Thermodynamic Limit for Mean-Field Spin Models

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Abstract

If the Boltzmann-Gibbs state ω_N of a mean-field N-particle system with Hamiltonian H_N verifies the condition

$$\omega_N(H_N) \;\geq\; \omega_N(H_{N_1} + H_{N_2})$$

for every decomposition $N_1 + N_2 = N$, then its free energy density increases with N. We prove such a condition for a wide class of spin models which includes the Curie-Weiss model, its p-spin generalizations (for both even and odd p), its random field version and also the finite pattern Hopfield model. For all these cases the existence of the thermodynamic limit by subadditivity and boundedness follows.

1 Introduction

The rigorous theory of the thermodynamic limit which already in the sixties was a well established part of equilibrium statistical mechanics [Ru] received recently a new impulse thanks to the treatment of the Sherrington-Kirkpatrick [MPV] model of the mean field spin glass done by Guerra and Toninelli [GuTo]. Moreover in a sequel work [Gu] it became clear that a good control of the limit, especially when it is obtained by monotonicity through subadditivity arguments, may lead to sharp bounds for the model and carries important informations well beyond the existence of the limit itself. In this paper we build a theory of thermodynamic limit which apply to a family of cases including both random and non-random mean field models like the Curie Weiss model [Ba], its p-spin generalizations, its random field version [MP], and also the finite pattern Hopfield model [Ho]. Due to the explicit size dependence of the local interactions we stress that mean field models do not fall into the class for which standard techinques [Ru] can be applyed to prove the existence of the thermodynamic limit. Moreover even in the Curie Weiss model in which the exact solution is available it is interesting to obtain the existence of the thermodynamic quantities without exploiting the exact solution (see [EN, CGI]).

With respect to the theory relative to the random case [CDGG] we use here a different interpolation technique which works pointwise with respect to the disorder. The novelty of our approach relies on the fact that while in the previous case the condition for the existence of the limit is given in terms of a suitably deformed quenched measure, in the class of models we treat here we are able to give a condition with a direct thermodynamic meaning: the Bolztmann-Gibbs state for a large system provides a good approximation for the subsystems. The fact that our existence condition is fully independent from the interpolation parameter relies on the convexity of the interpolating functional, a property still under investigation for the spin glass models [CG1, CG2].

2 Definitions and Results.

We consider a system of N sites: $\{1, 2, ..., N\}$, to each site we associate a spin variable σ_i taking values in $\{\pm 1\}$. A spin configuration is specified by the sequence $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and we denote the sets of all spin configurations by $\Sigma_N = \{\pm 1\}^N$. We will study models defined by a *mean field* Hamiltonian, i.e. for a given bounded function $g : [-1, 1] \to \mathbb{R}$,

$$H_N(\sigma) = -N g(m_N) \tag{1}$$

where

$$m_N(\sigma) \; = \; \frac{1}{N} \sum_{i=1}^{N} \sigma_i \; . \tag{2}$$

Definition 1 *For each N and a given inverse temperature β we introduce the partition function*

$$Z_N \; = \sum_{\sigma \in \Sigma_N} e^{-\beta H_N(\sigma)} \, , \tag{3}$$

the free energy density (and the auxiliary function α_N)

$$-\, \beta f_N \; = \; \frac{1}{N} \ln Z_N \; = \; \alpha_N \, , \tag{4}$$

and, for a generic observable $O(\sigma)$, the Boltzmann-Gibbs state

$$\omega_N(O) \; = \; \frac{\sum_{\sigma \in \Sigma_N} O(\sigma) e^{-\beta H_N(\sigma)}}{Z_N} . \tag{5}$$

Remark 1 *Obviously from (3) and (4), one has that if* g *and* g' *are two functions from* $[-1,1]$ *to* $\mathbb{R}$ *such that*

$$\|g - g'\| := \sup\{|g(x) - g'(x)| : -1 \leq x \leq 1\} \tag{6}$$

is bounded, then one has

$$|\alpha_N - \alpha'_N| \leq \beta\|g - g'\| \tag{7}$$

for every N*.*

We can now state our main result:

