Mean-Field Monomer-Dimer models. A review.

Diego Alberici, Pierluigi Contucci, Emanuele Mingione

University of Bologna - Department of Mathematics
Piazza di Porta San Donato 5, Bologna (Italy)
E-mail: diego.alberici2@unibo.it, pierluigi.contucci@unibo.it, emanuele.mingione2@unibo.it

To Chuck Newman, on his 70th birthday

Abstract: A collection of rigorous results for a class of mean-field monomer-dimer models is presented. It includes a Gaussian representation for the partition function that is shown to considerably simplify the proofs. The solutions of the quenched diluted case and the random monomer case are explained. The presence of the attractive component of the Van der Waals potential is considered and the coexistence phase coexistence transition analysed. In particular the breakdown of the central limit theorem is illustrated at the critical point where a non Gaussian, quartic exponential distribution is found for the number of monomers centered and rescaled with the volume to the power 3/4.

1. Introduction

The monomer-dimer models, an instance in the wide set of interacting particle systems, have a relevant role in equilibrium statistical mechanics. They were introduced to describe, in a simplified yet effective way, the process of absorption of monoatomic or diatomic molecules in condensed-matter physics [15,16,39] or the behaviour of liquid solutions composed by molecules of different sizes [25]. Exact solutions in specific cases (e.g. the perfect matching problem) have been obtained on planar lattices [24,26,34,36,41] and the problem on regular lattices is also interesting for the liquid crystals modelling [1,20,27,31,37]. The impact and the interest that monomer-dimer models have attracted has progressively grown beyond physics. Their thermodynamic behaviour has indeed proved to be useful in computer science for the matching problem [11,33] or for the applications of statistical physics methods to the social sciences [10].

From the physical point of view monomers and dimers cannot occupy the same site of a lattice due to the strong repulsion generated by the Pauli exclusion principle. Beside this interaction though, as already noticed by Peierls [38] in the first theoretical physics accounts, the attractive component of the Van der Waals potentials might influence the phase structure of the model and its thermodynamic behaviour. With the contemporary presence of those two interactions the global physical observables become particularly difficult to study. Generic Gaussian fluctuations on each ergodic component can still be expected but the nature of the critical point, if any, is a priori not obvious.

Here we focus on a set of monomer-dimer models in the mean field setting, i.e. on the complete graph where every site interacts in average with any other, and present a review of recent results. Section 2 introduces the general properties of the monomer-dimer systems that we approach with the help of a Gaussian representation for their partition function. This representation and its combinatorial features help to embed and ease part of the classical difficulties of their studies. The celebrated Heilmann and Lieb relation, so rich of rigorous consequences, emerges as the formula of integration by parts for Gaussian random vectors. The absence of phase transition for the pure hard-core case is therefore derived. Section 3 treats two quenched cases, namely the diluted complete graph of Erdős-Rényi type as well as the diluted random monomer field activity. For both cases we compute the exact solution. The diluted graph is treated with the help of correlation inequalities and the representation of the monomer density as the solution of an iterative distributional equation. The random monomer activity model is solved by reducing the computation of the equilibrium state to a standard variational problem, again with the help of the Gaussian representation. Section 4 introduces a genuine deterministic mean field model with and without the attractive interaction. It is shown how the model with attraction displays a phase space structure similar to the mean field ferromagnet but without the usual plus-minus symmetry. The model has a coexistence line bounded by a critical point with standard mean-field critical exponents. In Section 5 we show that while outside the critical point the central limit theorem holds, at criticality it breaks down and the limiting distribution is found at a scale of $N^{3/4}$ and turns out to be a quartic exponential, like in the well known results by Newman and Ellis [21, 22] for the ferromagnet.

2. Definition and general properties

Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set $E \subseteq \{ij \equiv \{i,j\} \,|\, i \in V, \, j \in V, \, i \neq j\}$.

Definition 2.1 (Monomer-dimer configurations). *A set of edges $D \subseteq E$ is called a monomer-dimer configuration, or a matching, if the edges in D are pairwise non-incident. The space of all possible monomer-dimer configurations on the graph G is denoted by $\mathscr{D}_G$.*

Given a monomer-dimer configuration D, we say that every edge in D is occupied by a *dimer*, while every vertex that does not appear in D is occupied by a *monomer*. The set of monomers associated to D is denoted by $M(D)$.
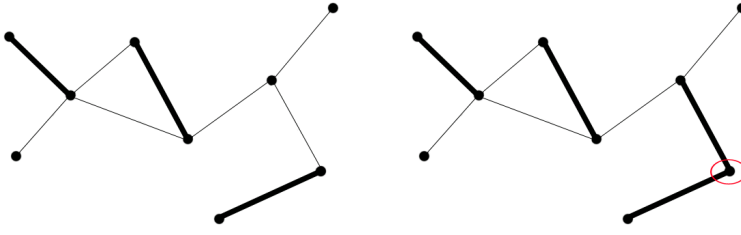
Fig. 1. The bold edges in the left figure form a monomer-dimer configuration on the graph, while those in the right figure do not because two of them share a vertex.

Remark 2.2. We can associate the dimer occupation variable $\alpha_{ij} \in \{0,1\}$ to each edge $ij \in E$: the edge ij is occupied by a dimer if and only if α_{ij} takes the value 1. It is clear that monomer-dimer configurations are in one-to-one correspondence with vectors $\alpha \in \{0,1\}^E$ satisfying the following constraint:

$$\sum_{j \sim i} \alpha_{ij} \le 1, \quad \forall\, i \in V \tag{2.1}$$

where $j \sim i$ means that $ij \in E$. Therefore, with a slight abuse of notation, we denote by $\mathscr{D}_G$ also the set of $\alpha \in \{0,1\}^E$ that satisfy eq. (2.1). The condition (2.1) guarantees that at most one dimer can be incident to a given vertex i, namely two dimers cannot be incident. This fact is usually referred as *hard-core interaction* or *hard-core constraint* or *monogamy constraint*. We also introduce an auxiliary variable, the monomer occupation variable,

$$\alpha_i := 1 - \sum_{j \sim i} \alpha_{ij} \in \{0,1\} \tag{2.2}$$

for each vertex $i \in V$: the vertex i is occupied by a monomer if and only if α_i takes the value 1.

The definition of monomer-dimer configurations already allows to raise non-trivial combinatorial questions as "How many monomer-dimer configurations, for a fixed number of dimers, exist on given a graph G?". This combinatorial problem is known to be NP-hard in general, but there are polynomial algorithms and exact solutions for specific cases [24, 29, 33, 34, 41]. In Statistical Mechanics a further structure is introduced and the previous problem becomes a specific limit case. We consider a Gibbs probability measure on the set of monomer-dimer configurations. There are several choices for the measure, depending on how we decide to model the interactions in the system.

2.1. Pure hard-core interaction. This amounts to take into account only the hard-core interaction among particles and assign a dimer activity $w_{ij} \ge 0$ to each edge $ij \in E$ and a monomer activity $x_i > 0$ to each vertex $i \in V$.

Definition 2.3 (Monomer-dimer models with pure hard-core interaction). *A pure monomer-dimer model on G is given by the following probability*

measure on $\mathscr{D}_G$:

$$\mu_G(D) := \frac{1}{Z_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \quad \forall\, D \in \mathscr{D}_G \,, \qquad (2.3)$$

where the normalizing factor, called partition function, is

$$Z_G := \sum_{D \in \mathscr{D}_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \,. \qquad (2.4)$$

We denote by $\langle \cdot \rangle_G$ the expectation w.r.t. the measure μ_G. The dependence of the measure on the activities w_{ij}, x_i is usually implicit in the notations.

