Multi-species mean-field spin-glasses. Rigorous results.

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July 22, 2013

Abstract

We study a multi-species spin glass system where the density of each species is kept fixed at increasing volumes. The model reduces to the Sherrington-Kirkpatrick one for the single species case. The existence of the thermodynamic limit is proved for all densities values under a convexity condition on the interaction. The thermodynamic properties of the model are investigated and the annealed, the replica symmetric and the replica symmetry breaking bounds are proved using Guerra's scheme. The annealed approximation is proved to be exact under a high temperature condition. We show that the replica symmetric solution has negative entropy at low temperatures. We study the properties of a suitably defined replica symmetry breaking solution and we optimise it within a *ziggurat ansatz*. The generalized order parameter is described by a Parisi-like partial differential equation.

Keywords: Multi-species spin glasses, annealed region, replica symmetric solution, replica symmetry breaking bounds.

1 Introduction

Multi-species spin systems at different densities are often encountered in nature. The bipartite case without disordered made its appearance since the work on meta-magnets by Cohen and Kinkaid [1]. When several types of magnetic particles like iron and manganese are diluted into a nonmagnetic metallic host the Ruderman-Kittel-Kasuya-Yosida interactions generate a multi-species spin glass phase [2]. The rich complex behaviour emerging from those physical systems revealed to be useful in a variety of applications ranging from biology to social sciences and several models were proposed and studied in the mean field approximation without disorder [11, 9, 10] and with disorder as well [6, 7, 8]. In this paper we introduce and study the multi-species mean field spin glass model i.e. a system composed by spins belonging to a finite number of different species and we study its thermodynamic behaviour at fixed species densities. Spin couples interact through a centered gaussian variable whose variances depend only on the two species. We rigorously prove bounds for the model pressure and control their properties in a region of convexity defined in terms of the variance matrix of the interactions, namely when the interactions within a group dominate the inter-groups interactions (see [4, 6, 8] for a similar conditions in neural network theory).

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The paper is organised as follows. Section 2 introduces the notations and some technical tools. Section 3 defines the model and the observables. The thermodynamic limit proof is illustrated in section 4. Section 5 studies the annealed region with the second moment method. Section 6 considers the replica symmetric bound and shows that at low temperature it has a negative entropy. Finally section 7 investigate the replica symmetry breaking interpolation within the *ziggurat* ansatz, a multi-dimensional generalisation of the Parisi ansatz.

2 Notation and technical tools

Much of the recent progresses in the study of mean field spin glass models is based on methods and arguments introduced by Guerra in a series of works (see e.g. [12, 13, 14, 15, 5]), constituting the so called *interpolation method*. Beyond the original works, the interested reader can found a detailed and complete exposition with several applications of this method in [16], while in [3] these techniques are shown at work on the simpler Curie-Weiss model. In order to present a self-consistent exposition, hereafter we outline briefly the basic ideas.

Let N be an integer, and for $i \in I = \{1, ..., N\}$, let U_i and $\widetilde{U}_i$ be two families of centered Gaussian random variables, independent each other, uniquely determined by the respective covariance matrices $\mathbb{E}(U_i U_j) = C_{ij}$ and $\mathbb{E}(\widetilde{U}_i \widetilde{U}_j) = \widetilde{C}_{ij}$. We treat the set of indices i as configuration space for some statistical mechanics system. Let $a_i \in \mathbb{R}^+$ for each $i \in I$ be an arbitrary (finite) weight. We define the Hamiltonian interpolating function as the following random variable

$$H_i(t) := \sqrt{t} U_i + \sqrt{1-t} \tilde{U}_i,$$

where $t \in [0, 1]$ is the real parameter used for interpolation.

Let us introduce also the so-called *quenched measures*. First, we define the random partition function of the system as

$$Z(t) := \sum_i a_i e^{H_i(t)},$$

and the random Gibbs measure as

$$G_i(t) := \frac{a_i e^{H_i(t)}}{Z(t)}.$$

Then let $F : (I \times I) \longrightarrow \mathbb{R}$ be an observable defined in the duplicated configuration space, we define the *quenched measure* as

$$\langle F \rangle_t := \mathbb{E}\Big(\Omega_t(F)\Big), \tag{1}$$

where

$$\Omega_t(F) := \sum_{i,j} G_i(t) G_j(t) F. \tag{2}$$

The measure Ω_t is called the duplicated random Gibbs measure.
Keeping in mind definition (1), it is possible to prove (see [16]) the following

Proposition 1. *Consider the functional*

$$\varphi(t) := \mathbb{E} \log Z(t) \tag{3}$$

then for its t-derivative the following expression holds

$$\varphi'(t) = \frac{1}{2}\langle C_{ii} - \tilde{C}_{ii}\rangle_t - \frac{1}{2}\langle C_{ij} - \tilde{C}_{ij}\rangle_t. \tag{4}$$