THEOREM 1 *Let* $H_N(\sigma)$ *be a mean field Hamiltonian (see eq. (1), (2)). If for every*

partition of the set $\{1, 2, ..., N\}$ *into* $\{1, 2, ..., N_1\}$ *and* $\{N_1+1, ..., N\}$ *with* $N = N_1 + N_2$ *and*

$$H_{N_1} = H_{N_1}(\sigma_1, ..., \sigma_{N_1}) \, , \qquad H_{N_2} = H_{N_2}(\sigma_{N_1+1}, ..., \sigma_N) \, , \tag{8}$$

the condition

$$\omega_N(H_N) \ \geq \ \omega_N(H_{N_1} + H_{N_2}) \, , \tag{9}$$

is verified, then the thermodynamic limit exists in the sense:

$$\lim_{N\to\infty} \alpha_N \ = \ \inf_N \alpha_N \ = \ \alpha. \tag{10}$$

3 Proof

Definition 2 *Let us define the interpolating Hamiltonian as a function of the parameter* $t \in [0,1]$*:*

$$H_N(t) \ = \ tH_N + (1-t)[H_{N_1} + H_{N_2}] \, , \tag{11}$$

and consider its relative partition function $Z_N(t)$, free energy density $f_N(t)$ and Boltzmann state $\omega_{N,t}$.

The interpolation method that we are going to use is based on the sign control for both the first and second derivative of $\alpha_N(t)$. More precisely the following holds:

Lemma 1 *Let H_N be the mean field Hamiltonian and $H_N(t)$ its relative interpolation.*
If

$$\frac{d}{dt}\alpha_N(t) \leq 0 \tag{12}$$

for all $t \in [0,1]$, then

$$\alpha_N \leq \frac{N_1}{N}\alpha_{N_1} + \frac{N_2}{N}\alpha_{N_2} , \tag{13}$$

for each decomposition $N = N_1 + N_2$.

Proof: trivially follows from the fundamental theorem of calculus and from the observation that definition (11) implies:

$$Z_N(1) = Z_N , \tag{14}$$

$$\alpha_N(1) = \alpha_N , \tag{15}$$

$$Z_N(0) = Z_{N_1} Z_{N_2} , \tag{16}$$

and

$$\alpha_N(0) = \frac{N_1}{N}\alpha_{N_1} + \frac{N_2}{N}\alpha_{N_2} . \tag{17}$$

$\square$

Lemma 2 *Computing the t derivative of $\alpha_N(t)$, we get:*

$$\begin{aligned}\alpha'(t) = \frac{d}{dt}\frac{1}{N}\log Z_N(t) &= -\frac{\beta}{N}\sum_{\sigma\in\Sigma_N}[H_N - H_{N_1} - H_{N_2}]\,\frac{e^{-\beta H_N(t)}}{Z_N(t)} \\ &= -\frac{\beta}{N}\omega_{N,t}[H_N - H_{N_1} - H_{N_2}]. \end{aligned} \tag{18}$$

Lemma 3 *The second derivative of $\alpha_N(t)$ is positive:*

$$\alpha''_N(t) \;=\; \frac{d^2}{dt^2}\alpha_N(t) \geq 0\,, \tag{19}$$

Proof: a direct computation gives

$$\alpha''_N(t) = \frac{d}{dt}\left(-\frac{\beta}{N}\omega_{N,t}[H_N - H_{N_1} - H_{N_2}]\right)$$

$$= \frac{\beta^2}{N}\Big(\omega_{N,t}\left[(H_N - H_{N_1} - H_{N_2})^2\right] - \omega^2_{N,t}\left[H_N - H_{N_1} - H_{N_2}\right]\Big)\,. \tag{20}$$

From Jensen's inequality applied to the convex function $x \mapsto x^2$, it follows that $\alpha''_N(t) \geq 0$. □

We are now able to prove the statement of Theorem 1.