Remark 2.4. Equivalently, one can think the measure (2.3) as a Gibbs measure on the space of occupancy variables α (see Remark 2.2), namely

$$\mu_G(\alpha) = \frac{1}{Z_G} e^{-H_G(\alpha)} \quad \forall\, \alpha \in \mathscr{D}_G \,, \qquad (2.5)$$

where

$$H_G(\alpha) := -\sum_{ij \in E} h_{ij}\alpha_{ij} - \sum_{i \in V} h_i\alpha_i \quad \forall\, \alpha \in \mathscr{D}_G \qquad (2.6)$$

is the *Hamiltonian function* and $h_i := \log x_i$, $h_{ij} := \log w_{ij}$ are called *monomer, dimer field* respectively. The partition function (2.4) rewrites

$$Z_G = \sum_{\alpha \in \mathscr{D}_G} \exp\Big(\sum_{ij \in E} h_{ij}\alpha_{ij} + \sum_{i \in V} h_i\alpha_i \Big) \,. \qquad (2.7)$$

Remark 2.5. It is worth to notice that the definition 2.3 is redundant for two reasons. First one can consider without loss of generality monomer-dimer models on complete graphs only: a monomer-dimer model on the graph $G = (V, E)$ coincides with a monomer-dimer model on the complete graph with $N = |V|$ vertices, by taking $w_{ij} = 0$ for all pairs $ij \notin E$. In this case we denote the partition function (2.7) with Z_N. Secondly, one can set without loss of generality all the monomer activities equal to 1: the monomer-dimer model with activities (w_{ij}, x_i) coincides with the monomer-dimer model with activities $(\frac{w_{ij}}{x_i x_j}, 1)$, since the relation

$$\prod_{i \in M(D)} x_i = \Big(\prod_{i \in V} x_i \Big) \prod_{ij \in D} \frac{1}{x_i\, x_j} \,.$$

shows that the partition function is multiplied by an overall constant and therefore the probability measure is left unchanged. The same argument shows also that if the dimer activity is uniform on the graph then it can be set equal to 1: the monomer-dimer model with activities (w, x_i) coincides with the monomer-dimer model with activities $(1, \frac{x_i}{\sqrt{w}})$, since

$$w^{|D|} = w^{N/2} \left(\frac{1}{\sqrt{w}} \right)^{|M(D)|} \,.$$

Remark 2.6. The following bounds for the pressure (logarithm of the partition function) will be useful:

$$\sum_{i \in V} \log x_i \; \leq \; \log Z_G \; \leq \; \sum_{i \in V} \log x_i + \sum_{ij \in E} \log \left(1 + \frac{w_{ij}}{x_i \, x_j} \right) . \qquad (2.8)$$

The lower bound is obtained considering only the configuration with no dimers, while the upper bound is obtained by eliminating the hard-core constraint.

An interesting fact about monomer-dimer models is that they are strictly related to Gaussian random vectors.

Proposition 2.7 (Gaussian representation [5, 43]). *The partition function of any monomer-dimer model over N vertices can be written as*

$$Z_N \; = \; \mathbb{E}_\xi \left[\prod_{i=1}^{N} (\xi_i + x_i) \right] , \qquad (2.9)$$

where $\xi = (\xi_1, \ldots, \xi_N)$ is a Gaussian random vector with mean 0 and covariance matrix $W = (w_{ij})_{i,j=1,\ldots,N}$ and $\mathbb{E}_\xi[\,\cdot\,]$ denotes the expectation with respect to ξ. The diagonal entries w_{ii} are arbitrary numbers, chosen in such a way that W is a positive semi-definite matrix.

Proof. The monomer-dimer configurations on the complete graph are all the partitions into pairs of any set $A \subseteq \{1, \ldots, N\}$, hence

$$Z_N = \sum_{D \in \mathscr{D}_N} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i = \sum_{A \subseteq \{1,\ldots,N\}} \sum_{\substack{P \text{ partition} \\ \text{of } A \text{ into pairs}}} \prod_{ij \in P} w_{ij} \prod_{i \in A^c} x_i .$$
$$(2.10)$$

Now choose w_{ii} for $i = 1, \ldots, N$ such that the matrix $W = (w_{ij})_{i,j=1,\ldots,N}$ is positive semi-definite[1]. Then there exists an (eventually degenerate) Gaussian vector $\xi = (\xi_1, \ldots, \xi_N)$ with mean 0 and covariance matrix W. And by the Isserlis-Wick rule

$$\mathbb{E}_\xi \left[\prod_{i \in A} \xi_i \right] \; = \; \sum_{\substack{P \text{ partition} \\ \text{of } A \text{ into pairs}}} \prod_{ij \in P} w_{ij} . \qquad (2.11)$$

Substituting (2.11) into (2.10) one obtains (2.9). □

We notice that the representation (2.9) allows to express average values w.r.t. the measure (2.3) as Gaussian averages. For example, given a vertex $i \in V$, its monomer probability by (2.7) writes

$$\langle \alpha_i \rangle_N \; = \; \frac{\partial}{\partial h_i} \log Z_N \qquad (2.12)$$

then, using the representation (2.9) in r.h.s. of (2.12) together with the identity $\frac{\partial}{\partial h_i} \equiv x_i \frac{\partial}{\partial x_i}$, one obtains

$$\langle \alpha_i \rangle_N \; = \; x_i \, \mathbb{E}_\xi \left[\frac{1}{\xi_i + x_i} g_N(\xi, x) \right] \qquad (2.13)$$

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[1] For example one can choose $w_{ii} \geq \sum_{j \neq i} w_{ij}$ for every $i = 1, \ldots, N$.

where $g_N(\xi, x) = \dfrac{1}{Z_N} \prod_{i=1}^{N} (\xi_i + x_i)$.

Heilmann and Lieb [29] provided a recursion for the partition functions of monomer-dimer models. As we will see this is a fundamental tool to obtain exact solutions and to prove general properties.

Proposition 2.8 (Heilmann-Lieb recursion [29]). *Fixing any vertex $i \in V$ it holds:*

$$Z_G = x_i \, Z_{G-i} + \sum_{j \sim i} w_{ij} \, Z_{G-i-j} \,. \tag{2.14}$$

Here $G - i$ denotes the graph obtained from G deleting the vertex i and all its incident edges.

The Heilmann-Lieb recursion can be obtained directly from the definition (2.4), exploiting the hard-core constraint: the first term on the r.h.s. of (2.14) corresponds to a monomer on i, while the following terms correspond to a dimer on ij for some j neighbour of i. Here we show a different proof that uses Gaussian integration by parts.

Proof (see [5]). Set $N := |V|$. Introduce zero dimer weights $w_{hk} = 0$ for all the pairs $hk \notin E$, so that $Z_G \equiv Z_N$. Following the proposition 2.7, introduce an N-dimensional Gaussian vector ξ with mean 0 and covariance matrix W. Then write the identity (2.9) isolating the vertex i:

$$Z_G = \mathbb{E}_\xi \left[\prod_{k=1}^{N} (\xi_k + x_k) \right] = x_i \, \mathbb{E}_\xi \left[\prod_{k \neq i} (\xi_k + x_k) \right] + \mathbb{E}_\xi \left[\xi_i \prod_{k \neq i} (\xi_k + x_k) \right]. \tag{2.15}$$

Now apply the Gaussian integration by parts to the second term on the r.h.s. of (2.15):

$$\mathbb{E}_\xi \left[\xi_i \prod_{k \neq i} (\xi_k + x_k) \right] = \sum_{j=1}^{N} \mathbb{E}_\xi [\xi_i \xi_j] \, \mathbb{E}_\xi \left[\frac{\partial}{\partial \xi_j} \prod_{k \neq i} (\xi_k + x_k) \right] = \sum_{j \neq i} w_{ij} \, \mathbb{E}_\xi \left[\prod_{k \neq i, j} (\xi_k + x_k) \right]. \tag{2.16}$$

Notice that summing over $j \neq i$ in the r.h.s. of (2.16) is equivalent to sum over $j \sim i$, since by definition $w_{ij} = 0$ if $ij \notin E$. Substitute (2.16) into (2.15):

$$Z_G = x_i \, \mathbb{E}_\xi \left[\prod_{k \neq i} (\xi_k + x_k) \right] + \sum_{j \sim i} w_{ij} \, \mathbb{E}_\xi \left[\prod_{k \neq i, j} (\xi_k + x_k) \right]. \tag{2.17}$$

To conclude the proof observe that $(\xi_k)_{k \neq i}$ is an $(N-1)$-dimensional Gaussian vector with mean 0 and covariance $(w_{hk})_{h,k \neq i}$. Hence by proposition 2.7

$$Z_{G-i} = \mathbb{E}_\xi \left[\prod_{k \neq i} (\xi_k + x_k) \right]. \tag{2.18}$$

And similarly

$$Z_{G-i-j} = \mathbb{E}_\xi \left[\prod_{k \neq i, j} (\xi_k + x_k) \right]. \tag{2.19}$$

The main general result about monomer-dimer models is the absence of phase transitions, proved by Heilmann and Lieb [29, 30]. This result is obtained by localizing the complex zeros of the partition functions far from the positive real axes. A different probabilistic approach has been later proposed by van den Berg [12].