The generalization to multi-partite systems requires only minor modifications. Suppose that the system is composed by a finite number S of species indexed by $s \in \mathcal{S}$, then $|\mathcal{S}| = S$. Consider a generic statistical mechanic system as before and assume that:

- the configuration space is decomposed in a disjoint union $I = \bigcup_{s\in\mathcal{S}} I^{(s)}$,

- the U's are also decomposed in the following way

$$U_i = \sum_{s,p\in\mathcal{S}} U_i^{(sp)}, \tag{5}$$

where $U_i^{(sp)}$ is a family of gaussian r.v. such that the covariance matrix is of the form

$$\mathbb{E}(U_i^{(sp)} U_j^{(s'p')}) = \Delta^2_{sp}\delta_{ss'}\delta_{pp'}C_{ij}^{(s)}C_{ij}^{(p)} \tag{6}$$

where $C_{ij}^{(s)}$ is a covariance matrix defined on $I^{(s)} \times I^{(s)}$.

Notice that the covariance matrix defined in (6) is the Schur-Hadamard product of the $C_{ij}^{(s)}$ and then is positive definite. The family of positive parameters $(\Delta^2_{sp})_{s,p\in\mathcal{S}}$ tunes the interactions between the various species.
For a fixed couple (i,j) we can think at each $C_{ij}^{(s)}$ as a component of a vector in the space $\mathbb{R}^S$ and then, thanks to (5) and (6), the covariance matrix of the entire system can be rewritten, with a little abuse of notation, as a quadratic form in $\mathbb{R}^S$, namely as

$$C_{ij} = \mathbb{E}(U_iU_j) = \sum_{s,p\in\mathcal{S}} C_{ij}^{(s)}\Delta^2_{sp}C_{ij}^{(p)} = \Big(\mathbf{C}, \mathbf{\Delta C}\Big), \tag{7}$$

where $\mathbf{C} := (C_{ij}^{(s)})_{s\in\mathcal{S}}$ is a vector in $\mathbb{R}^S$ and $\mathbf{\Delta}$ is the real symmetric matrix defined by the entries

$$\mathbf{\Delta} := (\Delta^2_{sp})_{s,p\in\mathcal{S}}.$$

Suppose for simplicity that $C_{ii}^{(s)} = \sqrt{c}$ for some $c \in \mathbb{R}^+$ for each $i \in I, s \in \mathcal{S}$, that is

$$C_{ii} = c(\mathbf{1}, \mathbf{\Delta 1}), \tag{8}$$

where

$$\mathbf{1} := (1)_{s\in\mathcal{S}}.$$

Under the assumption that an analogous decomposition holds for the $\widetilde{U}$'s too, then

$$\widetilde{C}_{ij} = \mathbb{E}(\widetilde{U}_i\widetilde{U}_j) = \sum_{s,p\in\mathcal{S}} \widetilde{C}_{ij}^{(s)}\Delta^2_{sp}\widetilde{C}_{ij}^{(p)} = \Big(\widetilde{\mathbf{C}}, \mathbf{\Delta}\widetilde{\mathbf{C}}\Big), \tag{9}$$

and

$$\widetilde{C}_{ii} = \widetilde{c}(\mathbf{1}, \mathbf{\Delta 1}). \tag{10}$$

In the multipartite framework, by (7,8,9,10), Proposition 1 becomes

Proposition 2. *Consider the functional defined in (3), then for its t-derivative the following holds*

$$\varphi'(t) = \frac{1}{2}(c-\widetilde{c})(\mathbf{1},\mathbf{\Delta 1}) - \frac{1}{2}\langle(\mathbf{C},\mathbf{\Delta C}) - (\widetilde{\mathbf{C}},\mathbf{\Delta}\widetilde{\mathbf{C}})\rangle_t. \tag{11}$$

In order to separate the contribution of the various species, let us introduce the operator $\mathcal{P}_s$ as the canonical projector in $\mathbb{R}^S$.
For any $s \in \mathcal{S}$ and for any vector $\mathbf{u} = (u^{(s)})_{s\in\mathcal{S}}$ in $\mathbb{R}^S$, we have that

$$\mathcal{P}_s\Big(\mathbf{u}\Big) := u^{(s)}. \tag{12}$$

Clearly, for two vectors $\mathbf{u},\mathbf{v}$, the following relation holds

$$\Big(\mathbf{u},\mathbf{\Delta v}\Big) = \sum_{s\in\mathcal{S}} \mathcal{P}_s\Big(\mathbf{u}\Big)\mathcal{P}_s\Big(\mathbf{\Delta v}\Big) = \sum_{s\in\mathcal{S}} \mathcal{P}_s\Big(\mathbf{\Delta u}\Big)\mathcal{P}_s\Big(\mathbf{v}\Big). \tag{13}$$

