Dilution Robustness for Mean Field Ferromagnets

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Abstract

In this work we compare two different random dilution of a mean field ferromagnet: the first model is built on a Bernoulli-diluted network while the second lives on a Poisson-diluted network. While it is known that the two models have in the thermodynamic limit the same free energy we investigate on the structural constraints that the two models must fulfill. We rigorously derive for each model the set of identities for the multi-overlaps distribution using different methods for the two dilutions: constraints in the former model are obtained by studying the consequences of the self-averaging of the internal energy density, while in the latter are obtained by a stochastic-stability technique. Finally we prove that the identities emerging in the two models are the same, showing *robustness* of the ferromagnetic properties of diluted networks with respect to the details of dilution.

Keywords: diluted networks, spin glasses, polynomial identities.

1 Introduction

In the past decades an increasing interest has been shown in statistical mechanics models living on diluted networks (see i.e. [3][10][12][26][27][29]). For diluted spin glasses [23][7] this interest is at least double motivated: Despite their mean field nature, they share with finite-dimensional models the fact that each spin interact with a finite number of other spins. Secondly, they are mathematically equivalent to some random optimization problems (i.e. K-SAT or X-OR-SAT depending on the size of the instantaneous interaction [24][25]).

Simpler ferromagnets [4][5][18], even thought not interesting from hard satisfiability viewpoints, are still interesting for their finite connectivity nature and as a benchmark for testing different variations on the topology of the underlying graph where they live.

With this aim we consider two different ways in diluting the network [13]: In the first ferromagnet, links are distributed accordingly to a Bernouilli probability distribution, in the second ferromagnet, links are distributed accordingly to a Poisson probability distribution.

For these models we compared the properties of a family of linear constraints for the order parameters (often known as Aizenman-Contucci polynomials [2][6] in the case of spin glasses).

We choose to investigate these relations as in the earlier development of these constraints [2][20] in the spin glass framework in which they were obtained as a result of the stability of the quenched measure with respect to random perturbation or equivalently through the bound on the fluctuation properties of the internal energy. Here we propose them as a test for robustness of the dilution.

The method to approach the identities for the two model is structurally different. For the Poisson case in fact the additive law of Poissonian random variables make possible the direct exploitation of the stochastic stability property. The same strategy is not available for the Bernoulli random variables but for those we derive the set of identities from the general bound on the quenched fluctuations (even though, for the sake of completeness, in the Appendix we derive the constraints within this general framework of this general framework for the Poisson case too). The methods we use are generalizations of those appearing in [8][11][16][17][22][28][15][19][6][1].

Our main result is a rigorous proof of the identities and the fact that they coincide for the two dilutions.

2 The mean field diluted ferromagnet

We introduce a large number N of sites, labeled by the index $i = 1, ..., N$, where Ising variables $\sigma_i = \pm 1$ are paste.
Then we introduce two families of discrete independent random variables $\{i_\nu\}$,$\{j_\nu\}$, uniformly distributed on $1, 2, ..., N$.
The Hamiltonian $H_N(\sigma)$ of the diluted ferromagnet is expressed trough

$$H_N(\sigma) = -\sum_{\nu=1}^{x} \sigma_{i_\nu}\sigma_{j_\nu} \tag{1}$$

where x does depend on the dilution probability distribution.
For the Bernoullian dilution case, we denote x as k, where k is a random variable distributed respecting the Bernouilli average

$$E_B[\cdot] = \sum_{k=0}^{M} \frac{M!}{(M-k)!k!}(\frac{\alpha}{N})^k(1-\frac{\alpha}{N})^{M-k}[\cdot], \tag{2}$$

$M = N(N-1)/2$ being the maximum amount of couples $\sigma_i\sigma_j$ existing in the model and α/N the probability that two spins interact.
$\alpha > 0$ plays the role of the connectivity.
The mean and the variance of k are obtained as

$$E_B[k] = \frac{M\alpha}{N} \tag{3}$$

$$E_B[k^2] - E_B^2[k] = \frac{M\alpha}{N}(1-\frac{\alpha}{N}). \tag{4}$$

Furthermore, for the sake of clearness, we remember that the Bernoulli distribution has the following properties

$$E_B[kg(k)] = \frac{M\alpha}{N}E_B[g(k+1)], \tag{5}$$

$$E_B[k^2g(k)] = \frac{M(M-1)\alpha^2}{N^2}E_B[g(k+2)] - \frac{M\alpha}{N}E_B[g(k+1)], \tag{6}$$

$$\frac{d}{d\alpha}E_B[g(k)] = \frac{M}{N}E_B[g(k+1)-g(k)]. \tag{7}$$

For the Poissonian dilution case, we denote x as $\xi_{\alpha N}$, which is a Poisson random variable of mean αN, for some $\alpha > 0$ (again defining the connectivity of the model), such that

$$P(\xi_{\alpha N} = k) = \pi(k, \alpha N) = \exp\Big(- \alpha N\Big) \frac{(\alpha N)^k}{k!}, \;\; k = 0, 1, 2, ... \tag{8}$$

Furthermore, we stress that the Poisson distribution obeys the following properties

$$k\pi(k, \lambda) = \lambda\pi(k-1, \lambda) \tag{9}$$

$$\frac{d}{d\lambda}\pi(k, \lambda) = -\pi(k, \lambda) + \pi(k-1, \lambda)(1 - \delta_{k,0}). \tag{10}$$

As for the Bernoulli case, the average with respect to the Poisson measure will be denoted by a proper index

$$E_P = \sum_{k=0}^{\infty} \frac{e^{\alpha N}(\alpha N)^k}{k!}. \tag{11}$$

We define further the expectation with respect to all the quenched variables $\mathbf{E}$ as the product of the expectation over the dilution distribution and the expectation over the uniformly distributed variables

$$\mathbf{E} = E_{B,P} \cdot \frac{1}{N^2} \sum_{i,j}^{1,N} .$$

The thermodynamic objects, we deal with, are the partition function

$$Z_N(\alpha, \beta) = \sum_{\{\sigma\}} e^{-\beta H_N(\alpha)}, \tag{12}$$

the quenched intensive free energy

$$A_N(\alpha, \beta) = \frac{1}{N}\mathbf{E} \ln Z_N(\alpha, \beta), \tag{13}$$

the Boltzmann state

$$\omega(g(\sigma)) = \frac{1}{Z_N(\alpha,\beta)} \sum_{\{\sigma_N\}} g(\sigma) e^{-\beta H_N(\alpha)}, \tag{14}$$

the replicated Boltzmann state

$$\Omega(g(\sigma)) = \prod_s \omega^{(s)}(g(\sigma^{(s)})) \tag{15}$$

and the global average $\langle g(\sigma)\rangle$ defined as

$$\langle g(\sigma)\rangle = \mathbf{E}[\Omega(g(\sigma))]. \tag{16}$$

As we lack the Gaussian framework of symmetric spin-glasses, a priori, the order parameter of the theory is the infinite series of multioverlaps q_n, defined as

$$q_{1\cdots n} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \cdots \sigma_i^{(n)},$$

where particular emphasis is due to the magnetization $m = q_1 = (1/N)\sum_{i=1}^N \sigma_i$ and to the two replica overlap $q_{12} = (1/N)\sum_{i=1}^N \sigma_i^1 \sigma_i^2$.