Theorem 2.9 (Zeros of the partition function [29]). *Consider uniform monomer activity x on the graph and arbitrary dimer activities w_{ij}. The partition function Z_G is a polynomial of degree N in x, where $N = |V|$. The complex zeros of Z_G are purely imaginary:*

$$\{x \in \mathbb{C} \,|\, Z_G(w_{ij}, x) = 0\} \subset {}_1\mathbb{R} \,. \qquad (2.20)$$

Furthermore they interlace the zeros of Z_{G-i} for any given $i \in V$, that is:

$$a_1 \le a_1' \le a_2 \le a_2' \le \cdots \le a_{N-1}' \le a_N \,, \qquad (2.21)$$

where $-{}_1a_1, \ldots, -{}_1a_N$ are the zeros of Z_G and $-{}_1a_1', \ldots, -{}_1a_{N-1}'$ are the zeros of Z_{G-i}. The relation (2.21) holds with strict inequalities if $w_{ij} > 0$ for all $i, j \in V$.

Corollary 2.10 (Absence of phase transitions). *Consider dimer activities $w\,w_{ij}^{(N)}$ and monomer activities $x\,x_i^{(N)}$ and assume they are chosen in such a way that $p := \lim_{N\to\infty} \frac{1}{N} \log Z_N$ exists. Then the function p is analytic in the variables $(w, x) \in (0, \infty)^2$ and the derivatives $\frac{\partial^{h+k}}{\partial^h w \, \partial^k x}$ can be interchanged with the limit $N \to \infty$.*

2.2. Hard-core and imitative interactions. Beyond the hard-core constraint it is possible to enrich monomer-dimer models with other kinds of interaction. For example in this work we consider, for a given $D \in \mathscr{D}_G$, the set of edges connecting particles of the same kind

$$I(D) = \{ij \in E \,|\, i, j \in M(D) \text{ or } i, j \notin M(D)\} \qquad (2.22)$$

and we introduce an interaction between any pair of vertices $ij \in I(D)$ tuned by a coupling $J_{ij} \in \mathbb{R}$. More precisely

Definition 2.11 (Monomer-dimer models with imitative interactions). *An imitative monomer-dimer model on G is given by the following Gibbs probability measure on $\mathscr{D}_G$:*

$$\mu_G(D) := \frac{1}{Z_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \prod_{ij \in I(D)} e^{J_{ij}} \qquad (2.23)$$

for all $D \in \mathscr{D}_G$. The partition function is

$$Z_G := \sum_{D \in \mathscr{D}_G} \prod_{ij \in D} w_{ij} \prod_{i \in M(D)} x_i \prod_{ij \in I(D)} e^{J_{ij}} \qquad (2.24)$$

The dependence of the measure on the coefficients w_{ij}, x_i, J_{ij} is usually implicit in the notations.

When all the J_{ij}'s take the value zero this model is the pure hard-core model introduced in the previous section. Positive values of the J_{ij}'s favour the configurations with clusters of dimers and clusters of monomers.

Remark 2.12. The usual Gibbs form $\frac{1}{Z_G}\, e^{-H_G(\alpha)}$ for the measure (2.24) is obtained by setting $x_i =: e^{h_i}$, $w_{ij} =: e^{h_{ij}}$ and taking as Hamiltonian function

$$H_G(\alpha) := -\sum_{ij \in E} h_{ij}\alpha_{ij} - \sum_{i \in V} h_i\alpha_i - \sum_{ij \in E} J_{ij}\big(\alpha_i\alpha_j + (1-\alpha_i)(1-\alpha_j)\big) \quad (2.25)$$

for all $\alpha \in \mathscr{D}_G$.

The Gaussian representation and the recursion relation found for the pure hard-core case can be extended to the imitative case.

Proposition 2.13. *The partition function of any monomer-dimer model over N vertices can be written as*

$$Z_N = \mathbb{E}_\xi\Big[\sum_{A \subset \{1,\ldots,N\}} \prod_{i \in A} \xi_i \prod_{i \in A^c} x_i \prod_{\substack{i,j \in A \ or \\ i,j \in A^c}} e^{J_{ij}/2} \Big], \qquad (2.26)$$

where $\xi = (\xi_1, \ldots, \xi_N)$ is a Gaussian random vector with mean 0 and covariance matrix $W = (w_{ij})_{i,j=1,\ldots,N}$ and $\mathbb{E}_\xi[\cdot]$ denotes the expectation with respect to ξ. The diagonal entries w_{ii} are arbitrary numbers, chosen in such a way that W is a positive semi-definite matrix. Moreover we set $J_{ii} = 0$.

The proof is the same as proposition 2.7. It is interesting to observe that, when all the ξ_i's are positive, the sum inside the expectation on the r.h.s. of (2.26) is the partition function of an Ising model.

Proposition 2.14. *Fixing any vertex $i \in V$ it holds:*

$$Z_G = x_i\, \widetilde{Z}_{G-i} + \sum_{j \sim i} w_{ij}\, \widetilde{Z}_{G-i-j}, \qquad (2.27)$$

where:

  – in the partition function $\widetilde{Z}_{G-i}$ the monomer activity x_k is replaced by $x_k\, e^{J_{ik}}$ for every vertex k (notice that only the neighbours of i actually change their activities);
  – in the partition function $\widetilde{Z}_{G-i-j}$ the dimer activity $w_{kk'}$ is replaced by $w_{kk'}\, e^{J_{ik}+J_{ik'}+J_{jk}+J_{jk'}}$ for all vertices k, k' (notice that only the neighbours of i or j actually change their activities).

The relation (2.27) can be obtained directly from the definition: the first term on the r.h.s. corresponds to a monomer on i, while the following terms correspond to a dimer on ij for some j neighbour of i.

The hard-core interaction is not sufficient to cause a phase transition, but adding also the imitative interaction the system can have phase transitions [6, 15, 16, 31]: in sections 4 we will study this phase transition on the complete graph. The location of the zeros of the partition function in the complex plane in presence of imitation is an open problem.

3. Quenched models: Erdős-Rényi and random field

In this section we consider monomer-dimer models with pure hard-core interactions in some random environment: the randomness is either in the structure of the graph or in the activities. In the first case we considered a class of random graphs that have locally tree-like structure and finite variance degree distribution [2]: this is the same for which the ferromagnetic Ising model was rigorously solved by Dembo and Montanari [17,19], using the local weak convergence strategy developed in [9]. For the sake of clarity, in this review we have chosen to present the results on the Erdős-Rényi random graph, but they easily extend for example to random regular graphs and configuration models.

3.1. Self-averaging for monomer-dimer models. One of the most important property describing the effects of randomness in statistical mechanics models is the *self-averaging* of physical quantities. Under quite general hypothesis a monomer-dimer model with independent random weights has a self-averaging pressure density [5].

Let $w_{ij}^{(N)} \geq 0$, $1 \leq i < j \leq N$, $N \in \mathbb{N}$, and $x_i > 0$, $i \in \mathbb{N}$, be *independent* random variables and consider the (random!) partition function

$$Z_N = \sum_{D \in \mathscr{D}_N} \prod_{ij \in D} w_{ij}^{(N)} \prod_{i \in M_N(D)} x_i . \qquad (3.1)$$

Since the dimer weights may depend on N and may take the value zero, this framework is very general. Denote simply by $\mathbb{E}[\,\cdot\,]$ the expectation with respect to all the weights and assume

$$\sup_N \sup_{ij} \mathbb{E}[w_{ij}^{(N)}] =: C_1 < \infty , \quad \sup_i \mathbb{E}[x_i] =: C_2 < \infty , \quad \sup_i \mathbb{E}[x_i^{-1}] =: C_3 < \infty .$$
$$(3.2)$$

The pressure density $p_N := \frac{1}{N} \log Z_N$ is a random variable with finite expectation, indeed

$$N\,p_N \begin{cases} \geq \log \prod_{i=1}^{N} x_i = \sum_{i=1}^{N} \log x_i \geq \sum_{i=1}^{N}(1 + x_i^{-1}) \in L^1(\mathbb{P}) \\ \leq Z_N - 1 \in L^1(\mathbb{P}) \end{cases} .$$

The following theorem shows that in the limit $N \to \infty$ the pressure density p_N concentrates around its expectation.

Theorem 3.1 (see [5]). *For all $t > 0$, $N \in \mathbb{N}$, $q \geq 1$*

$$\mathbb{P}\big(|p_N - \mathbb{E}[p_N]| \geq t\big) \leq 2 \exp\left(-\frac{t^2 N}{4\,q^2\,\log^2 N}\right) + (a + b\,N)\,N^{1-q} , \qquad (3.3)$$

where $a := 4 + 2C_2 C_3$, $b := 2C_1 C_3^2$. As a consequence, choosing $q > 3$,

$$|p_N - \mathbb{E}[p_N]| \xrightarrow[N \to \infty]{} 0 \quad \mathbb{P}\text{-almost surely} . \qquad (3.4)$$

If the random variables $w_{ij}^{(N)}$, x_i, x_i^{-1} are bounded, then one can obtain an exponential rate of convergence instead of (3.3).

