z_{-i}, Z_{-i-j} the partitions function of the model over the vertices $\{1,\dots,N\}\setminus\{i\}$, $\{1,\dots,N\}\setminus\{i,j\}$ respectively, with weights $\boldsymbol{x}_{i-1}$, $\boldsymbol{x}^{i+1}$, $\tilde{\boldsymbol{w}}_{i-1}$, $\tilde{\boldsymbol{w}}^{i+1}$. It is important (for the first inequality in (65)) to notice that these partition functions do not depend on the weights w_i', w_i''. In the same way one finds

$$p_N'' - p_N' \;\le\; \frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N} \frac{w_{ij}''}{x_i\,x_j}\right) . \tag{66}$$

Denote by $\mathbb{E}''$ the expectation with respect to the random vector w_i'' only. Then the inequalities (65), (66) provide respectively the following random bounds

$$p_N' - \mathbb{E}''[p_N''] \;=\; \mathbb{E}''[p_N' - p_N''] \;\overset{(66)}{\le}\; \frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N} \frac{w_{ij}'}{x_i\,x_j}\right) ; \tag{67}$$

$$\begin{aligned} \mathbb{E}''[p_N''] - p_N' \;&=\; \mathbb{E}''[p_N'' - p_N'] \;\overset{(67)}{\le}\; \mathbb{E}''\left[\frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N} \frac{w_{ij}''}{x_i\,x_j}\right)\right] \;\le \\ &\le\; \frac{1}{N}\log\left(1 \,+\, \sum_{j=i+1}^{N} \frac{\mathbb{E}[w_{ij}]}{x_i\,x_j}\right) . \end{aligned} \tag{68}$$

Choose $q > 0$ and the previous inequalities provide a bound for $|M_{N+i} - M_{N+i-1}|$ that holds true "with high probability":

$$\begin{aligned} &\mathbb{P}\left(|M_{N+i} - M_{N+i-1}| > \frac{q}{N}\log N\right) \;\overset{(59)}{\le}\; \mathbb{P}\left(\sup_{\tilde{\boldsymbol{w}}_{i-1},\,\tilde{\boldsymbol{w}}^{i+1}} \left|p_N' - \mathbb{E}''[p_N'']\right| > \frac{q}{N}\log N\right) \;\le \\ &\le\; \mathbb{P}\left(\sup_{\tilde{\boldsymbol{w}}_{i-1},\,\tilde{\boldsymbol{w}}^{i+1}} \big(p_N' - \mathbb{E}''[p_N'']\big) > \frac{q}{N}\log N\right) \,+\, \mathbb{P}\left(\sup_{\tilde{\boldsymbol{w}}_{i-1},\,\tilde{\boldsymbol{w}}^{i+1}} \big(\mathbb{E}''[p_N''] - p_N'\big) > \frac{q}{N}\log N\right) \;\overset{(66),(67)}{\le} \\ &\le\; \mathbb{P}\left(\frac{1}{N}\log\big(1 + \sum_{j=i+1}^{N} \frac{w_{ij}}{x_i\,x_j}\big) > \frac{q}{N}\log N\right) \,+\, \mathbb{P}\left(\frac{1}{N}\log\big(1 + \sum_{j=i+1}^{N} \frac{\mathbb{E}[w_{ij}]}{x_i\,x_j}\big) > \frac{q}{N}\log N\right) \;= \\ &\le\; \mathbb{P}\left(1 + \sum_{j=i+1}^{N} \frac{w_{ij}}{x_i\,x_j} > N^q\right) \,+\, \mathbb{P}\left(1 + \sum_{j=i+1}^{N} \frac{\mathbb{E}[w_{ij}]}{x_i\,x_j} > N^q\right) \;\le \\ &\le\; \mathbb{E}\left[1 + \sum_{j=i+1}^{N} \frac{w_{ij}}{x_i\,x_j}\right] N^{-q} \,+\, \mathbb{E}\left[1 + \sum_{j=i+1}^{N} \frac{\mathbb{E}[w_{ij}]}{x_i\,x_j}\right] N^{-q} \;\le \\ &\le\; 2\,(1 + N\,C_1 C_3^2)\,N^{-q} \;; \end{aligned} \tag{69}$$

here at the penultimate step we have applied the Markov inequality.