3 Bernoullian diluted case

Identities in the Bernoullian model will be obtained as an indirected consequence of the internal energy self-average [14][19]; before focusing on this procedure, let us recall that

Definition 1 *A quantity $A(\sigma)$ is called self-averaging if*

$$\lim_{N\to\infty} \langle \Big(A(\sigma) - \langle A(\sigma)\rangle \Big)^2 \rangle = \lim_{N\to\infty} \Big(\langle A^2(\sigma)\rangle - \langle A(\sigma)\rangle^2 \Big) = 0, \tag{17}$$

by which we can recall the following

Proposition 1 *Given two well-behaved functions $A(\sigma)$ and $B(\sigma)$, if at least one on them is self-averaging, then the following relation holds*

$$\lim_{N\to\infty}\langle A(\sigma)B(\sigma)\rangle = \lim_{N\to\infty}\langle A(\sigma)\rangle\langle B(\sigma)\rangle \tag{18}$$

Proof
The proof is really simple. Let us suppose the self-averaging quantity is $B(\sigma)$ and use $A(\sigma)$ as a trial function. Then we have

$$\begin{aligned} 0 &\leq |\langle A(\sigma)B(\sigma)\rangle - \langle A(\sigma)\rangle\langle B(\sigma)\rangle| \\ &= |\langle A(\sigma)B(\sigma) - A(\sigma)\langle B(\sigma)\rangle + \langle A(\sigma)\rangle B(\sigma) - \langle A(\sigma)\rangle\langle B(\sigma)\rangle| \\ &= |\langle A(\sigma)(\langle B(\sigma) - \langle B(\sigma)\rangle\rangle)\rangle| \leq \sqrt{\langle A^2(\sigma)\rangle}\sqrt{\langle (B(\sigma)-\langle B(\sigma)\rangle)^2\rangle}, \end{aligned} \tag{19}$$

where, the last passage used Cauchy-Schwartz relation.
In the thermodynamic the proof becomes completed. □

The scheme to follow is then clear: Using the above proposition as the underlying backbone in the derivation of the constraints in this section, we must, at first, show that the internal energy density of the model self-averages and subsequently use as trial functions suitably chosen quantities of the order parameters.
The identities follow by evaluating explicitly both the terms of eq. (18): This operation produces several terms, all involving the order parameters, among which massive cancelations happen and the resting part gives the identities.

3.1 Self-averaging of the internal energy density

Once defined $h_l = H(\sigma^{(l)})/N$ as the density of the Hamiltonian evaluated on the generic l^{th} replica and

$$\theta = \tanh(\beta) \tag{20}$$

$$\alpha' = M\alpha/N^2 \quad \stackrel{N\to\infty}{\longrightarrow} \quad \alpha/2 \tag{21}$$

for the sake of simplicity, let us start the plan with the following

Theorem 1 *In the thermodynamic limit, and in β-average the internal energy density self-averages*

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}\mathbb{E}\Big(\Omega(h^2)-\Omega(h)^2\Big)d\beta=0. \tag{22}$$

Proof
Starting from the thermodynamic relation

$$\mathbf{E}[\Omega(h^2)-\Omega^2(h)]=-\frac{1}{N}\frac{d}{d\beta}\mathbf{E}[\Omega(h)] \tag{23}$$

we evaluate explicitly the term $E[\Omega(h)]$ as

$$\begin{aligned}
\mathbf{E}[\Omega(h)] &= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^{k}\sigma_{i_\nu}\sigma_{j_\nu}e^{-\beta H}}{Z_N(\alpha,\beta)}\Big]= & (24)\\
&= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}k\sigma_{i_0}\sigma_{j_0}e^{-\beta H}}{Z_N(\alpha,\beta)}\Big]= & (25)\\
&= -\alpha'\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}e^{\beta\sigma_{i_0}\sigma_{j_0}}e^{-\beta H}}{\sum_{\{\sigma\}}e^{\beta\sigma_{k_0}\sigma_{l_0}}e^{-\beta H}}\Big]= & (26)\\
&= -\alpha'\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}(\cosh\beta+\sigma_{i_0}\sigma_{j_0}\sinh\beta)e^{-\beta H}}{\sum_{\{\sigma\}}(\cosh\beta+\sigma_{k_0}\sigma_{l_0}\sinh\beta)e^{-\beta H}}\Big]= & (27)\\
&= -\alpha'\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}\sigma_{j_0}(1+\sigma_{i_0}\sigma_{j_0}\theta)e^{-\beta H}}{\sum_{\{\sigma\}}(1+\sigma_{k_0}\sigma_{l_0}\theta)e^{-\beta H}}\Big]= & (28)\\
&= -\alpha'\mathbf{E}\Big[\frac{\omega(\sigma_{i_0}\sigma_{j_0})+\theta}{1+\omega(\sigma_{k_0}\sigma_{l_0})\theta}\Big], & (29)
\end{aligned}$$

where in (25) we fixed the index ν, in (26) we used the property (5) of the Bernoulli distribution and we introduced two further families of random variables $\{k_\nu\}$,$\{l_\nu\}$, and in (27) we used $e^{\beta\sigma_{i_0}\sigma_{j_0}}=\cosh\beta+\sigma_{i_0}\sigma_{j_0}\sinh\beta$.
Let us now expand the denominator of (29) taking in mind the relation

$$\frac{1}{(1+\tilde{\omega}_t\theta)^p}=1-p\tilde{\omega}_t\theta+\frac{p(p+1)}{2!}\tilde{\omega}_t^2\theta^2-\frac{p(p+1)(p+2)}{3!}\tilde{\omega}_t^3\theta^3+...$$

such that, by posing $p = 1$, we obtain

$$\mathbf{E}[\Omega(h)] = -\alpha' \mathbf{E}\Big[\theta + \sum_{n=1}^{\infty} (-1)^n \theta^n (1-\theta^2) \langle q_{1\dots n}^2 \rangle \Big]. \tag{30}$$

By applying the modulus function to the equation above we can proceed further with the following bound

$$|\mathbf{E}[\Omega(h)]| \leq \alpha' \mathbf{E}\Big[|\theta| + \sum_{n=1}^{\infty} |\theta^n (1-\theta^2) \langle q_{1\dots n}^2 \rangle| \Big]. \tag{31}$$