Proof of THEOREM 1. From Lemma (2) we notice that the hypothesis (9)

$$\omega_N\left(H_N\right) \geq \omega_N\left(H_{N_1} + H_{N_2}\right)$$

is equivalent to the condition $\alpha'_N(1) \leq 0$. On the other hand from Lemma (3) it follows that $\alpha'_N(t)$ is an increasing function of t. This means that the determination of the sign of $\alpha'_N(t)$ can be in general established by the evaluation of the sign in the extremes of the interval [0,1]. In particular we have:

$$\alpha'_N(1) \leq 0 \implies \alpha'_N(t) \leq 0, \quad \forall t \in [0,1] \tag{21}$$

Using now Lemma (1), the subadditivity property (13) holds for α_N and then, by standard arguments [Ru],

$$\lim_{N\to\infty} \alpha_N \;=\; \inf_N \alpha_N \tag{22}$$

The existence of thermodynamic limit finally follows from boundedness of the function g in Eq. (1). Indeed, calling K the maximum of $g(x)$ on the interval $[-1, 1]$, we have

$$\alpha_N = \frac{1}{N} \ln \sum_{\sigma\in\Sigma_N} e^{\beta N g(m_N)} \geq \frac{1}{N} \ln e^{\beta N K} = \beta K \; . \tag{23}$$

4 Applications

In this Section we identify a class of mean field models for which the hypotheses of our theorem are verified. Specifically these will be all models such that the function g of formula (1) is convex or polynomial.

Corollary 1 *Let the Hamiltonian be of the form*

$$H_N(\sigma) \;=\; -Ng(m_N) \tag{24}$$

with $g : [-1, 1] \to \mathbb{R}$ *a bounded* convex *function. Then the thermodynamic limit of the free energy exists.*

Proof: For a given $\sigma \in \Sigma_N$ and for every decomposition $N = N_1 + N_2$ we define the quantities

$$m_{N_1}(\sigma) \;=\; \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i \qquad m_{N_2}(\sigma) \;=\; \frac{1}{N_2} \sum_{i=N_1+1}^{N} \sigma_i \; , \tag{25}$$

so that the total magnetization is a convex linear combination of the two:

$$m_N \; = \; \frac{N_1}{N} m_{N_1} + \frac{N_2}{N} m_{N_2} \; . \tag{26}$$

Using this definitions the hypothesis (9) of Theorem (1) is verified:

$$\omega_N \left(H_N - H_{N_1} - H_{N_2} \right) = -N \omega_N \left(g(m_N) - \frac{N_1}{N} g(m_{N_1}) - \frac{N_2}{N} g(m_{N_2}) \right) \; \geq 0 \tag{27}$$

where the last inequality follows from convexity of g. □

Remark 2 *The previous Corollary can be obviously generalized to the case where the function g is a* convex *bounded function of many variables, each of them fulfilling the property (25).*

Corollary 2 *Let the Hamiltonian be of the form*

$$H_N(\sigma) \; = \; -N g(m_N) \, , \tag{28}$$

with $g : [-1, 1] \to \mathbb{R}$ a polynomial *function of degree $n \in \mathbb{N}$. Then the thermodynamic limit exists.*

Proof: First of all we consider the case $g(x) = x^k$ (the generalization will follow in a simple way) with associated Hamiltonian

$$H_N(\sigma) \; = \; -N m_N^k \; = \; -\frac{1}{N^{k-1}} \sum_{i_1, i_2, \ldots, i_k = 1}^{N} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \; . \tag{29}$$