If we denote by $(\mathbf{e}_s)_{s\in\mathcal{S}}$ the canonical basis of $\mathbb{R}^S$, the canonical projection can be expressed as a scalar product, that is

$$\mathcal{P}_s\Big(\mathbf{u}\Big) = (\mathbf{e}_s,\mathbf{u}). \tag{14}$$

Let us recall briefly the Guerra's RSB scheme. Let U_i be a family of centered Gaussian random variables uniquely determined by the covariance matrix $\mathbb{E}(U_iU_j) = C_{ij}$ and let us introduce the integer K, associated to the number of levels of Replica Symmetry Breaking (RSB in the following). For each couple $(l,i) \in \{1,2,...,K\} \times I$, let us introduce further the family of centered Gaussian random variables B_i^l independent from the U_i and uniquely defined through the covariances

$$\mathbb{E}(B_i^l B_j^{l'}) = \delta_{ll'}\widetilde{B}_{ij}^l,$$

and point out that there is independence between different l, l' levels of symmetry breaking.
Further, we need some preliminary definitions:
For the average with respect to B_i^l and U_i we use the following notation

$$\begin{aligned}
\mathbb{E}_l(\cdot) &= \int\prod_i d\mu(B_i^l)(\cdot), \qquad \forall l = 1,...,K, && (15)\\
\mathbb{E}_0(\cdot) &= \int\prod_i d\mu(U_i)(\cdot), && (16)\\
\mathbb{E}(\cdot) &= \mathbb{E}_0\mathbb{E}_1...\mathbb{E}_K(\cdot). && (17)
\end{aligned}$$

We need also a sequence of non negative real numbers $(m_0, m_1, ..., m_K, m_{K+1})$ with $m_0 = 0,\ m_{K+1} = 1$ and we define recursively the following the random variables

$$\begin{aligned}
Z_K(t) &:= \sum_i \omega_i \exp\big(\sqrt{t}U_i + \sqrt{1-t}\sum_{l=1}^K B_i^l\big), && (18)\\
Z_{l-1}^{m_l} &:= \mathbb{E}_l(Z_l^{m_l}), && (19)\\
f_l &:= \frac{Z_l^{m_l}}{\mathbb{E}_l(Z_l^{m_l})}, && (20)
\end{aligned}$$

and the following modified Gibbs states,

$$\begin{aligned}
\widetilde{\omega}_{K,t}(\cdot) &:= \omega_t(\cdot) \qquad \widetilde{\Omega}_{K,t} = \Omega_t(\cdot), && (21)\\
\widetilde{\omega}_{l,t}(\cdot) &:= \mathbb{E}_{l+1}...\mathbb{E}_K(f_{l+1}...f_K\omega_t(\cdot)) \qquad \forall l = 0,...,K, && (22)\\
\widetilde{\Omega}_{l,t}(\cdot) &:= \mathbb{E}_{l+1}...\mathbb{E}_K(f_{l+1}...f_K\Omega_t(\cdot)) \qquad \forall l = 0,...,K, && (23)\\
\langle\cdot\rangle_{l,t} &:= \mathbb{E}(f_1...f_K\widetilde{\Omega}_{l,t}(\cdot)) \qquad \forall l = 0,...,K. && (24)
\end{aligned}$$

Bearing in mind the previous definitions, it is possible to prove (see [12]) the following

Proposition 3. *Consider the functional*

$$\varphi(t) = \mathbb{E}_0 \log(Z_0(t)), \tag{25}$$

then for its t-derivative the following relation holds

$$\varphi'(t) = \frac{1}{2}\langle C_{ii} - \widehat{B}^K_{ii}\rangle_{K,t} - \frac{1}{2}\sum_{l=0}^{K}(m_{l+1} - m_l)\langle C_{ij} - \widehat{B}^l_{ij}\rangle_{l,t} \tag{26}$$

where $\widehat{B}^0_{ij} = 0$ and $\widehat{B}^l_{ij} = \sum_{l'=1}^{l}\widetilde{B}^l_{ij}$.