As an immediate consequence of (64) and (69),

$$\begin{aligned} &\mathbb{P}\left(\exists\, i = 1,\dots,2N-1 \text{ s.t. } |M_i - M_{i-1}| > \frac{q}{N}\log N\right) \;\le \\ &\le\; N\,\big(2\,(1 + C_2C_3)\,N^{-q}\big) \,+\, (N-1)\,\big(2\,(1 + N\,C_1C_3^2)\,N^{-q}\big) \\ &\le\; 2\,\big(2 + C_2C_3 + C_1C_3^2\,N\big)\,N^{1-q} \,. \end{aligned} \tag{70}$$

Therefore by the extended Azuma's inequality (theorem A4), for all $t>0$ it holds

$$\mathbb{P}\big(|M_{N-1}-M_0|\geq t\big)\;\leq\;2\,\exp\left(-\frac{t^2}{2}\,\frac{N}{2\,q^2\,\log^2 N}\right)\,+\,2\left(2+C_2C_3+C_1C_3^2\,N\right)N^{1-q} \tag{71}$$

and the proof of (56) is concluded. Choosing $q>3$ the r.h.s. of (56) is summable with respect to $N\in\mathbb{N}$, hence (57) follows by a standard application of the Borel-Cantelli lemma. $\square$

Appendix

In this appendix we state the main technical results used in the paper. We omit their proofs, that can be found in the literature.

Theorem A1 (Gaussian integration by parts; Wick-Isserlis formula)**.** *Let* $(\xi_1,\dots,\xi_n)$ *be a Gaussian random vector with mean* 0 *and positive semi-definite covariance matrix* $C=(c_{ij})_{i,j=1,\dots,n}$ *. Let* $f\colon\mathbb{R}^{n-1}\to\mathbb{R}$ *be a differentiable function such that* $\mathbb{E}\big[\big|\xi_1\,f(\xi_2,\dots,\xi_n)\big|\big]<\infty$ *and* $\mathbb{E}\big[\big|\frac{\partial f}{\partial\xi_j}(\xi_2,\dots,\xi_n)\big|\big]<\infty$ *for all* $j=2,\dots,n$*. Then:*

$$\mathbb{E}\big[\xi_1\,f(\xi_2,\dots,\xi_n)\big]\;=\;\sum_{j=2}^{n}c_{1j}\,\mathbb{E}\left[\frac{\partial f}{\partial\xi_j}(\xi_2,\dots,\xi_n)\right]. \tag{A1}$$

As a consequence one can prove the following:

$$\mathbb{E}\left[\prod_{i=1}^{n}\xi_i\right]\;=\;\sum_{\substack{P\text{ partition of}\\\{1,\dots,n\}\text{ into pairs}}}\;\prod_{\{i,j\}\in P}c_{ij}\;. \tag{A2}$$

The Gaussian integration by parts (A1) can be found in [16]. The Wick-Isserlis formula (A2) follows by (A1) using an induction argument; but it appeared for the first time in [10].

Theorem A2 (Laplace's method)**.** *Let* $\phi\colon[a,b]\to\mathbb{R}$ *be a function of class* C^2*. Suppose that there exists* $x_0\in\,]a,b[$ *such that*

i. $\phi(x_0)>\phi(x)$ *for all* $x\in[a,b]$ *(i.e.* x_0 *is the only global maximum point of* ϕ*);*

ii. $\phi''(x_0)<0$ *.*

Then as $n\to\infty$

$$\int_a^b e^{n\,\phi(x)}\,\mathrm{d}x\;=\;e^{n\,\phi(x_0)}\,\frac{\sqrt{2\pi}}{\sqrt{-n\,\phi''(x_0)}}\,\big(1+o(1)\big)\;. \tag{A3}$$

A formal proof of the Laplace's method can be found in [7].

Theorem A3 (uniform weak law of large numbers)**.** *Let* $\mathcal{X}$*,* Θ *be metric spaces. Let* X_i*,* $i\in\mathbb{N}$ *be i.i.d. random variables taking values in* $\mathcal{X}$*. Let* $f\colon\mathcal{X}\times\Theta\to\mathbb{R}$ *be a function such that* $f(\cdot,\theta)$ *is measurable for all* $\theta\in\Theta$*. Suppose that:*

i. Θ *is compact;*

ii. $\mathbb{P}\big(f(X_1,\cdot)\text{ is continuous at }\theta\big)=1$ *for all* $\theta\in\Theta$ *;*

iii. $\exists\, F\colon \mathcal{X} \to [0,\infty]$ *such that* $\mathbb{P}\big(|f(X_1,\theta)| \leq F(X_1)\big) = 1$ *for all* $\theta \in \Theta$ *and* $\mathbb{E}[F(X_1)] < \infty$ *.*

Then for all $\varepsilon > 0$

$$\mathbb{P}\bigg(\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^{n} f(X_i,\theta) - \mathbb{E}[f(X,\theta)]\bigg| \geq \varepsilon\bigg) \xrightarrow[n\to\infty]{} 0\ . \tag{A4}$$

The uniform law of large number appeared in [11]. It is based on the (standard) law of large numbers and on a compactness argument.