By splitting the summation into two pieces, the first containing the summation with indexes all different among themselves, the second containing the remaining terms, we have

$$H_N(\sigma) \; = -\frac{1}{N^{k-1}} \left[\sum_{i_1 \neq \ldots \neq i_k} \sigma_{i_1} \cdots \sigma_{i_k} + \sum_{i_1, \ldots, i_k}^{*} \sigma_{i_1} \cdots \sigma_{i_k} \right] \tag{30}$$

where the second summation $\sum^*$ includes all terms with at least two equal indexes. A simple computation shows that

$$\frac{1}{N^{k-1}} \sum_{i_1,\ldots,i_k}^{*} \sigma_{i_1} \cdots \sigma_{i_k} \; = \; \mathcal{O}(1) \tag{31}$$

Defining now the model with Hamiltonian

$$\tilde{H}_N(\sigma) = -\frac{1}{(N-1)(N-2)\cdots(N-k+1)} \sum_{i_1 \neq i_2 \neq \ldots \neq i_k = 1}^{N} \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\,, \tag{32}$$

it follows that

$$\begin{aligned} H_N(\sigma) &= \frac{(N-1)(N-2)\cdots(N-k+1)}{N^{k-1}} \tilde{H}_N(\sigma) + \mathcal{O}(1) \\ &= \tilde{H}_N(\sigma) + \mathcal{O}(1) \end{aligned} \tag{33}$$

Using Remark 1 one has that the two models H_N and $\tilde{H}_N$ have the same thermodynamic limit (if any). On the other hand for the model $\tilde{H}_N$ we have

$$\begin{aligned} \omega_N(\tilde{H}_N) &= -\frac{1}{(N-1)(N-2)\cdots(N-k+1)} \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} \omega_N(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}) \\ &= -N\omega_N(\sigma_1\sigma_2\cdots\sigma_k) \end{aligned} \tag{34}$$

where the last equality follows from permutation invariance ($\omega_N(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k})$ does not depend on the choice of the indexes). Analogously one can repeat the same computation for $\tilde{H}_{N_1}$ and $\tilde{H}_{N_2}$ in the state ω_N. By permutation symmetry, this yields that Hypothesis (9) of Theorem (1) is verified as an equality

$$\omega_N\left(\tilde{H}_N - \tilde{H}_{N_1} - \tilde{H}_{N_2}\right) = 0\,, \tag{35}$$

implying the existence of thermodynamic limit for the model $\tilde{H}_N$, and so for the model H_N.

Since we have proved the Corollary for $g(x) = x^k$, the case of a generic polynomial function of degree n

$$g(x) = \sum_{k=0}^{n} a_k x^k \tag{36}$$

is treated by the same argument, applied to each monomial of the sum. $\square$

Corollary 3 *Using Remark 1 and the Stone-Weierstrass theorem, the thermodynamic limit of α_N exists if, instead of g being polynomial, g is merely continuous up to the boundary of* $[-1, 1]$.

4.1 Examples

1. **The Curie-Weiss models.**

   For every integer p, with $p < N$, consider the model defined by

   $$H_N(\sigma) := -\frac{1}{N^{p-1}} \sum_{i_1, i_2, \dots, i_p = 1}^{N} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p} \tag{37}$$

   which represents the generalized Curie Weiss model with p-spin interaction. The standard Curie-Weiss model corresponds to the case $p = 2$. From eq. (2) the previous Hamiltonian can be written as

   $$H_N = -N m_N^p \tag{38}$$

   and the existence of the thermodynamic limit is then implied by Corollary (2). Moreover, the same result holds for any linear combination (see Eq.(36)) of generalized Curie Weiss models with p-spin interaction, both ferromagnetic and antiferromagnetic.