Both $|\theta|$ and $|\langle q_{1\dots n}^2 \rangle|$ belong to $[0,1]$ so we get

$$|E[\Omega(h)]| \leq \alpha' \Big[1 + (1-\theta^2) \sum_{n=1}^{\infty} \theta^n \Big], \tag{32}$$

whose harmonic series converges to $1/(1-\theta)$, $|\theta| < 1$;
The fact that the convergence is not guaranteed at zero temperature with this technique is not a problem because, first the identities we are looking for hold in β-average, secondly the zero temperature has been intensively investigated elsewhere [21].
For each finite β, then, we can write

$$\begin{aligned} |\mathbf{E}[\Omega(h)]| &\leq \alpha' \Big[1 + \frac{(1-\theta^2)}{1-\theta}\Big] = && (33) \\ &= \alpha' \Big[1 + \frac{(1-\theta)(1+\theta)}{1-\theta}\Big] = && (34) \\ &= \alpha' \Big[1 + (1+\theta)\Big] \leq && (35) \\ &\leq 3\alpha', && (36) \end{aligned}$$

and consequently

$$\begin{aligned} \int_{\beta_1}^{\beta_2} \mathbf{E}[\Omega(h^2) - \Omega^2(h)] d\beta &\leq \int_{\beta_1}^{\beta_2} \mid \mathbf{E}[\Omega(h^2) - \Omega^2(h)] \mid d\beta = \\ &= \frac{1}{N} \int_{\beta_1}^{\beta_2} \mid \frac{d}{d\beta} \mathbf{E}[\Omega(h)] \mid d\beta \quad \leq \\ &\leq 3\frac{\alpha'}{N}(\beta_2 + \beta_1) \end{aligned} \tag{37}$$

$$\Rightarrow \qquad \lim_{N\to\infty}\int_{\beta_1}^{\beta_2}\mathbf{E}[\Omega(h^2)-\Omega^2(h)]d\beta = 0 \tag{38}$$

and the proof is closed. □

We can now introduce the following lemma.

Lemma 1 *Let us consider for simplicity the quantity*

$$\Delta G = \sum_{l=1}^{s}\Big[E\big(\Omega(h_l G)-\Omega(h_l)\Omega(G)\big)\Big]. \tag{39}$$

For every smooth, well behaved, function G, in β-average, we have

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}|\Delta G|d\beta = 0 \tag{40}$$

Proof

$$\begin{aligned}
\int_{\beta_1}^{\beta_2}|\Delta G|d\beta &\leq \int_{\beta_1}^{\beta_2}\sum_{l=1}^{s}|\,\mathbf{E}[\Omega(h_l G)-\Omega(h_l)\Omega(G)]\,|\,d\beta & (41)\\
&\leq \int_{\beta_1}^{\beta_2}\sum_{l=1}^{s}\sqrt{\mathbf{E}[(\Omega(h_l G)-\Omega(h_l)\Omega(G))^2]}d\beta & (42)\\
&\leq s\int_{\beta_1}^{\beta_2}\sqrt{\mathbf{E}[\Omega(h^2)-\Omega^2(h)]}d\beta & (43)\\
&\leq s\sqrt{\beta_2-\beta_1}\sqrt{\int_{\beta_1}^{\beta_2}\mathbf{E}[\Omega(h^2)-\Omega^2(h)]d\beta} \stackrel{N\to\infty}{\longrightarrow} 0 & (44)
\end{aligned}$$

where (41) comes from triangular inequality; (42) is obtained via the Jensen inequality applied to the measure $\mathbf{E}[\cdot]$. In the same way (43) comes from Schwarz inequality applied on the measure $\Omega(\cdot)$ (being G well behaved, in particular bounded), while (44) is obtained via Jensen inequality applied on the measure $(\beta_2-\beta_1)^{-1}\int_{\beta_1}^{\beta_2}(\cdot)d\beta$. □

Now we can state the main theorem for the linear constraints.

We are going to introduce directly specific trial function that we call $f_G(\alpha,\beta)$.

Theorem 2 *Let us consider the following series of functions G and of multi-overlaps acting, in complete generality, on s replicas*

$$\begin{aligned} f_G(\alpha,\beta) &= \alpha'\Big[\Big(\sum_{l=1}^{s}\langle Gm_l^2\rangle - s\langle Gm_{s+1}^2\rangle\Big)\Big(1-\theta^2\Big) + \\ &+2\theta\Big(\sum_{a<l}^{1,s}\langle Gq_{al}^2\rangle - s\sum_{l}^{1,s}\langle Gq_{l,s+1}^2\rangle + \frac{s(s+1)}{2}\langle Gq_{s+1,s+2}^2\rangle\Big) + \\ &+3\theta^2\Big(\sum_{l<a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle - s\sum_{l<a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{s(s+1)}{2}\sum_{l}^{1,s}\langle Gq_{l,s+1,s+2}^2\rangle + \\ &-\frac{s(s+1)(s+2)}{3!}\langle Gq_{s+1,s+2,s+3}^2\rangle\Big) + O(\theta^3)\Big], \end{aligned} \tag{45}$$

in the thermodynamic limit the following generator of linear constraints holds:

$$\lim_{N\to\infty}\int_{\beta_1}^{\beta_2} d\beta|f_G(\alpha,\beta)| = 0. \tag{46}$$

Proof
So far the proof is a straightforward application of the backbone we outlined. Let us consider explicitly the quantities encoded in (39). For the sake of clearness all the calculations are reported in appendix, here we present just the results.

$$\begin{aligned} \mathbf{E}[\Omega(h_lG)] &= -\alpha'\Big[\langle Gm_l^2\rangle + \theta\Big(\sum_{a=1}^{s}\langle Gq_{a,l}^2\rangle - s\langle Gq_{l,s+1}^2\rangle\Big) + \\ &+\theta^2\Big(\sum_{a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle - s\sum_{a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{s(s+1)}{2}\langle Gq_{l,s+1,s+2}^2\rangle\Big) \\ &+O(\theta^2)\Big], \end{aligned} \tag{47}$$

$$
\begin{aligned}
\mathbf{E}[\Omega(h_l)\Omega(G)] \;=\; & -\alpha'\Big[\langle Gm_l^2\rangle + \theta\Big(\sum_{a=1}^{s+1}\langle Gq_{a,l}^2\rangle - (s+1)\langle Gq_{l,s+1}^2\rangle\Big) + \\
& +\theta^2\Big(\sum_{a}^{1,s}\langle Gm_a^2\rangle - (s+1)\langle Gm_l^2\rangle + \sum_{a<b}^{1,s}\langle Gq_{l,a,b}^2\rangle + \\
& -(s+1)\sum_{a}^{1,s}\langle Gq_{l,a,s+1}^2\rangle + \frac{(s+1)(s+2)}{2}\langle Gq_{l,s+1,s+2}^2\rangle\Big) \\
& +O(\theta^2)\Big]. \qquad (48)
\end{aligned}
$$

Subtracting the last equation from the former, immediately we conclude that

$$\Delta G = -f_G(\alpha,\beta), \qquad (49)$$

from which theorem thesis follows. □

3.2 Linear constraints for multi-overlaps

From a practical viewpoint it is impossible to show the whole set of identities, and we restrict ourselves in showing just the first ones, (as usually happens even in spin-glasses counterpart [6] or in neural network [9]).