Theorem A4 (extension of the Azuma's inequality)**.** *Let* $M = (M_i)_{i=0,\dots,n}$ *be a real martingale with respect to a filter. Suppose that there exist constants* $\varepsilon > 0$ *and* $c_1,\dots,c_n < \infty$ *such that*

$$\mathbb{P}\big(\exists\, i = 1,\dots,n \text{ s.t. } |M_i - M_{i-1}| > c_i\big) \,\leq\, \varepsilon\ .$$

Then for all $t > 0$

$$\mathbb{P}\big(|M_n - M_0| > t\big) \,\leq\, 2\,\exp\bigg(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\bigg) + \varepsilon\ . \tag{A5}$$

The Azuma's inequality is a useful tool in the martingale theory that allows to obtain concentration results. Its usual formulation is given with $\varepsilon = 0$. The extension with $\varepsilon > 0$ can be found in [6]; but it can be proven also starting from the usual formulation and introducing a suitable stopping time, following the ideas in [18].

References

[1] A. Aharony, "Tricritical points in systems with random fields", *Physical Review B* **18**(7), 3318-3327 (1978)

[2] D. Alberici, P. Contucci, "Solution of the monomer-dimer model on locally tree-like graphs. Rigorous results", *Communications in Mathematical Physics*, doi 10.1007/s00220-014-2080-3 (2014)

[3] D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with attractive interaction. Exact solution and rigorous results", *Journal of Mathematical Physics* **55**, 063301 (2014)

[4] D. Alberici, P. Contucci, E. Mingione, "The exact solution of a mean-field monomer-dimer model with attractive potential", *Europhysics Letters* **106**, 10001-10005 (2014)

[5] T.S. Chang, "Statistical theory of the adsorption of double molecules", Proceedings of the Royal Society of London A **169**, 512-531 (1939)

[6] F. Chung, L. Lu, "Concentration inequalities and martingale inequalities - a survey", *Internet Mathematics* **3**(1) 79-127 (2006)

[7] N.G. De Bruijn, *Asymptotic methods in Analysis* 2nd ed., pp.63-65, North-Holland, Amsterdam, 1961

[8] O.J. Heilmann, E.H. Lieb, "Theory of monomer-dimer systems", *Communications in Mathematical Physics* **25**, 190-232 (1972)

[9] O.J. Heilmann, E.H. Lieb, "Monomers and dimers", *Physical Review Letters* **24**(25), 1412-1414 (1970)

[10] L. Isserlis, "On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables", *Biometrika* **12**, 134-139 (1918)

[11] R.I. Jennrich, "Asymptotic properties of non-linear least squares estimators", *The Annals of Mathematical Statistics* **40**(2), 633-643 (1969)

[12] J.K. Roberts, "Some properties of mobile and immobile adsorbed films", *Proceedings of the Cambridge Philosophical Society* **34**, 399-411 (1938)

[13] W. Rudin, *Principles of Mathematical Analysis* 3rd ed., pp. 223-228, McGraw-Hill International, USA, 1976

[14] S.R. Salinas, W.F. Wreszinski, "On the mean-field Ising model in a random external field", Journal of Statistical Physics **41**(l/2), 299-313 (1985)

[15] T. Schneider, E. Pytte, "Random-field instability of the ferromagnetic state", *Physical Review B* **15**(3), 1519-1522 (1977)

[16] M. Talagrand, *Spin Glasses: A Challange for mathematicians. Cavity and Mean Field Models*, pp. 574-575, Springer, Berlin, 2003

[17] I.G. Vladimirov, "The monomer-dimer model and Lyapunov exponents of homogeneous gaussian random fields", *Discrete and Continuous Dynamical Systems B* **18**(2), 575-600 (2013)

[18] N.C. Wormald, "The differential equation method for random processes and greedy algorithms", in *Lectures on Approximation and Randomized Algorithms*, pp. 73-155, M. Karonski and H.J. Proemel editors, PWN, Warsaw, 1999.