3.2. Erdős-Rényi random graph. Let G_N be a Erdős-Rényi random graph over N vertices: each pair of vertices is connected by an edge independently with probability $c/N > 0$. Denote by $Z_N(x)$ the partition function of a monomer-dimer model with monomer activity $x > 0$ and pure hard-core interaction on the graph G_N:

$$Z_N(x) \ = \ \sum_{D \in \mathscr{D}_{G_N}} x^{N-2|D|} \ , \qquad (3.5)$$

$\langle \, \cdot \, \rangle_{G_N,x}$ will be the corresponding Gibbs expected value. The pressure density is

$$p_N(x) \ := \ \frac{1}{N} \, \log Z_N(x) \ , \qquad (3.6)$$

and the monomer density is

$$m_N(x) \ := \ \Big\langle \frac{N - 2|D|}{N} \Big\rangle_{N,x} \ = \ x \, \frac{\partial p_N}{\partial x}(x) \ . \qquad (3.7)$$

Since the set of configurations $\mathscr{D}_{G_N}$ is random, the partition function, the pressure density and the monomer density are random variables and the Gibbs measure is a random measure. This randomness is treated as quenched with respect to the thermal fluctuations.

Theorem 3.2 (see [2, 40]). *Almost surely and for all $x > 0$ the monomer density and the pressure density converge in the thermodynamical limit. Precisely:*

$$m_N(x) \ \xrightarrow[N\to\infty]{a.s.} \ \mathbb{E}[M(x)] \qquad (3.8)$$

$$p_N(x) \ \xrightarrow[N\to\infty]{a.s.} \ \mathbb{E}\big[\log \frac{M(x)}{x}\big] \ - \ \frac{c}{2} \, \mathbb{E}\big[\log\big(1 + \frac{M_1(x)}{x} \, \frac{M_2(x)}{x}\big)\big] \ . \qquad (3.9)$$

The law of the random variable $M(x)$ is the only solution supported in $[0, 1]$ of the following fixed point distributional equation:

$$M \ \overset{d}{=} \ \frac{x^2}{x^2 + \sum_{i=1}^{\Delta} M_i} \qquad (3.10)$$

where $(M_i)_{i\in\mathbb{N}}$ are i.i.d. copies of M and Δ is an independent Poisson(c)-distributed random variable. The limit monomer density and the limit pressure density are analytic functions of the activity $x > 0$.

The expression for the pressure on the right hand side of (3.9) was provided by Zdeborová and Mézard [45] via the theoretical physics method of cavity fields. This theorem provides a complete rigorous proof of their conjecture, partially studied in [13, 40].

The proof of theorem 3.2 relies on the locally tree-like structure of the Erdős-Rényi random graphs. Precisely fix a radius $r \in \mathbb{N}$ and for any vertex v denote by $[G_N, v]_r$ the ball of center v and radius r in the graph G_N; then consider a random tree T rooted at the vertex o and with independent Poisson(c)-distributed offspring sizes; it holds (see [18]):

$$\frac{1}{N} \sum_{v \in G_N} F\big([G_N, v]_r\big) \ \xrightarrow[n\to\infty]{a.s.} \ \mathbb{E}F\big([T, o]_r\big) \qquad (3.11)$$
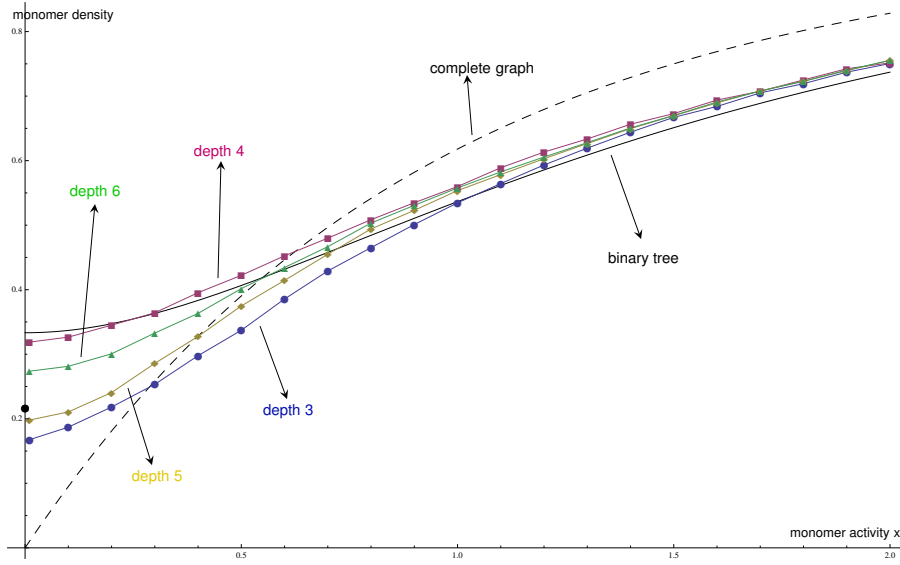
Fig. 2. Upper (even depths) and lower (odd depths) bounds for the limit monomer density $m(x) = \lim_{N \to \infty} m_N(x)$ versus the monomer activity x, in the Erdős-Rényi case with $c = 2$. The binary tree (continuous line) and the complete graph (dashed line) cases are also shown. The distributional recursion (3.10) is iterated a finite number r of times with initial values $M_i \equiv 1$: the obtained random variable $M_x(r)$ represents the root monomer probability of the random tree $[T, o]_r$. For values of $x = 0.01, 0.1, 0.2, \ldots, 2$, the random variables $M_x(r)$, $r = 3, 4, 5, 6$ are simulated numerically 10000 times and an empirical mean is taken in order to approximate $\mathbb{E}[M_x(r)]$. $\mathbb{E}[M_x(r)]$ provides an upper/low approximation of $m(x)$ when r is even/odd.

for every bounded real function F invariant under rooted graph isomorphisms.

Clearly the monomer density rewrites as an average over the vertices:

$$m_N(x) = \frac{1}{N} \sum_{v \in G_N} \mathcal{M}_x(G_N, v) \ , \quad \text{where}$$

$$\mathcal{M}_x(G_N, v) := \langle \mathbb{1}(v \text{ is a monomer}) \rangle_{G_N, x} \ . \tag{3.12}$$

A priori $\mathcal{M}_x(G_N, v)$ depends on the whole graph G_N, but it can be substituted by local quantities thanks to the following correlation inequalities:

Lemma 3.3 (Correlation inequalities). *Let (G, o) be a rooted graph, let $r \in \mathbb{N}$. If $[G, o]_{2r+1}$ is a tree, then*

$$\mathcal{M}_x([G, o]_{2r+1}) \leq \mathcal{M}_x(G, o) \leq \mathcal{M}_x([G, o]_{2r}) \ . \tag{3.13}$$

Therefore one can deduce that

$$m_N(x) \xrightarrow[N \to \infty]{a.s.} \lim_{r \to \infty} \mathbb{E}\mathcal{M}_x\big([T, o]_r\big) \tag{3.14}$$

provided the existence of the $\lim_{r \to \infty}$. In this way the problem on random graphs is reduce to the study of the root monomer probability on a random tree. As usual in Statistical Mechanics working on trees is much easier since there are no loops in the interactions.