Proposition 2 *The first class of multi-overlap constraints is obtained by choosing $G = m^2$.*

In fact, if we set $G = q_1^2 = m^2$, the function $f_G(\alpha,\beta)$ becomes

$$
\begin{aligned}
f_{m^2}(\alpha,\beta) \;=\; & \alpha'\Big[\Big(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\Big)\Big(1-\theta^2\Big) + \\
& -2\theta\Big(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\Big) + \\
& +3\theta^2\Big(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle\Big) + O(\theta^3)\Big],
\end{aligned}
$$

from which, changing the Jacobian $d\theta=(1-\theta^2)d\beta$, we get

$$
\begin{aligned}
\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}|f_{m^2}(\alpha,\beta)|d\beta &= \frac{\alpha}{2}\int_{\theta_1}^{\theta_2} d\frac{\theta}{(1-\theta^2)}\Big[|\Big(\langle m_1^4\rangle-\langle m_1^2m_2^2\rangle\Big)+ \qquad (50)\\
&-2\theta\Big(\langle m_1^2q_{12}^2\rangle-\langle m_1^2q_{23}^2\rangle\Big)+\\
&+3\theta^2\Big(\langle m_1^2q_{123}^2\rangle-\langle m_1^2q_{234}^2\rangle\\
&+O(\theta^3)\,|\Big]=0,
\end{aligned}
$$

where, the (not interesting) breakdown at $\theta=1$, of the expression above, reflect the lacking of convergence of the harmonic series we used in eq. (31).

Proposition 3 *The second class of multi-overlap constraints is obtained by choosing* $G=q_{12}^2$.

In fact, if we set $G=q_{12}^2$ the function $f_G(\alpha,\beta)$ becomes

$$
\begin{aligned}
f_{q^2}(\alpha,\beta) &= \alpha'\Big[\Big(2\langle m_1^2q_{12}^2\rangle-2\langle m_3^2q_{12}^2\rangle\Big)\Big(1-\theta^2\Big)+\\
&+2\theta\Big(\langle q_{12}^4\rangle-4\langle q_{12}^2q_{23}^2\rangle+3\langle q_{12}^2q_{34}^2\rangle\Big)+\\
&-6\theta^2\Big(\langle q_{12}^2q_{123}^2\rangle-3\langle q_{12}^2q_{234}^2\rangle+2\langle q_{12}^2q_{345}^2\rangle\Big)+O(\theta^3)\Big].
\end{aligned}
$$

Again

$$
\begin{aligned}
\lim_{N\to\infty}\int_{\beta_1}^{\beta_2}|f_{q^2}(\alpha,\beta)|d\beta &= \frac{\alpha}{2}\int_{\theta_1}^{\theta_2} d\frac{\theta}{(1-\theta^2)}\Big[|\Big(\langle m_1^2q_{12}^2\rangle-\langle m_3^2q_{12}^2\rangle\Big)+ \qquad (51)\\
&+\theta\Big(\langle q_{12}^4\rangle-4\langle q_{12}^2q_{23}^2\rangle+3\langle q_{12}^2q_{34}^2\rangle\Big)+\\
&-3\theta^2\Big(\langle q_{12}^2q_{123}^2\rangle-3\langle q_{12}^2q_{234}^2\rangle+2\langle q_{12}^2q_{345}^2\rangle\Big)\\
&+O(\theta^3)\,|\Big]=0,
\end{aligned}
$$

from which the constraints are obtained as the r.h.s. members of (73,74) set to zero. □

4 Poissonian diluted case

The idea beyond the cavity field technique, we are going to use, is that, calling $F(\beta)$ the extensive free energy, while dealing with the (self-averaging) intensive free energy density $f(\beta)$, a bridge among the two, in the large N limit, is offered simply by

$$\Big(- F_{N+1}(\beta) - F_N(\beta)\Big) = f(\beta) + O(N^{-1}). \tag{52}$$

As our system is topologically quenched disordered, the $N+1$ spin, acting as an "external cavity spin" for the former N, is a random field.
The identities which will be found unaffected by the tuning of this field, will be said stochastically stable.
For finding these polynomials the simplest way is finding order parameter monomials left invariant by the random field. Then by deriving them against this field we will obtain such polynomials, which can be set to zero as the derivative must be.

4.1 Cavity field decompositions for the pressure density

To start applying the plan let us decompose (in distribution) a Poissonian random Hamiltonian of $N+1$ spins in two Hamiltonians [1]: The former of the "inner" N interacting spins, the latter as the pasted spin interacting with the inner N spins of the cavity.

Forgetting corrections going to zero in the thermodynamic limit we can write in distribution

$$H_{N+1}(\alpha) = -\sum_{\nu=1}^{P_{\alpha(N+1)}} \sigma_{i_\nu}\sigma_{j_\nu} \quad \sim \quad -\sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} - \sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}\sigma_{N+1}, \tag{53}$$

or simply for compactness

$$H_{N+1}(\alpha) \sim H_N(\tilde{\alpha}) + \hat{H}_N(\tilde{\alpha})\sigma_{N+1} \tag{54}$$

where

$$\tilde{\alpha} = \frac{N}{N+1}\alpha \stackrel{N\to\infty}{\longrightarrow} \alpha, \qquad \hat{H}_N(\tilde{\alpha}) = -\sum_{\nu=1}^{P_{2\tilde{\alpha}}} \sigma_{i_\nu}. \tag{55}$$

For the sake of simplicity now it is convenient to paste an interpolating parameter $t \in [0,1]$ on the term encoding for the linear connectivity shift so to menage the derivative with respect to the random field by deriving with respect to this parameter.
To this task we state the next

Definition 2 *We define the t-dependent Boltzmann state $\tilde{\omega}_t$ as*

$$\tilde{\omega}_t(g(\sigma)) = \frac{1}{Z_{N,t}(\alpha,\beta)} \sum_{\{\sigma\}} g(\sigma) e^{\beta \sum_{\nu=1}^{P_{\tilde{\alpha}N}} \sigma_{i_\nu}\sigma_{j_\nu} + \beta \sum_{\nu=1}^{P_{2\tilde{\alpha}t}} \sigma_{i_\nu}}. \tag{56}$$

We stress the simplicity by which the t parameter switches among the system of $N+1$ spins and the one built just by the former N in the large N limit: In fact, being the two body Hamiltonian left invariant by the gauge symmetry $\sigma_i \to \epsilon\sigma_i$ for all $i \in (1,...,N)$ with $\epsilon = \pm 1$, by choosing $\epsilon = \sigma_{N+1}$ we have

$$Z_{N,t=1}(\tilde{\alpha},\beta) = Z_{N+1}(\alpha,\beta) \tag{57}$$

$$Z_{N,t=0}(\tilde{\alpha},\beta) = Z_N(\tilde{\alpha},\beta). \tag{58}$$

Note that $Z_{N,t}(\tilde{\alpha},\beta)$ is defined accordingly to (56) and coherently, dealing with the perturbed Boltzmann measure, we introduce an index t also to the global averages $\langle . \rangle_t$.

4.2 Stochastic stability via cavity fields

We are now ready to attack the problem.
We will divide the ensemble of overlap monomials in two big categories: stochastically stable monomials and (as a side results) not stochastically stable one. Then we find explicitly the family of the stochastically stable monomials, and by putting their t-derivative equal to zero we find the identities. To follow the plan let us start introducing the next

Definition 3 *We define as stochastically stable monomials the multi-overlap monomials where each replica appears an even number of times.*

We are ready to introduce the main theorem, which offers as a straightforward consequence a useful corollary, stated immediately after.