We discuss now a generalization of the previous scheme for multipartite systems.
Let K be an integer and consider an arbitrary sequence of points $\Gamma := (\mathbf{q}_l)_{l=1,...,K} \in [0,1]^S$. For each triple (l,i,s) with $l = 1,2,...,K$, $i \in I$, $s \in \mathcal{S}$, let us introduce the family of centered Gaussian random variables $B_i^{l,(s)}$ independent from the U_i and uniquely defined through the covariances

$$\mathbb{E}(B_i^{l,(s)}B_j^{l',(s')}) = \delta_{ss'}\delta_{ll'}\mathcal{P}_s\Big(\mathbf{\Delta u}_l(\Gamma)\Big)\mathcal{P}_s\Big(\widetilde{\mathbf{C}}_l\Big) \tag{27}$$

where, for each value of l, the component of the vector $\widetilde{\mathbf{C}}_l = (\widetilde{C}^{(s)}_{l,ij})_{s\in\mathcal{S}}$, are covariance matrix defined on $I^{(s)} \times I^{(s)}$ and $\mathbf{u}_l(\Gamma)$ is an arbitrary vector in $\mathbb{R}^S$ which depends on the choice of the sequence Γ .
Notice that (27) implies independence between two different $l^{(s)}, l'^{(s)}$ levels of symmetry breaking of each s-species. For each $l = 1,2,...,K$ and $i \in I$, we can define the following family of random variables

$$B_i^l := \sum_{s\in\mathcal{S}} B_i^{l,(s)}$$

then by (13) we have that

$$\mathbb{E}(B_i^l B_j^{l'}) = \delta_{ll'}\Big(\mathbf{u}_l(\Gamma), \mathbf{\Delta}\widetilde{\mathbf{C}}_l\Big). \tag{28}$$

Suppose for simplicity that $\widetilde{C}^{(s)}_{l,ii} = 1$ for each l,i,s, that is

$$\mathbb{E}(B_i^l B_i^{l'}) = \delta_{ll'}\Big(\mathbf{u}_l(\Gamma), \mathbf{\Delta 1}\Big).$$

Let us introduce the following notations for the average with respect to B_i^l, U_i,

$$\begin{aligned}
\mathbb{E}_l(\cdot) &= \int \prod_i d\mu(B_i^l)(\cdot) \qquad \forall l = 1, ..., K, & (29)\\
d\mu(B_i^l) &= \prod_{s\in\mathcal{A}} d\mu(B_i^{l,(s)}), & (30)\\
\mathbb{E}_0(\cdot) &= \int \prod_i d\mu(U_i)(\cdot), & (31)\\
\mathbb{E}(\cdot) &= \mathbb{E}_0\mathbb{E}_1...\mathbb{E}_K(\cdot). & (32)
\end{aligned}$$

Hence, the multi-species analogous of the Proposition 3, is the following

Proposition 4. *Consider the functional*

$$\varphi(t) = \mathbb{E}_0 \log(Z_0(t)), \tag{33}$$

then for its t-derivative the following relation holds

$$\varphi'(t) = \frac{1}{2}(\mathbf{1}, \mathbf{\Delta 1}) - \sum_{l=1}^{K} \Big(\mathbf{u}_l(\Gamma), \mathbf{\Delta 1}\Big) - \frac{1}{2}\sum_{l=0}^{K}(m_{l+1} - m_l)\langle\Big(\mathbf{C}, \mathbf{\Delta C}\Big) - \widehat{B}^l\rangle_{l,t} \tag{34}$$

where $\widehat{B}^0 = 0$ *and* $\widehat{B}^l = \sum_{l'=1}^{l} \Big(\mathbf{u}_l(\Gamma), \mathbf{\Delta}\widetilde{\mathbf{C}}_{l'}\Big)$.

3 The multi-species mean field spin glass

For each $s \in \mathcal{S}$ consider the set $\Lambda_N^{(s)} \subset \mathbb{Z}$ such that

$$\begin{aligned}
\bigcap_{s\in\mathcal{S}} \Lambda_N^{(s)} &= \emptyset, & (35)\\
|\Lambda_N^{(s)}| &= N^{(s)}, & (36)\\
N &= \sum_{s\in\mathcal{S}} N^{(s)}. & (37)
\end{aligned}$$

We consider a disordered model defined by a collection $(\sigma^{(s)})_{s\in\mathcal{S}}$ of Ising variables, meaning that $\sigma_i^{(s)} = \pm 1$ for each $\forall s \in \mathcal{S}, i \in \Lambda_N^{(s)}$.

We denote by Σ_N the family of possible configurations $\sigma = \{\sigma_i^{(s)}\}_{s\in\mathcal{S}, i\in\Lambda_N^{(s)}}$, then we have that $\mid \Sigma_N \mid= 2^N$. In the sequel the following definitions will be used.

1. *Hamiltonian*

For every $N \in \mathbb{N}$, let $\{H_N(\sigma)\}_{\sigma\in\Sigma_N}$ be a family of 2^N gaussian r.v. defined by

$$H_N(\sigma) := -\frac{1}{\sqrt{N}} \sum_{s,p\in\mathcal{S}} \sum_{i\in\Lambda_N^{(s)}} \sum_{j\in\Lambda_N^{(p)}} J_{ij}^{(sp)}\sigma_i^{(s)}\sigma_j^{(p)}, \tag{38}$$

where the J's are *Gaussian i.i.d.* r.v. such that for every s, p, i, j we have that

$$\mathbb{E}(J_{ij}^{(sp)}) = 0, \tag{39}$$

and

$$\mathbb{E}(J_{ij}^{(sp)} J_{i'j'}^{(s'p')}) = \delta_{ss'}\delta_{pp'}\delta_{ii'}\delta_{jj'}\Delta_{sp}^2, \tag{40}$$

with $\Delta_{sp}^2 = \Delta_{ps}^2$.