The problem is now approached by means of the Heilmann-Lieb recursion. By lemma 3.3, the sequences of monomer probabilities respectively at even and odd depths of the tree are monotonic:

$$\mathcal{M}_x\big([T,o]_{2r}\big) \nearrow M_{\text{even}}(x) , \quad \mathcal{M}_x\big([T,o]_{2r+1}\big) \searrow M_{\text{odd}}(x) \quad \text{as } r \to \infty . \quad (3.15)$$

The relation (2.14) for partition functions gives the following relation for root monomer probabilities:

$$\begin{pmatrix} M_{\text{even}}(x) \\ M_{\text{odd}}(x) \end{pmatrix} \overset{d}{=} \begin{pmatrix} \frac{x^2}{x^2 + \sum_{i=1}^{\Delta} M_{\text{odd}}^{(i)}(x)} \\ \frac{x^2}{x^2 + \sum_{i=1}^{\Delta} M_{\text{even}}^{(i)}(x)} \end{pmatrix} \quad (3.16)$$

where $(M_{\text{even}}^{(i)}, M_{\text{odd}}^{(i)})$, $i \in \mathbb{N}$, are i.i.d. copies of $(M_{\text{even}}, M_{\text{odd}})$. A direct computation from equation (3.16) shows that

$$\mathbb{E}[|M_{\text{even}}(x) - M_{\text{odd}}(x)|] \leq \frac{c^2}{x^4} \mathbb{E}[|M_{\text{even}}(x) - M_{\text{odd}}(x)|] \quad (3.17)$$

therefore $M_{\text{even}}(x) = M_{\text{odd}}(x)$ almost surely for every $x > \sqrt{c}$. Now allow the monomer activity to take complex values in $\mathbb{H}_+ = \{z \in \mathbb{C} \,|\, \Re(z) > 0\}$. This has no physical or probabilistic meaning, but it is a technique to obtain powerful results by exploiting complex analysis. Using the Heilmann-Lieb recursion one can prove that for any rooted graph (G,o), the function $\mathcal{M}_z(G,o)$ is analytic in $z \in \mathbb{H}_+$ and is uniformly bounded as $|\mathcal{M}_z(G,o)| \leq |z|/\Re(z)$. It follows that the limit functions $M_{\text{even}}(z)$ and $M_{\text{odd}}(z)$ are analytic on $\mathbb{H}_+$. Therefore by uniqueness of the analytic continuation

$$M_{\text{even}}(x) = M_{\text{odd}}(x) =: M(x) \quad \text{almost surely for every } x > 0 \quad (3.18)$$

and (3.8) follows by (3.14). $M(x)$ satisfies the distributional fixed point equation (3.10). The solution supported in $[0,1]$ is unique, since for any random variable $M' \in [0,1]$ that satisfies (3.10) it can be shown by induction on $r \in \mathbb{N}$ that

$$\mathcal{M}_x\big([T,o]_{2r+1}\big) \overset{d}{\leq} M' \overset{d}{\leq} \mathcal{M}_x\big([T,o]_{2r}\big) . \quad (3.19)$$

These are the ideas to prove the convergence of the monomer density. To complete the theorem 3.2 it remains to prove the convergence of the pressure density. The convergence of $p_N(x)$ to some function $p(x)$ is guaranteed by the convergence of its derivative $m_N(x)/x$ together with the bounds 2.8. Call $\tilde{p}(x)$ the function defined by the right hand side of (3.9), which can be "guessed" by the heuristic method of energy shifts. Direct computations show that $x \tilde{p}'(x) = m(x) = x p'(x)$ for every $x > 0$ and $\lim_{x\to\infty} \tilde{p}(x) - \log x = 0 = \lim_{x\to\infty} p(x) - \log x$. Therefore $p = \tilde{p}$.

3.3. Random Field. For the class of models described above the randomness is in the graph structure. The model below instead introduces a randomness in the monomer activities and is useful to describe impurities. Consider the pure hard-core monomer-dimer model defined in 2.3 and assume that $G = (V, E)$ is the complete graph with N vertices, the monomer activities $(x_i)_{i \in V}$ are *i.i.d.* positive random variables and the dimer activity is uniform $w_{ij} = w/N > 0 \; \forall i \neq j \in V$. The partition function is

$$Z_N = \sum_{D \in \mathscr{D}_N} \left(\frac{w}{N}\right)^{|D|} \prod_{i \in M(D)} x_i \,. \tag{3.20}$$

Notice that now the partition function and the pressure density $\frac{1}{N} \log Z_N$ are random variables. The first important fact is that under the assumptions of Theorem 3.1 the pressure density is self-averaging, namely it converges almost surely to its expectation usually called *quenched pressure density*. The Gaussian representation for the partition function (2.9) and a careful application of the Laplace method allows us to find its limiting value. More precisely the next theorem shows that thermodynamic limit the quenched pressure density exists and is given by a one-dimensional variational principle, which admits a unique solution.

Theorem 3.4 (see [5]). *Let $w > 0$. Let $x_i > 0$, $i \in \mathbb{N}$ be i.i.d. random variables with $\mathbb{E}_x[x] < \infty$ and $\mathbb{E}_x[(\log x)^2] < \infty$. Then:*

$$\exists \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{\boldsymbol{x}}[\log Z_N] = \sup_{\xi \geq 0} \Phi(\xi) \in \mathbb{R} \tag{3.21}$$

where

$$\Phi(\xi) := -\frac{\xi^2}{2w} + \mathbb{E}_x[\log(\xi + x)] \quad \forall \, \xi \geq 0 \,, \tag{3.22}$$

the function Φ reaches its maximum at a unique point ξ^* which is the only solution in $[0, \infty[$ of the fixed point equation

$$\xi = \mathbb{E}_x\left[\frac{w}{\xi + x}\right] \,. \tag{3.23}$$

Theorem 3.4 allows to compute the main macroscopic quantity of physical interest, that is the *dimer density*, in terms of the positive solution ξ^* of the fixed point equation (3.23).

Corollary 3.5. *In the hypothesis of the theorem 3.4 the limiting pressure per particle*

$$p(w) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{\boldsymbol{x}}\left[\log Z_N(w)\right] \tag{3.24}$$

exists and is a smooth function of $w > 0$. Moreover the limiting dimer density

$$d := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{\boldsymbol{x}}\left[\langle |D| \rangle_N\right] = w \frac{\mathrm{d}p}{\mathrm{d}w} = \frac{(\xi^*)^2}{2w} \,. \tag{3.25}$$

A detailed proof of Theorem 3.4 can be found in [5]. Here we mention the main ideas. The first step is to use the Gaussian representation (2.9) for the partition function (3.20) that gives

$$Z_N = \mathbb{E}_\xi \left[\prod_{i=1}^N (\xi + x_i) \right] , \tag{3.26}$$

where ξ is a one-dimensional Gaussian random variable with mean 0 and variance w/N. Indeed by proposition 2.7, $Z_N = \mathbb{E}_g \left[\prod_{i=1}^N (g_i + x_i) \right]$ where $g = (g_1, \ldots, g_N)$ is an N-dimensional Gaussian random vector with mean 0 and constant covariance matrix $(w/N)_{i,j=1,\ldots,N}$. It is easy to check that the vector g has the same joint distribution of the vector $(\xi, \ldots, \xi)$ and the identity (3.26) follows. It is important to notice how easily, in this mean-field framework, the Gaussian representation reduces the degrees of freedom of the system. By explicitly rewriting (3.26) as

$$Z_N = \frac{\sqrt{N}}{\sqrt{2\pi w}} \int_{\mathbb{R}} e^{-\frac{N}{2w}\xi^2} \prod_{i=1}^N (\xi + x_i) \, d\xi . \tag{3.27}$$

and considering the function

$$f_N(\xi) := e^{-\frac{N}{2w}\xi^2} \prod_{i=1}^N (\xi + x_i) \quad \forall \xi \in \mathbb{R} \tag{3.28}$$

one sees that Theorem 3.4 follows by approximating e^Φ in the integral (3.27) with the Laplace method.

4. The mean-field case

Let $h \in \mathbb{R}$ and $J \geq 0$ and consider the imitative monomer-dimer model in definition 2.11 within the follwing assumptions: $G = (V, E)$ is the complete graph with N vertices and $\forall i \neq j \in V$ we set $w_{ij} = 1/N$, $x_i \equiv e^h$ and $J_{ij} = J/N$. Since the number of edges is of order N^2, in order to keep the logarithm of the partition function of order N, a normalisation of the dimer activity as $1/N$ (see Remark 2.6) and the imitation coefficient as J/N are needed.