Theorem 3 *At $t = 1$ (where a proper Boltzmannfaktor can be built), and in the thermodynamic limit, we get*

$$\tilde{\omega}_{N,t}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}) = \tilde{\omega}_{N+1}(\sigma_{i_1}\sigma_{i_2}...\sigma_{i_n}\sigma^n_{N+1}). \tag{59}$$

Corollary 1 *In the thermodynamic limit, the averages $\langle\cdot\rangle_t$ of the stochastically stable monomials become t-independent in β-average.*

Proof
Let us focus on the proof of Theorem 3. Corollary 1 will be produced as a straightforward application of Theorem 3 on stochastically stable monomials. Let us start the proof. Let us assume for a generic multi-overlap monomial the following representation

$$Q = \prod_{a=1}^{s}\sum_{i_l^a}\prod_{l=1}^{n^a}\sigma^a_{i^a_l} I(\{i^a_l\})$$

where a labels replicas, the inner product accounts for the spins depicted by the index l which belong to the Boltzmann state a of the product state Ω for the multi-overlap $q_{a,a'}$ and flow on the whole natural numbers from 1 to the amount of times the replica a appears into the expression.
The external product multiplies all the terms coming from the internal one. The factor I fixes replica-bond constraints.
For example the monomial $Q = q_{12}q_{23}$ has $s = 3, n^1 = n^3 = 1, n^2 = 2$ and $I = N^{-2}\delta_{i^1_1,i^3_1}\delta_{i^2_1,i^3_2}$, there the δ-functions give the correlations $1,2 \rightarrow q_{1,2}$ and $2,3 \rightarrow q_{2,3}$.
By applying the Boltzmann and quenched-disordered expectations we get

$$\langle Q\rangle_t = \mathbf{E}\sum_{i^a_l} I(\{i^a_l\})\prod_{a=1}^{s}\omega_t(\prod_{l=1}^{n^a}\sigma^a_{i^a_l}).$$

Let us suppose now that Q is not stochastically stable (for otherwise the proof will be simply ended) and let us decompose it by factorizing the Boltzmann state ω and splitting the terms involving replicas appearing an even number of time from the ones involving replicas appearing odd number of times. Evaluate the whole receipt at $t = 1$.

$$\langle Q\rangle_t = \mathbf{E}\sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega_a(\prod_{l=1}^{n^a} \sigma_{i_l^a}^a) \prod_{b=u+1}^{s} \omega_b(\prod_{l=1}^{n^b} \sigma_{i_l^b}^b),$$

where u stands for the amount of replicas which appear an odd number of times inside Q.
In this way we split the measure Ω in two ensembles ω_a and ω_b. Replicas belonging to ω_b are in an even number while the ones in ω_a in odd numbers.
At this point, as the Hamiltonian has two body interaction and consequently is left unchanged by the symmetry $\sigma_i^a \to \sigma_i^a \sigma_{N+1}^a, \forall i \in (1, N)$ (as $\sigma_{N+1}^2 \equiv 1$), we apply such a symmetry globally to the whole set of N spins. The even measure is left unchanged by this symmetry while the odd one takes a multiplying term σ_{N+1}

$$\langle Q\rangle = \sum_{i_l^a, i_l^b} I(\{i_l^a\}, \{i_l^b\}) \prod_{a=1}^{u} \omega(\sigma_{N+1}^a \prod_{l=1}^{n^a} \sigma_{i_l^a}^a) \prod_{b=u+1}^{s} \omega(\sigma_{N+1}^b \prod_{l=1}^{n^b} \sigma_{i_l^b}^b).$$

The last trick is that, by noticing the arbitrariness of the $N+1$ label in σ_{N+1}, we can change it in a generical label k for each $k \neq \{i_l^a\}$ and multiply by $1 = N^{-1}\sum_{k=1}^{N}$. At finite N the thesis is recovered forgetting terms $O(1/N)$ and becomes exact in the thermodynamic limit. $\Box$

It is straightforward checking that the effect of Theorem 3 has no effects on stochastically stable multi-overlap monomials (Corollary 1) thanks to the dichotomy of the Ising spins ($\sigma_{N+1}^{2n} \equiv 1 \forall n \in \mathbf{N}$). $\Box$

The last point missing to obtain the identities is finding a streaming equation to work out the derivatives with respect to the random field of the stochastically stable monomials. To this task we introduce the following

Proposition 4 *Given F_s as a generic function of the spins of s replicas, the* *following streaming equation holds*

$$
\begin{aligned}
\frac{\partial \langle F_s\rangle_{t,\tilde\alpha}}{\partial t} &= 2\tilde\alpha\theta[\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\rangle_{t,\tilde\alpha} - s\langle F_s\sigma_{i_0}^{s+1}\rangle_{t,\tilde\alpha}] \quad + \\
&+ \quad 2\tilde\alpha\theta^2[\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\rangle_{t,\tilde\alpha} - s\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\rangle_{t,\tilde\alpha} + \\
&+ \frac{s(s+1)}{2!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde\alpha}] \quad + \\
&+ \quad 2\tilde\alpha\theta^3[\sum_{a<b<c}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^c\rangle_{t,\tilde\alpha} - s\sum_{a<b}^{1,s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^b\sigma_{i_0}^{s+1}\rangle_{t,\tilde\alpha} + \\
&+ \frac{s(s+1)}{2!}\sum_{a=1}^{s}\langle F_s\sigma_{i_0}^a\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\rangle_{t,\tilde\alpha} + \\
&+ \frac{s(s+1)(s+2)}{3!}\langle F_s\sigma_{i_0}^{s+1}\sigma_{i_0}^{s+2}\sigma_{i_0}^{s+3}\rangle_{t,\tilde\alpha}] + O(\theta^3) \qquad (60)
\end{aligned}
$$

Proof
The proof follows by direct calculations

$$
\begin{aligned}
\frac{\partial \langle F_s\rangle_{t,\tilde\alpha}}{\partial t} &= \frac{\partial}{\partial t}\mathbf{E}[\frac{\sum_{\{\sigma\}} F_s e^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{P_{\tilde\alpha N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a + \beta\sum_{\nu=1}^{P_{2\tilde\alpha t}}\sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}} e^{\sum_{a=1}^{s}(\beta\sum_{\nu=1}^{P_{\tilde\alpha N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a + \beta\sum_{\nu=1}^{P_{2\tilde\alpha t}}\sigma_{i_\nu}^a)}}] = \qquad (61) \\
&= 2\tilde\alpha\mathbf{E}[\frac{\sum_{\{\sigma\}} F_s e^{\sum_{a=1}^{s}(\beta\sigma_{i_0}^a + \beta\sum_{\nu=1}^{P_{\tilde\alpha N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a + \beta\sum_{\nu=1}^{P_{2\tilde\alpha t}}\sigma_{i_\nu}^a)}}{\sum_{\{\sigma\}} e^{\sum_{a=1}^{s}(\beta\sigma_{i_0}^a + \beta\sum_{\nu=1}^{P_{\tilde\alpha N}}\sigma_{i_\nu}^a\sigma_{j_\nu}^a + \beta\sum_{\nu=1}^{P_{2\tilde\alpha t}}\sigma_{i_\nu}^a)}}] - 2\tilde\alpha\langle F_s\rangle_{t,\tilde\alpha} = \\
&= 2\tilde\alpha\mathbf{E}[\frac{\tilde\Omega_t(F_s e^{\sum_{a=1}^{s}\beta\sigma_{i_0}^a})}{\tilde\Omega_t(e^{\sum_{a=1}^{s}\beta\sigma_{i_0}^a})}] - 2\tilde\alpha\langle F_s\rangle_{t,\tilde\alpha} = \\
&= 2\tilde\alpha\mathbf{E}[\frac{\tilde\Omega_t(F_s\Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0}^a\sinh\beta))}{\tilde\Omega_t(\Pi_{a=1}^{s}(\cosh\beta + \sigma_{i_0}^a\sinh\beta))}] - 2\tilde\alpha\langle F_s\rangle_{t,\tilde\alpha} = \\
&= 2\tilde\alpha\mathbf{E}[\frac{\tilde\Omega_t(F_s\Pi_{a=1}^{s}(1 + \sigma_{i_0}^a\theta))}{(1 + \tilde\omega_t(\sigma_{i_0}^a)\theta)^s}] - 2\tilde\alpha\langle F_s\rangle_{t,\tilde\alpha},
\end{aligned}
$$