2. *Covariance matrix*

The covariance matrix of the system is

$$C_N(\sigma, \tau) := \mathbb{E}(H_N(\sigma)H_N(\tau)) \tag{41}$$

and, thanks to (39) and(40), a simple computation shows that

$$C_N(\sigma, \tau) = \frac{1}{N} \sum_{s,p \in \mathcal{S}} \Delta_{sp}^2 \Big(\sum_{i \in \Lambda_N^{(s)}} \sigma_i^{(s)} \tau_i^{(s)} \Big) \Big(\sum_{j \in \Lambda_N^{(p)}} \sigma_j^{(p)} \tau_j^{(p)} \Big). \tag{42}$$

To show explicitly the dependence trough the choice of the various sizes we can define for every $s \in \mathcal{S}$ the relative density

$$\alpha_N^{(s)} := \frac{N^{(s)}}{N}, \tag{43}$$

and the relative overlap

$$q_N^{(s)}(\sigma, \tau) = \frac{1}{N^{(s)}} \sum_{i \in \Lambda_N^{(s)}} \sigma_i^{(s)} \tau_i^{(s)}, \tag{44}$$

then the covariance matrix can be write in the form

$$C_N(\sigma, \tau) = N \sum_{s \in \mathcal{S}} \sum_{p \in \mathcal{S}} \Delta_{sp}^2 \alpha_N^{(s)} \alpha_N^{(p)} q_N^{(s)}(\sigma, \tau) q_N^{(p)}(\sigma, \tau). \tag{45}$$

In the vector notation introduced in Section 2, we rewrite (45) as

$$C_N(\sigma, \tau) = N \Big(\mathbf{q}_N, \mathbf{\Delta} \mathbf{q}_N \Big), \tag{46}$$

where

$$\mathbf{q}_N = \Big(q_N^{(s)}(\sigma, \tau) \Big)_{s \in \mathcal{S}} \tag{47}$$

Example 1. *For example, in the case of two species, namely $\mathcal{S} = \{a, b\}$, the covariance matrix $\mathbf{q}_N$ is a 2-dimensional vector and $\mathbf{\Delta}$ is a 2×2 matrix defined by the entries*

$$\begin{bmatrix} \alpha_N^{(a)} \alpha_N^{(a)} \Delta_{aa}^2 & \alpha_N^{(a)} \alpha_N^{(b)} \Delta_{ab}^2 \\ \alpha_N^{(a)} \alpha_N^{(b)} \Delta_{ab}^2 & \alpha_N^{(b)} \alpha_N^{(b)} \Delta_{bb}^2 \end{bmatrix}$$

3. *Thermodynamic stability and normalized covariance*

We define the normalized covariance by

$$c_N(\sigma,\tau) := \frac{C_N(\sigma,\tau)}{N}. \tag{48}$$

An Hamiltonian is said *thermodynamically stable* if it exists a constant $\bar{c} < \infty$ such that

$$\lim_{N\to\infty} c_N(\sigma,\sigma) \leq \bar{c}. \tag{49}$$

In our case, by (45), we get

$$c_N(\sigma,\sigma) = \sum_{s,p\in\mathcal{S}} \Delta_{sp}^2 \alpha_N^{(s)} \alpha_N^{(p)} q_N^{(s)}(\sigma,\sigma) q_N^{(p)}(\sigma,\sigma) \leq \sum_{s,p\in\mathcal{S}} \alpha_N^{(s)} \alpha_N^{(p)} \Delta_{sp}^2 < \infty$$

and then the Hamiltonian (38) is *thermodynamically stable*.

4. *Random partition function*

The random partition function extends standard partition function for disordered systems and reads off as

$$Z_N := \sum_{\sigma} a_N(\sigma,\mathbf{h}) e^{-H_N(\sigma)}, \tag{50}$$

where

$$a_N(\sigma,\mathbf{h}) := \exp\Big(\sum_{s\in\mathcal{A}} h^{(s)} \sum_{i\in\Lambda_N^{(s)}} \sigma^{(s)}\Big), \tag{51}$$

and $\mathbf{h} := (h^{(s)})_{s\in\mathcal{A}}$ is a vector which represents an external (magnetic in the physical literature) field acting in each party separately.