One can express the Hamiltonian in terms of occupancy variables as

$$H_N(\alpha) := -h \sum_{i=1}^N \alpha_i - \frac{J}{N} \sum_{1 \leq i < j \leq N} \left(\alpha_i \alpha_j + (1 - \alpha_i)(1 - \alpha_j) \right) \tag{4.1}$$

for every monomer-dimer configuration on the complete graph $\alpha \in \mathscr{D}_N$. The partition function is

$$Z_N := \sum_{\alpha \in \mathscr{D}_N} N^{-D_N} \exp(-H_N) , \tag{4.2}$$

where $D_N := \sum_{1 \le i < j \le N} \alpha_{ij}$ represents the total number of dimer for a given configuration $\alpha \in \mathscr{D}_N$. Observe that the only relevant quantity in this setting is actually the total number of monomers in a given monomer-dimer configuration

$$M_N := \sum_{i=1}^{N} \alpha_i \tag{4.3}$$

indeed the hardcore constraint (2.2) implies that $M_N + 2D_N = N$ and the Hamiltonian (4.1) is actually a function of M_N only. We denote the corresponding Gibbs measure as

$$\mu_N(\alpha) := \frac{N^{-D_N(\alpha)} \exp(-H_N(\alpha))}{Z_N} \quad \forall \alpha \in \mathscr{D}_N \tag{4.4}$$

and the expectation with respect to the measure μ_N is denoted by $\langle \cdot \rangle_N$. In particular, setting $m_N := \frac{1}{N} \sum_{i=1}^{N} \alpha_i$, the average monomer density is

$$\langle m_N \rangle_N = \sum_{\alpha \in \mathscr{D}_N} \frac{\sum_{i=1}^{N} \alpha_i}{N} \frac{\exp(-H_N(\alpha))}{Z_N} = \frac{\partial}{\partial h} \frac{\log Z_N}{N} . \tag{4.5}$$

This model has been initially studied in [6, 7], where the behaviour of the pressure and monomer densities in the thermodynamic limit is analysed.

Theorem 4.1 (see [6]). *Let $h \in \mathbb{R}$, $J \ge 0$. Then there exists*

$$p := \lim_{N \to \infty} \frac{\log Z_N}{N} = \sup_m \psi(m) \tag{4.6}$$

the sup *can be taken indifferently over $m \in [0,1]$ or $m \in \mathbb{R}$ and*

$$\psi(m, h, J) := -Jm^2 + \frac{J}{2} + p^{(0)}(2Jm + h - J) \tag{4.7}$$

where for all $t \in \mathbb{R}$

$$p^{(0)}(t) := -\frac{1}{2}\big(1 - g(t)\big) - \frac{1}{2} \log \big(1 - g(t)\big) , \tag{4.8}$$

$$g(t) := \frac{1}{2} \left(\sqrt{e^{4t} + 4\,e^{2t}} - e^{2t} \right) . \tag{4.9}$$

Furthermore the function $\psi(m)$ attains its maximum in (at least) one point $m^* = m^*(h, J) \in (0, 1)$, which is a solution of the the consistency equation

$$m = g\big((2m - 1)\,J + h\big) . \tag{4.10}$$

At each value of the parameters (h, J) such that $h \mapsto m^*(h, J)$ is differentiable, the monomer density admits thermodynamic limit

$$\lim_{N \to \infty} \langle m_N \rangle_N = m^* . \tag{4.11}$$

In order to prove Theorem 4.1, first we need to deal with the case $J = 0$, then the limit (4.6) with $J > 0$ follows by a convexity argument introduced by Guerra [28] for the Curie-Weiss model. At $J = 0$ the model reduces to the pure monomer-dimer model of definition 2.3 on the complete graph with $x_i = x = e^h > 0$, $w_{ij} = 1/N > 0 \ \forall i \neq j \in V$. Let us denote by $Z_N^{(0)}$ and $\langle m_N \rangle_N^{(0)}$ respectively the partition function and the average monomer density at $J = 0$; it holds

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N^{(0)} = p^{(0)}(h) \tag{4.12}$$

$$\lim_{N \to \infty} \langle m_N \rangle_N^{(0)} = g(h) . \tag{4.13}$$

The function $p^{(0)}$ is analytic thus at $J = 0$ there are no phase transition in agreement with the general result of Heilman-Lieb [29]. The limit (4.12) can be obtained in two different ways:

1) by a combinatorial computation, since on the complete graph it is possible to compute explicitly the number of monomer-dimer configurations with a given number of monomers;
2) by using the Gaussian representation (2.9) of the partition function and the Laplace method.

The latter method furnish a better estimation of the convergence (4.12)

$$Z_N^{(0)}(h) \underset{N \to \infty}{\sim} \frac{\exp\left(N p^{(0)}(h)\right)}{\sqrt{2 - g(h)}} , \tag{4.14}$$

which will be fundamental in the study of the fluctuations of M_N (Section 5).

Remark 4.2. The limiting pressure density p can also be expressed as a different variational problem, equivalent to that of Theorem 4.1:

$$p = \sup_m \left(s(m) - \varepsilon(m)\right) \tag{4.15}$$

with

$$s(m) := -m \log m - \frac{1-m}{2} \log(1-m) + \frac{1+m}{2} \tag{4.16}$$

$$\varepsilon(m) := -J m^2 - (h - J) m - \frac{J}{2} . \tag{4.17}$$

The variational problem (4.15) can be obtained directly by the combinatorial computation mentioned before. The function s and ε in (4.16) are the *entropy and energy densities* respectively.

The properties of the solution(s) of the one-dimensional variational problem (4.6) appearing in theorem 4.1 determine the thermodynamic properties of the model. In particular we are interested in the value(s) of $m = m^*(h, J)$ where the maximum is reached, since it can be interpreted as the limiting value of the monomer density.

The function m^* (see [6]) is single-valued and smooth on the plane (h, J) with the exception of an implicitly defined open curve $h = \gamma(J)$ and its endpoint (h_c, J_c). Instead on γ there are two global maximum points $m_1 < m_2$ that

correspond to the *dimer* phase and the *monomer* phase respectively. Crossing the curve γ in the phase plane the order parameter m^* presents a jump discontinuity: in other words a *second order phase transition* occurs and γ is the coexistence curve. The point (h_c, J_c) is the *critical point* of the system, where m^* is continuous but not differentiable.
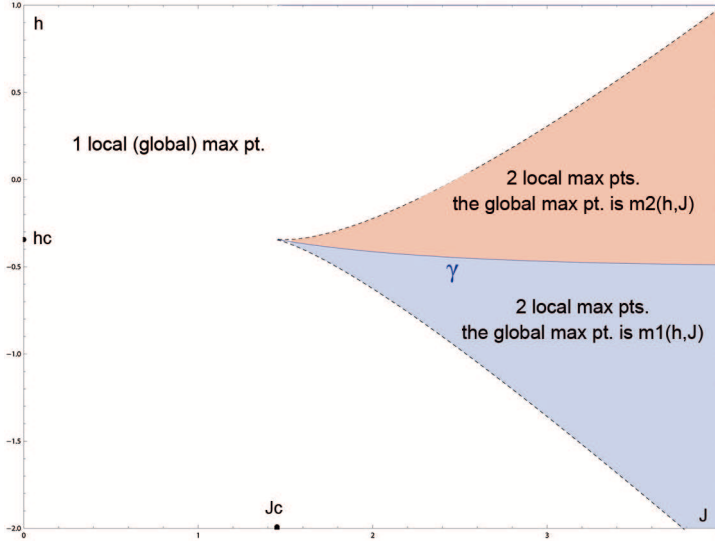

Fig. 3. Phase space (h, J). The curve γ separates the values (h, J) for which the global maximum point $m^*(h, J)$ of $m \mapsto \widetilde{p}(m, h, J)$ jumps between two values $m_1 < m_2$. This entails a discontinuity of $m^*(h, J)$ along the coexistence curve γ.

Remark 4.3. We notice that the techniques developed in [6] do not allow us to conclude the existence of the limiting monomer density on the coexistence curve γ. In the standard mean-field ferromagnetic model (Curie-Weiss) the existence of the magnetization on the coexistence curve $(h = 0)$ follows directly by the global spin flip symmetry, a property that we do not have in the present case.

The non analytic behaviour of $m^*(h, J)$ near the critical point is described by its *critical exponents*.

Theorem 4.4 (see [6]). *Consider the global maximum point $m^*(h, J)$ of the function $m \mapsto \widetilde{p}(m, h, J)$ defined by (4.15). Set $m_c := m^*(h_c, J_c)$. The critical exponents of m^* at the critical point (h_c, J_c) are:*

$$\boldsymbol{\beta} = \lim_{J \to J_c+} \frac{\log |m^*(\delta(J), J) - m_c|}{\log(J - J_c)} = \frac{1}{2}$$

along any curve $h = \delta(J)$ such that $\delta \in C^2([J_c, \infty[)$, $\delta(J_c) = h_c$, $\delta'(J_c) = \gamma'(J_c)$ (i.e. if the curve δ has the same tangent of the coexistence curve γ at the critical

point);

$$\frac{1}{\delta} = \lim_{J \to J_c} \frac{\log|m^*(\delta(J), J) - m_c|}{\log|J - J_c|} = \frac{1}{3}$$

$$\frac{1}{\delta} = \lim_{h \to h_c} \frac{\log|m^*(h, \delta(h)) - m_c|}{\log|h - h_c|} = \frac{1}{3}$$

along any curve $h = \delta(J)$ such that $\delta \in C^2(\mathbb{R}_+)$, $\delta(J_c) = h_c$, $\delta'(J_c) \neq \gamma'(J_c)$ or along a curve $J = \delta(h)$ such that $\delta \in C^2(\mathbb{R})$, $\delta(h_c) = J_c$, $\delta'(h_c) = 0$ (i.e. if the curve is not tangent to γ at the critical point).