then, by noticing that

$$\Pi_{a=1}^{s}(1+\sigma_{i_0}^{a}\theta)=1+\sum_{a=1}^{s}\sigma_{i_0}^{a}\theta+\sum_{a<b}^{1,s}\sigma_{i_0}^{a}\sigma_{i_0}^{b}\theta^2+\sum_{a<b<c}^{1,s}\sigma_{i_0}^{a}\sigma_{i_0}^{b}\sigma_{i_0}^{c}\theta^3+...$$

$$\frac{1}{(1+\tilde{\omega}_t\theta)^s}=1-s\tilde{\omega}_t\theta+\frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2-\frac{s(s+1)(s+2)}{3!}\tilde{\omega}_t^3\theta^3+...$$

we get

$$\frac{\partial\langle F_s\rangle_{t,\tilde{\alpha}}}{\partial t}=2\tilde{\alpha}\mathbf{E}[\tilde{\Omega}_t(F_s(1+\sum_{a=1}^{s}\sigma_{i_0}^{a}\theta+\sum_{a<b}^{1,s}\sigma_{i_0}^{a}\sigma_{i_0}^{b}\theta^2+\sum_{a<b<c}^{1,s}\sigma_{i_0}^{a}\sigma_{i_0}^{b}\sigma_{i_0}^{c}\theta^3+...))\times$$

$$\times(1-s\tilde{\omega}_t\theta+\frac{s(s+1)}{2!}\tilde{\omega}_t^2\theta^2-\frac{s(s+1)(s+2)}{3!}\tilde{\omega}_t^3\theta^3+...)]-2\tilde{\alpha}\langle F_s\rangle_{t,\tilde{\alpha}}.$$

from which the thesis follows. □

4.3 Linear constraints for multi-overlaps

We saw the stochastically stable multi-overlap monomials becomes asymptotically independent by the t parameter upon increasing the size of the system. Calling for simplicity $G_N(q)$ a stochastically stable multi-overlap monomial, identities follow as a consequence of Corollary 1 and are encoded in the following relation

$$\lim_{N\to\infty}\partial_t\langle G_N(q)\rangle_t=0.$$

As we did when we investigated the Bernoullian model, we analyze the stability of $\langle m^2\rangle$ and $\langle q_{12}^2\rangle$ up to the third order in θ so to compare the results at the end.

$$\begin{aligned}\partial_t\langle m_1^2\rangle_t &= 2\tilde{\alpha}\theta\Big(\langle m_1^3\rangle_t-\langle m_1^2m_2\rangle_t\Big)-2\tilde{\alpha}\theta^2\Big(\langle m_1^2q_{12}\rangle_t-\langle m_1^2q_{23}\rangle_t\Big)\\ &+ 2\tilde{\alpha}\theta^3\Big(\langle m_1^2q_{123}\rangle_t-\langle m_1^2q_{234}\rangle_t\Big)+O(\theta^3)\end{aligned}$$

$$\Rightarrow \quad \Big[2\alpha\theta\Big(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\Big) - 2\alpha\theta^2\Big(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\Big) \\ + 2\alpha\theta^3\Big(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle\Big) + O(\theta^3)\Big] = 0 \tag{62}$$

$$\begin{aligned} \partial_t\langle q_{12}^2\rangle_t &= 4\tilde{\alpha}\theta\Big(\langle m_1 q_{12}^2\rangle_t - \langle m_3 q_{12}^2\rangle_t\Big) + 2\tilde{\alpha}\theta^2\Big(\langle q_{12}^3\rangle_t - 4\langle q_{12}^2 q_{13}\rangle_t + 3\langle q_{12}^2 q_{34}\rangle_t\Big) + \\ &- 4\tilde{\alpha}\theta^3\Big(\langle q_{12}^2 q_{123}\rangle_t - 3\langle q_{12}^2 q_{134}\rangle_t + 4\langle q_{12}^2 q_{345}\rangle_t\Big) + O(\theta^3) \end{aligned}$$

$$\Rightarrow \quad \Big[4\alpha\theta\Big(\langle m_1^2 q_{12}^2\rangle - \langle m_3^2 q_{12}^2\rangle\Big) + 2\alpha\theta^2\Big(\langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{13}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle\Big) + \\ - 4\alpha\theta^3\Big(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{134}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle\Big) + O(\theta^3)\Big] = 0 \tag{63}$$

5 Discussion and outlook

Let us starting this section by comparing the results we get from the two models.
We have to compare respectively eq.s (73) versus (62) and (74) versus (63).
We see that the details of the dilution do not affect the constraints: the series show the same set of identities.

$$0 = \langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle, \tag{64}$$
$$0 = \langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle, \tag{65}$$
$$0 = \langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle, \tag{66}$$

when investigating the magnetization as a trial function and

$$0 = \langle m_1^2 q_{12}^2\rangle - \langle m_3^2 q_{12}^2\rangle, \tag{67}$$
$$0 = \langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{23}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle, \tag{68}$$
$$0 = \langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{134}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle, \tag{69}$$

when investigating the two replica overlap.
Even if a minor point, we stress that the global (α, β)-coefficients (not shown

here) clearly are related to the differences in the two method involved (in the former the constraints appear under the integral over the temperature, while in the latter this β-average is already worked out), furthermore the limiting connectivity in the Bernoulli dilution is $\alpha/2$, while is 2α in the Poisson model; so there is an overall factor 4 of difference among the results (for the sake of clearness we worked out in the appendix also the constraints in the Poisson diluted case via the first method, to check explicitly the coherence among the two model).

Despite we are not generally allowed in setting to zero each term in the expressions (62,63,73,74) (as we did to obtain alone the identities (64-69)), at least close to the critical line, where different multi-overlaps have different scaling laws [1], i.e. $q_n^2 \propto (\alpha\theta - 1)^n$, such a spreading is possible and we can forget each single (α, β)-coefficient as it does not affect the identities (it is never involved into the averages $\langle . \rangle$).