Notice that, to lighten the notation, in the *l.h.s.* of (50), and in the rest of the paper, we do not write explicitly the dependence on $\mathbf{h}$. With the same aim, the physical inverse temperature β, which appears in the standard definition of the partition function, in our case is set equal to 1 with no loss of generality as it can be recovered in every moment simply by properly rescaling the interactions parameters.

5. *Random pressure*

The random pressure mirrors the classical thermodynamical pressure of statistical mechanics, suitably extended to disordered systems and reads off as

$$P_N := \log Z_N. \tag{52}$$

6. *Quenched pressure density*

The main thermodynamic observable, whose extremization results in the self-consistencies of the theory is the quenched pressure density, defined as

$$p_N := \frac{1}{N}\mathbb{E}P_N = \frac{1}{N}\mathbb{E}\log Z_N \tag{53}$$

7. *Thermodynamic limit*
In order to preserve averages and avoid distributions, we will be interested -whenever possible- to the thermodynamic limit of the quenched pressure density, namely

$$p := \lim_{N\to\infty} p_N \tag{54}$$

8. *Random Gibbs measure and Gibbs state*

For every bounded function $f : \sigma \to \mathbb{R}$ we call the Gibbs state the following r.v.

$$\omega_N(f) := \sum_{\sigma} f(\sigma) G_N(\sigma) \tag{55}$$

where

$$G_N(\sigma) := \frac{a_N(\sigma, \mathbf{h}) e^{-H_N(\sigma)}}{Z_N} \tag{56}$$

is called Gibbs measure.

9. *n-product random Gibbs state*

We consider n copies of the configuration space, denoted by $\sigma^1, \dots, \sigma^n$ and, for every bounded function $f : (\sigma^1, \dots, \sigma^n) \to \mathbb{R}$, we call the random n-Gibbs state the following r.v.

$$\Omega_N(f) := \sum_{\sigma^1,\dots,\sigma^n} f(\sigma^1, \dots, \sigma^n) G_N(\sigma^1) \dots G_N(\sigma^n) \tag{57}$$

10. *Quenched equilibrium state*
Lastly we define

$$\langle f \rangle := \lim_{N\to\infty} \mathbb{E}\Omega_N(f). \tag{58}$$

4 The thermodynamical limit at fixed densities

In this section we prove, under a suitable condition, the existence of the thermodynamical limit for the pressure per particle when the species densities are kept constant (54), i.e. the limit of $N \to \infty$ is defined such that $\forall s \in \mathcal{S}$ the quantity $\alpha_N^{(s)} = \frac{N^{(s)}}{N} = \alpha^{(s)}$ is independent of N. the main result of this section is the following:

Theorem 1. *If the matrix* $\mathbf{\Delta}$ *is positive semi-definite, then*

$$\lim_{N\to\infty} p_N = \sup_N p_N,$$

where the limit is taken at fixed densities.

Notice that, since the relative densities are kept constants, the condition that $\mathbf{\Delta}$ is positive semi-definite is independent of the α's.

Proof of the Theorem. The strategy of the proof follows classical Guerra-Toninelli arguments. Let us consider two non interacting and *i.i.d.* copies of the original system defined by the Hamiltonian (38) of sizes respectively $N_1,\ N_2$. Clearly this implies that we have to consider $\forall s \in \mathcal{S}$ the relative subsets $\Lambda_{N_1}^{(s)}, \Lambda_{N_2}^{(s)}$ defined by the equations (35),(36),(37) and such that

$$\begin{aligned} \Lambda_{N_1}^{(s)} \cup \Lambda_{N_2}^{(s)} &= \Lambda_N^{(s)}, & (59)\\ \Lambda_{N_1}^{(s)} \cap \Lambda_{N_2}^{(s)} &= \emptyset, & (60)\\ |\Lambda_{N_1}^{(s)}| &= N_1^{(s)}, & (61)\\ |\Lambda_{N_2}^{(s)}| &= N_2^{(s)}, & (62)\\ N_1^{(s)} + N_2^{(s)} &= N^{(s)}. & (63) \end{aligned}$$

More explicitly, we can define, $\forall s \in \mathcal{S}$, the following

$$\begin{aligned} \Lambda_N^{(s)} &= \{1, \dots, N^{(s)}\}, & (64)\\ \Lambda_{N_1}^{(s)} &= \{1, \dots, N_1^{(s)}\}, & (65)\\ \Lambda_{N_2}^{(s)} &= \{N_1^{(s)} + 1, \dots, N^{(s)}\}. & (66) \end{aligned}$$