Theorem 4.4 proves that the model belongs to the same *universality class* of the mean-field ferromagnet.

5. Distributional limit theorems at the critical point

In this section we study the distributional limit of the random variable *number of monomers* with respect to the Gibbs measure on the complete graph [3, 4]. We show that a *law of large numbers* holds outside the coexistence curve γ, whereas on γ the limiting distribution is a convex combination of two Dirac deltas representing the two phases (theorems 5.1, 5.2). Moreover we show that a *central limit theorem* holds outside $\gamma \cup (h_c, J_c)$, while at the critical point a normalisation of order $N^{-3/4}$ for the fluctuations is required and the limiting distribution is $Ce^{-cx^4}\mathrm{d}x$ (theorems 5.1, 5.4).

In [3] we follow the Gaussian convolution method introduced by Ellis and Newman for the mean-field Ising model (Curie-Weiss) in [21–23] in order to deal with the imitative potential. An additional difficulty stems from the fact that even in the absence of imitation the system keeps an interacting nature due to the presence of the hard-core interaction: we use the Gaussian representation 2.7 to *decouple* the hard-core interaction.

We focus on the behaviour of the random variable $M_N = \sum_{i=1}^{N} \alpha_i$ (number of monomers) with respect to the Gibbs measure (4.4) with a suitable scaling when $N \to \infty$. From now on δ_x is the Dirac measure centred at x, $\mathcal{N}(x, \sigma^2)$ denotes the Gaussian distribution with mean x and variance σ^2 and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution with respect to the Gibbs measure μ_N as $N \to \infty$.

At $J = 0$ the law of large numbers and the central limit theorem hold true:

Theorem 5.1 (see [3]). *At $J = 0$ the following results hold:*

$$\frac{M_N}{N} \xrightarrow{\mathcal{D}} \delta_{g(h)} \tag{5.1}$$

and

$$\frac{M_N - N\,g(h)}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\partial g}{\partial h}(h)\right) \tag{5.2}$$

where g is the function defined by (4.9).

Notice that, even if $J = 0$, (5.2) is not a consequence of the standard central limit theorem, indeed M_N is not a sum of i.i.d. random variables because of the presence of the hard-core interaction. The theorem 5.1 follows from the recent results of Lebowitz-Pittel-Ruelle-Speer [35]. A different proof is presented here which includes also the general value of $J > 0$. We should mention that a slightly improvement of the result presented has been obtained with different methods in [44].

Consider the asymptotic behaviour of the distribution of the number of monomers M_N with respect to the Gibbs measure μ_N. The *law of large numbers* holds outside the coexistence curve γ, on γ instead it breaks down in a convex combination of two Dirac deltas. Precisely it holds

Theorem 5.2 (see [3]).

i) In the uniqueness region $(h, J) \in \big(\mathbb{R} \times \mathbb{R}_+\big) \setminus \gamma$, denoting by m^* the unique global maximum point of the function $\tilde{p}(m)$ defined by (4.7), it holds

$$\frac{M_N}{N} \xrightarrow{\mathcal{D}} \delta_{m^*} \tag{5.3}$$

ii) On the coexistence curve $(h, J) \in \gamma$, denoting by m_1, m_2 the two global maximum points of $\tilde{p}(m)$, it holds

$$\frac{M_N}{N} \xrightarrow{\mathcal{D}} \varrho_1\, \delta_{m_1} + \varrho_2\, \delta_{m_2} \ , \tag{5.4}$$

where $\varrho_l = \frac{b_l}{b_1 + b_2}$, $b_l = (-\lambda_l(2 - m_l))^{-1/2}$ and $\lambda_l = \frac{\partial^2 \widetilde{p}}{\partial m^2}(m_l)$, for $l = 1, 2$.

Remark 5.3. We notice that, on the contrary of what happens for the Curie-Weiss model, the statistical weights ϱ_1 and ϱ_2 on the coexistence curve are in general different, furthermore they are not simply given in terms of the second derivative of the variational pressure $\widetilde{p}$.

The first fact can be seen numerically, and analytically one can compute

$$\lim_{J \to \infty} \frac{\varrho_1(J)}{\varrho_2(J)} = \frac{1}{\sqrt{2}} \ . \tag{5.5}$$

The second fact can be interpreted as follows: the relative weights ϱ_l have two contributions reflecting the presence of two different kinds of interaction. The first contribution λ_l is given by the second derivative of the variational pressure (4.7), while the second contribution $2 - m_l$ comes from the second derivative of the pressure of the pure hard-core model.

The central limit theorem holds outside the union of the coexistence curve γ and the critical point (h_c, J_c). At the critical point its breakdown results in a different scaling $N^{3/4}$ and in a different limiting distribution $Ce^{-cx^4}\mathrm{d}x$. Precisely

Theorem 5.4 (see [3]).

i) Outside the coexistence curve and the critical point $(h, J) \in \big(\mathbb{R} \times \mathbb{R}^+\big) \setminus \big(\gamma \cup (h_c, J_c)\big)$, it holds

$$\frac{M_N - Nm^*}{N^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \sigma^2\Big) \tag{5.6}$$

where $\sigma^2 = -\lambda^{-1} - (2J)^{-1} > 0$ and $\lambda = \frac{\partial^2 \widetilde{p}}{\partial m^2}(m^*) < 0$.

ii) At the critical point (h_c, J_c), it holds

$$\frac{M_N - Nm_c}{N^{3/4}} \xrightarrow{\mathcal{D}} C \, \exp\left(\frac{\lambda_c}{24} x^4\right) \mathrm{d}x \qquad (5.7)$$

where $\lambda_c = \frac{\partial^4 \widetilde{p}}{\partial m^4}(m_c) < 0$, $m_c \equiv m^*(h_c, J_c)$ and $C^{-1} = \int_{\mathbb{R}} \exp(\frac{\lambda_c}{24} x^4)\mathrm{d}x$.

The first step to obtain these results is to perform a Gaussian convolution, following the ideas of Ellis and Newmann [21,22], in order to decouple the imitative interaction. Precisely taking $W \sim \mathcal{N}(0, (2J)^{-1})$ a random variable independent of M_N for all $N \in \mathbb{N}$, for all $\eta \geq 0$ and $u \in \mathbb{R}$, a direct computation shows that the distribution of

$$\frac{W}{N^{1/2-\eta}} + \frac{M_N - Nu}{N^{1-\eta}} \qquad (5.8)$$

is

$$C_N \, \exp\left(N \, \tilde{p}_N \left(\frac{x}{N^\eta} + u\right)\right) dx \,, \qquad (5.9)$$

where $C_N^{-1} = \int_{\mathbb{R}} \exp\left(N \, \tilde{p}_N(\frac{x}{N^\eta} + u)\right) dx$,

$$\tilde{p}_N(x) := -Jx^2 + \frac{J}{2} + p_N^{(0)}(2Jx + h - J) \qquad (5.10)$$

and $p_N^{(0)}(t) = \frac{1}{N} \log Z_N^{(0)}(t)$ denotes the pressure density of the monomer-dimer system at imitation potential $J = 0$ and monomer field t. Therefore we are interested in the limit as $N \to \infty$ of quantities like

$$\int_{\mathbb{R}} \exp\left(N \, \tilde{p}_N \left(\frac{x}{N^\eta} + u\right)\right) \phi(x) \, dx \quad , \quad \phi \text{ bounded continuous} \qquad (5.11)$$

which depends crucially on the scaling properties of $\tilde{p}_N$ near its global maximum point(s). Thanks to the Gaussian representation at $J = 0$, and precisely from eq.(4.14), we know that $\tilde{p}_N$ converges to $\tilde{p}$ in a very strong way, which allows to replace the Taylor expansion of $\tilde{p}_N$ by that of $\tilde{p}$.