Then, by looking explicitly at the constraints, several physical features can be recognized, in fact every term is well known: the first class (Eq.s 64,65,66) is the standard magnetization self-averaging on replica symmetric systems; In fact, by assuming replica equivalence eq. (64) turns out to be the standard internal energy self-averaging of the Curie-Weiss model. Eq.(65) and (66) contribute as higher order internal energy self-average by taking into account the dilution (in fact, they go to zero whenever $\alpha \to \infty$ because θ-powers higher than one go to zero and only the Curie-Weiss self-averaging for the internal energy survives as it should).

With a glance at the identities coming from the second constraint series (Eq.s 67,68,69) we recognize immediately the replica symmetry ansatz for the magnetization in the first identity, followed by the first and the second Aizenman-Contucci relation for systems with quenched disorder [2][4].

Interestingly these series are in agreement even with other models, apparently quite far away, as spin-glasses with Gaussian coupling $\mathcal{N}[1,1]$ instead of $\mathcal{N}[0,1]$ [15]. A very interesting conjecture may be that these constraints hold for systems whose interaction has on average positive strength and are affected by quenched disorder, independently if the disorder affects the strength of the interaction or the topology of the interaction. We plan to report soon on this topics.

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A Appendix

A.1 Details in Bernoulli dilution calculations

Here some technical calculations concerning the self-averaging technique on the Bernouilli diluted network are reported.

$$
\begin{aligned}
\mathbf{E}[\Omega(h_l G)] &= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l G\, e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}{\sum_{\{\sigma\}} e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}\Big] = \\
&= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}} k\sigma_{i_0}^l\sigma_{j_0}^l G\, e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}{\sum_{\{\sigma\}} e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}\Big],
\end{aligned}
$$

and remembering the properties of the Bernoullian distribution (5) we can continue writing

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}^l\sigma_{j_0}^l G\prod_{a=1}^{s} e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a}\, e^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^{s} e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a} e^{-\beta H_s}}\Big] = \\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}^l\sigma_{j_0}^l G\prod_{a=1}^{s}[\cosh\beta+\sigma_{i_0}^a\sigma_{j_0}^a\sinh\beta]\, e^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^{s}[\cosh\beta+\sigma_{i_0}^a\sigma_{j_0}^a\sinh\beta]\, e^{-\beta H_s}}\Big] =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}^l\sigma_{j_0}^l G\prod_{a=1}^{s}[1+\sigma_{i_0}^a\sigma_{j_0}^a\theta]\, e^{-\beta H_s}}{\sum_{\{\sigma\}}\prod_{a=1}^{s}[1+\sigma_{i_0}^a\sigma_{j_0}^a\theta]\, e^{-\beta H_s}}\Big] = \\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\Omega\Big(\sigma_{i_0}^l\sigma_{j_0}^l G\prod_{a=1}^{s}[1+\sigma_{i_0}^a\sigma_{j_0}^a\theta]\Big)}{\Big(1+\omega(\sigma_{i_0}^a\sigma_{j_0}^a)\theta\Big)^s}\Big] =
\end{aligned}
$$

Let us expand both the numerator and the denominator up to the second order in θ

$$
\begin{aligned}
= & -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Omega\Big((\sigma^l_{i_0}\sigma^l_{j_0}G)(1+\sum_a^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\theta+\sum_{a<b}^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\theta^2)\Big)\times\\
& \times\Big(1-s\omega(\sigma_{i_0}\sigma_{j_0})\theta+\frac{s(s+1)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\Big)\Big]=\\
= & -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Omega\Big(G\sigma^l_{i_0}\sigma^l_{j_0}+G\sum_a^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0}\theta+G\sum_{a<b}^{1,s}\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\sigma^l_{i_0}\sigma^l_{j_0}\theta^2\Big)\times
\end{aligned}
$$

$$
\begin{aligned}
& \times\Big(1-s\omega(\sigma^a_{i_0}\sigma^a_{j_0})\theta+\frac{s(s+1)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\Big)\Big]=\\
= & -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})+\theta\Big(\sum_a^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})-s\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})\omega(\sigma_{i_0}\sigma_{j_0})\Big)+\\
& +\theta^2\Big(\sum_{a<b}^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})-s\sum_a^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})\omega(\sigma_{i_0}\sigma_{j_0})+
\end{aligned}
$$

$$
\begin{aligned}
+ & \quad \frac{s(s+1)}{2}\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})\omega^2(\sigma^a_{i_0}\sigma^a_{j_0})\Big)\Big]=\\
= & -\frac{\alpha M}{N^2}\Big[\langle Gm_l^2\rangle+\theta\Big(\sum_{a=1}^{s}\langle Gq^2_{a,l}\rangle-s\langle Gq^2_{l,s+1}\rangle\Big)+\\
+ & \quad \theta^2\Big(\sum_{a<b}^{1,s}\langle Gq^2_{l,a,b}\rangle-s\sum_a^{1,s}\langle Gq^2_{l,a,s+1}\rangle+\frac{s(s+1)}{2}\langle Gq^2_{l,s+1,s+2}\rangle\Big)\Big].
\end{aligned}
$$

While the other term $\mathbf{E}[\Omega(h_l)\Omega(G)]$ can be worked out as follows:

$$
\mathbf{E}[\Omega(h_l)\Omega(G)]= \tag{70}
$$

$$
\begin{aligned}
&= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l G\, e^{\beta\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l}\ e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}{\sum_{\{\sigma\}} e^{\beta\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l} e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}\Big] = \\
&= -\frac{1}{N}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}} k\sigma_{i_0}^l\sigma_{j_0}^l G\, e^{\beta\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l}\ e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}{\sum_{\{\sigma\}} e^{\beta\sum_{\nu=1}^{k}\sigma_{i_\nu}^l\sigma_{j_\nu}^l} e^{(\beta\sum_{a=1}^{s}\sum_{\nu=1}^{k}\sigma_{i_\nu}^a\sigma_{j_\nu}^a)}}\Big] = \\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\sum_{\{\sigma\}}\sigma_{i_0}^l\sigma_{j_0}^l G\, e^{\beta\sigma_{i_0}^l\sigma_{j_0}^l}[\prod_{a=1}^{s} e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a}]\ e^{-\beta H_{s+1}}}{\sum_{\{\sigma\}} e^{\beta\sigma_{i_0}^l\sigma_{j_0}^l}[\prod_{a=1}^{s} e^{\beta\sigma_{i_0}^a\sigma_{j_0}^a}] e^{-\beta H_{s+1}}}\Big] = \\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\frac{\Omega\Big(\sigma_{i_0}^l\sigma_{j_0}^l G(1+\sigma_{i_0}^l\sigma_{j_0}^l\theta)[\prod_{a=1}^{s}(1+\sigma_{i_0}^a\sigma_{j_0}^a\theta)]\Big)}{\Big(1+\omega(\sigma_{i_0}\sigma_{j_0})\theta\Big)^{s+1}}\Big] = \\
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Omega\Big((\sigma_{i_0}^l\sigma_{j_0}^l G + G\theta)(1+\sum_{a}^{1,s}\sigma_{i_0}^a\sigma_{j_0}^a\theta + \sum_{a<b}^{1,s}\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^b\sigma_{j_0}^b\theta^2)\Big)\times \\
&\quad \times\Big(1-(s+1)\omega(\sigma_{i_0}\sigma_{j_0})\theta + \frac{(s+1)(s+2)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\Big)\Big] =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Big(\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l) + \theta\Big(\Omega(G) + \sum_{a}^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^l\sigma_{j_0}^l)\Big) + \\
&\quad + \theta^2\Big(\sum_{a}^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a) + \sum_{a<b}^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^b\sigma_{j_0}^b\sigma_{i_0}^l\sigma_{j_0}^l)\Big)\times \\
&\quad \times\Big(1-(s+1)\omega(\sigma_{i_0}\sigma_{j_0})\theta + \frac{(s+1)(s+2)}{2}\omega^2(\sigma_{i_0}\sigma_{j_0})\theta^2\Big)\Big] =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha M}{N^2}\mathbf{E}\Big[\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l) + \theta\Big(\Omega(G) + \sum_{a}^{1,s}\Omega(G\sigma_{i_0}^a\sigma_{j_0}^a\sigma_{i_0}^l\sigma_{j_0}^l) \\
&\quad -(s+1)\Omega(G\sigma_{i_0}^l\sigma_{j_0}^l)\omega(\sigma_{i_0}\sigma_{j_0})\Big) +
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
&+\theta^2\Big(\sum_{a}^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0})+\sum_{a<b}^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^b_{i_0}\sigma^b_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})\\
&-(s+1)\Omega(G)\omega(\sigma_{i_0}\sigma_{j_0})+\\
&\quad-(s+1)\sum_{a}^{1,s}\Omega(G\sigma^a_{i_0}\sigma^a_{j_0}\sigma^l_{i_0}\sigma^l_{j_0})\omega(\sigma_{i_0}\sigma_{j_0})\\
&+\frac{(s+1)(s+2)}{2}\Omega(G\sigma^l_{i_0}\sigma^l_{j_0})\omega^2(\sigma^a_{i_0}\sigma^a_{j_0})\Big)\Big]=
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{\alpha M}{N^2}\Big[\langle Gm_l^2\rangle+\theta\Big(\langle G\rangle+\sum_{a=1}^{s}\langle Gq^2_{a,l}\rangle-(s+1)\langle Gq^2_{l,s+1}\rangle\Big)+\\
+ \quad & \theta^2\Big(\sum_{a}^{1,s}\langle Gm_a^2\rangle-(s+1)\langle Gm_l^2\rangle+\sum_{a<b}^{1,s}\langle Gq^2_{l,a,b}\rangle+\\
- \quad & (s+1)\sum_{a}^{1,s}\langle Gq^2_{l,a,s+1}\rangle+\frac{(s+1)(s+2)}{2}\langle Gq^2_{l,s+1,s+2}\rangle\Big)\Big]
\end{aligned}
$$

A.2 Poisson identities via the self-averaging technique

For the sake of completeness we report also the constraints in the Poisson diluted model obtained by using the first method:

$$
\begin{aligned}
f_G(\alpha,\beta) \quad = \quad & \alpha\Big[\Big(\sum_{l=1}^{s}\langle Gm_l^2\rangle-s\langle Gm^2_{s+1}\rangle\Big)\Big(1-\theta^2\Big)+\\
+ \quad & 2\theta\Big(\sum_{a<l}^{1,s}\langle Gq^2_{al}\rangle-s\sum_{l}^{1,s}\langle Gq^2_{l,s+1}\rangle+\frac{s(s+1)}{2}\langle Gq^2_{s+1,s+2}\rangle\Big)+\\
+ \quad & 3\theta^2\Big(\sum_{l<a<b}^{1,s}\langle Gq^2_{l,a,b}\rangle-s\sum_{l<a}^{1,s}\langle Gq^2_{l,a,s+1}\rangle+\frac{s(s+1)}{2}\sum_{l}^{1,s}\langle Gq^2_{l,s+1,s+2}\rangle+\\
- \quad & \frac{s(s+1)(s+2)}{3!}\langle Gq^2_{s+1,s+2,s+3}\rangle\Big)+O(\theta^3)\Big],
\end{aligned}
\tag{72}
$$

by which, choosing as a trial function m^2 we have

$$\begin{aligned} f^P_{m^2}(\alpha,\beta) &= \alpha\Big[\Big(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\Big)\Big(1-\theta^2\Big) + \\ &\quad -2\theta\Big(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\Big) + \\ &\quad +3\theta^2\Big(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle\Big) + O(\theta^3)\Big], \end{aligned}$$

from which, changing the Jacobian $d\theta = (1-\theta^2)d\beta$, we get

$$\begin{aligned} \lim_{N\to\infty}\int_{\beta_1}^{\beta_2} |f^P_{m^2}(\alpha,\beta)|d\beta &= \alpha\int_{\theta_1}^{\theta_2} d\theta\Big[|\ \Big(\langle m_1^4\rangle - \langle m_1^2 m_2^2\rangle\Big) + \\ &- 2\frac{\theta}{(1-\theta^2)}\Big(\langle m_1^2 q_{12}^2\rangle - \langle m_1^2 q_{23}^2\rangle\Big) + \\ &+ 3\frac{\theta^2}{(1-\theta^2)}\Big(\langle m_1^2 q_{123}^2\rangle - \langle m_1^2 q_{234}^2\rangle \\ &+ O(\theta^3)\ |\Big] = 0. \end{aligned} \tag{73}$$

If we set $G = q_{12}^2$ as the trial function $f^P_G(\alpha,\beta)$ becomes

$$\begin{aligned} f^P_{q^2}(\alpha,\beta) &= \alpha\Big[\Big(2\langle m_1^2 q_{12}^2\rangle - 2\langle m_3^2 q_{12}^2\rangle\Big)\Big(1-\theta^2\Big) + \\ &+ 2\theta\Big(\langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{23}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle\Big) + \\ &- 6\theta^2\Big(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{234}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle\Big) + O(\theta^3)\Big]. \end{aligned}$$

Again

$$\begin{aligned} \lim_{N\to\infty}\int_{\beta_1}^{\beta_2} |f^P_{q^2}(\alpha,\beta)|d\beta &= 2\alpha\int_{\theta_1}^{\theta_2} d\theta\Big[|\ \Big(\langle m_1^2 q_{12}^2\rangle - \langle m_3^2 q_{12}^2\rangle\Big) + \\ &+ \frac{\theta}{(1-\theta^2)}\Big(\langle q_{12}^4\rangle - 4\langle q_{12}^2 q_{23}^2\rangle + 3\langle q_{12}^2 q_{34}^2\rangle\Big) + \\ &- 3\frac{\theta^2}{(1-\theta^2)}\Big(\langle q_{12}^2 q_{123}^2\rangle - 3\langle q_{12}^2 q_{234}^2\rangle + 2\langle q_{12}^2 q_{345}^2\rangle\Big) \\ &+ O(\theta^3)\ |\Big] = 0, \end{aligned} \tag{74}$$

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