Consider the following interpolating Hamiltonian

$$H_N(\sigma, t) = \sqrt{t} H_N(\sigma) + \sqrt{1-t}\Big(H_{N_1}(\sigma) + H_{N_2}(\sigma)\Big) \tag{67}$$

where

$$H_{N_1}(\sigma) = -\frac{1}{\sqrt{N_1}} \sum_{s,p\in\mathcal{S}} \sum_{i\in\Lambda_{N_1}^{(s)}} \sum_{j\in\Lambda_{N_1}^{(p)}} J_{ij}'^{(sp)} \sigma_i^{(s)} \sigma_j^{(p)}, \tag{68}$$

$$H_{N_2}(\sigma) = -\frac{1}{\sqrt{N_2}} \sum_{s,p\in\mathcal{S}} \sum_{i\in\Lambda_{N_2}^{(s)}} \sum_{j\in\Lambda_{N_2}^{(p)}} J_{ij}''^{(sp)} \sigma_i^{(s)} \sigma_j^{(p)}, \tag{69}$$

and where $J_{ij}'^{(sp)}$ and $J_{ij}''^{(sp)}$ are *i.i.d.* of $J_{ij}^{(sp)}$.
As usual we consider the interpolating pressure

$$P_N(t) = \mathbb{E} \log Z_N(t) = \mathbb{E} \log \sum_{\sigma\in\Sigma_N} a_N(\sigma, \mathbf{h}) e^{-H_N(\sigma,t)}, \tag{70}$$

whose boundaries values are

$$\begin{aligned} P_N(1) &\equiv P_N, & (71)\\ P_N(0) &\equiv P_{N_1} + P_{N_2}, & (72) \end{aligned}$$

since $\Sigma_N = \Sigma_{N_1} \cup \Sigma_{N_2}$ and $\Sigma_{N_1} \cap \Sigma_{N_2} = \emptyset$.

Proposition 5. *The t-derivative of the interpolating pressure is*

$$\frac{\partial}{\partial t}P_N(t) = -\frac{N}{2}\mathbb{E}\Omega_{N,t}\Big(Q_N\Big),$$

where

$$Q_N(\sigma,\tau) := \Big(\mathbf{q}_N, \boldsymbol{\Delta}\mathbf{q}_N\Big) - \frac{N_1}{N}\Big(\mathbf{q}_{N_1}, \boldsymbol{\Delta}\mathbf{q}_{N_1}\Big) - \frac{N_2}{N}\Big(\mathbf{q}_{N_2}, \boldsymbol{\Delta}\mathbf{q}_{N_2}\Big), \tag{73}$$

and the vectors $\mathbf{q}_{N_1}, \mathbf{q}_{N_2}$ are defined as in (47).

Proof of the proposition. The computation of the t-derivative works essentially in the same way exploited in Proposition 1 with the following identifications:

$$i \to \sigma, \quad a_i \to a_N(\sigma, \mathbf{h}), \quad U_i \to H_N(\sigma), \quad \widetilde{U}_i \to H_{N_1}(\sigma) + H_{N_2}(\sigma)$$

The key ingredient is that the diagonal term vanishes by the condition $N = N_1 + N_2$.□

Combining the Fundamental Theorem of Calculus and the previous proposition we have that

$$P_N - P_{N_1} - P_{N_2} = -\frac{N}{2}\int_0^1 dt \mathbb{E}\Omega_{N,t}\Big(Q_N\Big). \tag{74}$$

To finish the proof is sufficient to show that

Proposition 6. *If the matrix $\boldsymbol{\Delta}$ is positive semi-definite, then*

$$Q_N(\sigma,\tau) \le 0 \tag{75}$$

for every σ, τ and N.

Proof of the proposition. First at all, we write some fundamental relations.
By definitions (44), (64), (65), (66) we have that $\forall s \in \mathcal{S}$ the following hold

$$N^{(s)} q_N^{(s)}(\sigma,\tau) = \sum_{i=1}^{N^{(s)}} \sigma_i^{(s)}\tau_i^{(s)} = \sum_{i=1}^{N_1^{(s)}} \sigma_i^{(s)}\tau_i^{(s)} + \sum_{N^{(s)}+1}^{N^{(s)}} \sigma_i^{(s)}\tau_i^{(s)}$$

then

$$q_N^{(s)}(\sigma,\tau) = \frac{N_1^{(s)}}{N^{(s)}} q_{N_1}^{(s)}(\sigma,\tau) + \frac{N_2^{(s)}}{N^{(s)}} q_{N_2}^{(s)}(\sigma,\tau).$$

Now we observe that by (**??**) we have that

$$\frac{N_1^{(s)}}{N^{(s)}} = \frac{N_1^{(s)}}{N_1}\frac{N}{N^{(s)}}\frac{N_1}{N} = \frac{\alpha^{(s)}}{\alpha^{(s)}}\frac{N_1}{N} = \frac{N_1}{N},$$

and in a similar fashion

$$\frac{N_2^{(s)}}{N^{(s)}} = \frac{N_2}{N},$$

then $\forall s \in \mathcal{S}$ the following holds

$$q_N^{(s)}(\sigma,\tau) = \frac{N_1}{N} q_{N_1}^{(s)}(\sigma,\tau) + \frac{N_2}{N} q_{N_2}^{(s)}(\sigma,\tau). \tag{76}$$