6. Conclusions and outlooks

The relation of the class of models presented so far with the physically relevant ones in finite dimensional lattices represent an interesting research problem that can be carried out following the steps of the studies done for the ferromagnetic spin models [32, 42]. We want to point out, moreover, that the range of direct applications of mean-field models like these ones is quite developed and quickly expanding. To make a few examples: the diluted mean-field case studied in Section 3 is directly related to the matching problem studied in computer sciences [33]. The model with attractive interaction studied in Section 5 has been applied to the social sciences [10]. There is also a growing set of applications of monomer-dimer models to the study of socio-technical data from novel communication systems like voip conference calls and messaging [8]. At each single time every user cannot be in more than a call, i.e. the occupation number fullfills a hard-core constraint. While the old style phone calls were well described by a monomer-dimer system the novel technological devices needs a wider space of

higher dimensional polymers that allow the presence of multiple individuals in the same virtual room: the monomer correspond to a silent user, the dimer is a two-body conversation, the trimer a three-body and so on. The models to be investigated in this case are therefore polymer models with hard-core interaction on hypergraphs with no physical dimension, i.e. better described as some form of dilution of the complete hypergraph. The mean-field case and its diluted versions are therefore at the heart of the problem and not only mere approximations.

References

1. D. Alberici, "A cluster expansion approach to the Heilmann-Lieb liquid crystal model", *Journal of Statistical Physics* 162(3), 761-791 (2016)
2. D. Alberici, P. Contucci, "Solution of the monomer-dimer model on locally tree-like graphs. Rigorous results", *Communications in Mathematical Physics* **331**, 975-1003 (2014)
3. D. Alberici, P. Contucci, M. Fedele, E. Mingione, "Limit theorems for monomer-dimer mean-field models with attractive potential", *Communications in Mathematical Physics* **346**, 781-799 (2016)
4. D. Alberici, P. Contucci, M. Fedele, E. Mingione, "Limit theorems for monomer-dimer mean-field models with attractive potential", *Europhysics Letters* **114**, 10006 (2016)
5. D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with randomness. Exact solution and rigorous results", *Journal of Statistical Physics* **160**, 1721-1732 (2015)
6. D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with attractive interaction. Exact solution and rigorous results", *Journal of Mathematical Physics* **55**, 1-27 (2014)
7. D. Alberici, P. Contucci, E. Mingione, "The exact solution of a mean-field monomer-dimer model with attractive potential", *Europhysics Letters* **106**, 1-5 (2014)
8. D. Alberici, P. Contucci, E. Mingione, M. Molari, "Aggregation models on hypergraphs", to appear on *Annals of Physics*
9. D. Aldous, J.M. Steele, "The objective method: probabilistic combinatorial optimization and local weak convergence", *Encyclopaedia of Mathematical Sciences* **110**, 1-72 (2004)
10. A. Barra, P. Contucci, R. Sandell, C. Vernia, "An analysis of a large dataset on immigrant integration in Spain. The statistical mechanics perspective on social action", *Scientific Reports* **4**, 4174 (2014)
11. M. Bayati, C. Nair, "A rigorous proof of the cavity method for counting matchings", *Proceedings of the 44th Annual Allerton Conference on Communication, Control and Computing* (2006)
12. J. van den Berg, "On the absence of phase transition in the monomer-dimer model", *Perplexing Problems in Probability*, 185-195 (1999)
13. C. Bordenave, M. Lelarge, J. Salez, "Matchings on infinite graphs", *Probability Theory and Related Fields* **157**, 183-208 (2013)
14. R. Burioni, P. Contucci, M. Fedele, C. Vernia, A. Vezzani, "Enhancing participation to health screening campaigns by group interactions", *Scientific Reports* **5**, 9904 (2015)
15. Chang T.S., "Statistical theory of the adsorption of double molecules", *Proceedings of the Royal Society of London A* **169**, 512-531 (1939)
16. Chang T.S., "The number of configurations in an assembly and cooperative phenomena", *Proceedings of the Cambridge Philosophical Society* **38**, 256-292 (1939)
17. A. Dembo, A. Montanari, "Ising models on locally tree-like graphs", *Annals of Applied Probability* **20**, 565-592 (2010)
18. A. Dembo, A. Montanari, "Gibbs measures and phase transitions on sparse random graphs", *Brazilian Journal of Probability and Statistics* **24**, 137-211 (2010)
19. A. Dembo, A. Montanari, N. Sun, "Factor models on locally tree-like graphs", *Annals of Probability* **41**(6), 4162-4213 (2013)
20. M. Disertori, A. Giuliani, "The nematic phase of a system of long hard rods", *Communications in Mathematical Physics* **323**(1), 143-175 (2013)
21. R.S. Ellis, C.M. Newman, "The statistics of Curie-Weiss models", *Journal of Statistical Physics* **19**, 149-161 (1978)
22. R.S. Ellis, C.M. Newman, "Limit theorems for sums of dependent random variables occurring in statistical mechanics", *Probability Theory and Related Fields* **44**, 117-139 (1978)

23. R.S. Ellis, C.M. Newman, J.S. Rosen, "Limit theorems for sums of dependent random variables occurring in statistical mechanics", *Probability Theory and Related Fields* **51**, 153-169 (1980)
24. M.E. Fisher, "Statistical mechanics of dimers on a plane lattice", Physical Review **124**(6), 1664-1672 (1961)
25. R.H. Fowler, G.S. Rushbrooke, "An attempt to extend the statistical theory of perfect solutions", *Transactions of the Faraday Society* **33**, 1272-1294 (1937)
26. A. Giuliani, I. Jauslin, E.H. Lieb, "A Pfaffian formula for monomer-dimer partition functions", Journal of Statistical Physics **163**(2) (2016)
27. A. Giuliani, E.H. Lieb, "Columnar phase in quantum dimer models", *Journal of Physics A: Mathematical and Theoretical* **48**(23) (2015)
28. F. Guerra, "Mathematical aspects of mean field spin glass theory", *4th European Congress of Mathematics, Stockholm June 27-July 2 2004*, ed. A. Laptev, European Mathematical Society, 2005
29. O.J. Heilmann, E.H. Lieb, "Theory of monomer-dimer systems", *Communications in Mathematical Physics* **25**, 190-232 (1972)
30. O.J. Heilmann, E.H. Lieb, "Monomers and dimers", *Physical Review Letters* **24**, 1412-1414 (1970)
31. O.J. Heilmann, E.H. Lieb, "Lattice models for liquid crystals", *Journal of Statistical Physics* **20**, 680-693 (1979)
32. M. Kac, C.J. Thompson, "Critical Behavior of Several Lattice Models with Long-Range Interaction",*J. Math. Phys.*, **10**, 1373 (1969)
33. R. Karp, M. Sipser, "Maximum matchings in sparse random graphs", *Proceedings of the 22nd Annual Symposium on Foundations of Computer Science*, IEEE Computer Society Press, 364-375 (1981)
34. P.W. Kasteleyn, "The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice", *Physica* **27**(12), 1209-1225 (1961)
35. J.L. Lebowitz, B. Pittel, D. Ruelle, E.R. Speer, "Central limit theorems, Lee-Yang zeros, and graph-counting polynomials", preprint *arxiv:1408.4153* (2014)
36. E.H. Lieb, "The solution of the dimer problems by the transfer matrix method", *Journal of Mathematical Physics* **8**, 2339-2341 (1967)
37. L. Onsager, "The effects of shape on the interaction of colloidal particles", *Annals of the New York Academy of Sciences* **51**, 627-659 (1949)
38. R. Peierls, "On Ising's model of ferromagnetism", *Mathematical Proceedings of the Cambridge Philosophical Society* **32**(3), 477-481 (1936)
39. J.K. Roberts, "Some properties of mobile and immobile adsorbed films", *Proceedings of the Cambridge Philosophical Society* **34**, 399-411 (1938)
40. J. Salez, "Weighted enumeration of spanning subgraphs in locally tree-like graphs", *Random Structures and Algorithms* **43**, 377-397 (2013)
41. H.N.V. Temperley, M.E. Fisher, "Dimer problem in statistical mechanics - An exact result", *Philosophical Magazine* **6**(68), 1061-1063 (1961)
42. C.J., Thompson, "Ising Model in the High Density Limit", *Commun. Math. Phys.*, **36**, 255262 (1974)
43. I.G. Vladimirov, "The monomer-dimer model and Lyapunov exponents of homogeneous Gaussian random fields", *Discrete and Continuous Dynamical Systems B* **18**, 575-600 (2013)
44. W.-K. Chen, "Limit Theorems in the Imitative Monomer-Dimer Mean-Field Model via Stein's Method", *J. Math. Phys.*, **57**, 083302 (2016)
45. L. Zdeborová, M. Mézard, "The number of matchings in random graphs", *Journal of Statistical Mechanics* **5**, P05003 (2006)