2. **The random field Curie Weiss model.**

Here we consider the model defined by (see [MP] for a review)

$$H_N(h,\sigma) := -\frac{1}{N}\sum_{i,j=1}^{N}\sigma_i\sigma_j + \sum_{i=1}^{N} h_i\sigma_i \tag{39}$$

where $\{h_i\}_{i=1,\dots,N}$ is a family of i.i.d. Bernoulli random variables, with probability distribution

$$p(h_i) = \begin{cases} 1/2, & \text{if } h_i = 1, \\ 1/2, & \text{if } h_i = -1. \end{cases} \tag{40}$$

For a given realization of the random field h, we define the quantities

$$m_N^+(\sigma,h) = \frac{1}{N}\sum_{i=1}^{N}\frac{1+h_i}{2}\,\sigma_i \tag{41}$$

$$m_N^-(\sigma,h) = \frac{1}{N}\sum_{i=1}^{N}\frac{1-h_i}{2}\,\sigma_i \tag{42}$$

The Hamiltonian can be written in terms of these variables as

$$H_N = -Ng(m_N^+, m_N^-) \tag{43}$$

where

$$g(m_N^+, m_N^-) = (m_N^+ + m_N^-)^2 - (m_N^+ - m_N^-) \tag{44}$$

Since this function is obviously convex with respect to both m_N^+ and m_N^-, bounded by 2, and

$$m_N^{\pm}(\sigma,h) \;=\; \frac{N_1}{N} m_{N_1}^{\pm}(\sigma,h) + \frac{N_2}{N} m_{N_2}^{\pm}(\sigma,h) \tag{45}$$

the Corollary (1) can be applied and we find (pointwise in the h's)

$$\alpha_N(h) \;\leq\; \frac{N_1}{N}\alpha_{N_1}(h) + \frac{N_2}{N}\alpha_{N_2}(h)\;. \tag{46}$$

Averaging now over the h's, the subadditivity property for quenched α_N is proved and this yields (22).

3. The Hopfield model.

The Hamiltonian of the Hopfield model (see [Bo] for a review) is given by :

$$H_N(\xi,\sigma) := -\sum_{\mu=1}^{M} \frac{1}{N} \sum_{i,j=1}^{N} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j \tag{47}$$

where M is the (fixed) number of pattern and the $\{\xi_i^\mu\}_{i=1,\dots,N}^{\mu=1,\dots,M}$ is a family of i.i.d. Bernoulli variables with probability distribution

$$p(\xi_i^\mu) = \begin{cases} 1/2, & \text{if } \xi = 1, \\ 1/2, & \text{if } \xi = -1. \end{cases} \tag{48}$$

Defining the quantities

$$m_N^\mu(\sigma,\xi) = \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \sigma_i \quad , \forall \mu = 1, \dots, M \ , \tag{49}$$

the Hamiltonian (47) can be written as

$$H_N(\sigma,\xi) = -N \sum_{\mu=1}^{M} (m_N^\mu(\sigma,\xi))^2. \tag{50}$$

The model can be included in the general treatment of the previous section by considering a function g of M variables,

$$g(m_N^1(\sigma,\xi), \dots, m_N^M(\sigma,\xi)) = \sum_{\mu=1}^{M} (m_N^\mu(\sigma,\xi))^2 \tag{51}$$

such that

$$H_N \ = \ -N g(m_N^1, \dots, m_N^M). \tag{52}$$

Since

$$m_N^\mu(\sigma,\xi) \;=\; \frac{N_1}{N} m_{N_1}^\mu(\sigma,\xi) + \frac{N_2}{N} m_{N_2}^\mu(\sigma,\xi) \qquad \forall \mu = 1,\ldots,M \tag{53}$$

and the function g is convex with respect to every m_N^μ and bounded by M, using Corollary (1) we have (pointwise in the ξ's)

$$\alpha_N(\xi) \;\le\; \frac{N_1}{N}\alpha_{N_1}(\xi) + \frac{N_2}{N}\alpha_{N_2}(\xi)\;. \tag{54}$$

Averaging over the ξ's yields the (22).

Remark 3 *We want to notice that the method shown doesn't apply to the Hopfield model with a thermodynamically growing number of patterns defined for every positive constant γ by*

$$H_N(\xi,\sigma) := -\sum_{\mu=1}^{\gamma N} \frac{1}{N} \sum_{i,j=1}^{N} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j\;, \tag{55}$$

because this Hamiltonian is not of the form (1).

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