In vector notation we can write

$$\mathbf{q}_N = \frac{N_1}{N}\mathbf{q}_{N_1} + \frac{N_2}{N}\mathbf{q}_{N_2}. \tag{77}$$

It's easy to see that if $\mathbf{\Delta}$ is a positive semi-definite, real, symmetric matrix, hence the function

$$\mathbf{x} \to \Big(\mathbf{x}, \mathbf{\Delta x}\Big)$$

defined for $\mathbf{x} \in \mathbb{R}^S$ is convex and the conclusion follows straightforwardly from the relation (77).$\Box$

This proposition, combined with equation (74) gives immediately the superaddivity property of the pressure. As a consequence, since the quenched pressure density is bounded from the annealed one (see section 5), then by Fakete's lemma we get the statement of the Theorem.$\Box$

5 The annealed bound

As a first analysis we can study the annealed approximation for the pressure and investigate in which case it is exact. Using Jensen inequality and the concavity of the function $x \to \log(x)$ we define the annealed approximation as a bound, i.e.

$$p_N = \frac{1}{N}\mathbb{E}\log Z_N \leq \frac{1}{N}\log \mathbb{E}Z_N = p_N^A. \tag{78}$$

We can easily write p_N^A as

$$\begin{aligned} p_N^A &= \frac{1}{N}\log\sum_\sigma \mathbb{E}e^{-H_N(\sigma)} = \frac{1}{N}\log\sum_\sigma e^{\frac{1}{2}C_N(\sigma,\sigma)} = \frac{1}{N}\log\sum_\sigma e^{\frac{N}{2}(\mathbf{1},\mathbf{\Delta 1})} \\ &= \log 2 + \frac{1}{2}(\mathbf{1},\mathbf{\Delta 1}). \end{aligned} \tag{79}$$

We define the ergodic regime as the region of the phase space in which

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N = \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_N = p^A = \log 2 + \frac{1}{2}(\mathbf{1},\mathbf{\Delta 1}). \tag{80}$$

For this purpose we can use the second moment method checking when

$$\frac{\mathbb{E}(Z_N^2)}{\mathbb{E}^2(Z_N)} \leq C < \infty \tag{81}$$

for some constant $C \in \mathbb{R}$, uniformly in N. Since

$$\begin{aligned} \mathbb{E}(Z_N^2) &= \mathbb{E}\sum_{\sigma,\tau} e^{-H_N(\sigma)-H_N(\tau)} = \sum_{\sigma,\tau} e^{\frac{1}{2}\mathbb{E}(H_N(\sigma)+H_N(\tau))^2} \\ &= \sum_{\sigma,\tau} e^{N((\mathbf{1},\mathbf{\Delta 1})+(\mathbf{q}_N,\mathbf{\Delta q}_N))} = \mathbb{E}^2(Z_N)2^{-2N}\sum_{\sigma,\tau} e^{N(\mathbf{q}_N,\mathbf{\Delta q}_N)} \end{aligned} \tag{82}$$

and using the gauge transformation $\tau_i^{(s)} \to \sigma_i^{(s)}\tau_i^{(s)}$,

$$\frac{\mathbb{E}(Z_N^2)}{\mathbb{E}^2(Z_N)} = 2^{-2N}\sum_{\sigma,\tau} e^{N(\mathbf{m}_N(\tau),\mathbf{\Delta m}_N(\tau))} = 2^{-N}\sum_\tau e^{N(\mathbf{m}_N(\tau),\mathbf{\Delta m}_N(\tau))}, \tag{83}$$

where we define $\mathbf{m}_N(\tau) = \left(m_N^{(s)}(\tau)\right)_{s\in\mathcal{S}}$, with $m_N^{(s)}(\tau) = \frac{1}{N^{(s)}}\sum_{i=1}^{N^{(s)}}\tau_i^{(s)}$. If $\det\mathbf{\Delta} > 0$ we can linearize the quadratic form with a gaussian integration

$$\begin{aligned}
\frac{\mathbb{E}(Z_N^2)}{\mathbb{E}^2(Z_N)} &= \frac{2^{-N}}{\sqrt{\det\mathbf{\Delta}}}\int\frac{d\mathbf{z}}{2\pi}e^{-\frac{1}{2}(\mathbf{z},\mathbf{\Delta}^{-1}\mathbf{z})}\sum_{\tau}e^{\sqrt{2N}(\mathbf{m}_N(\tau),\mathbf{z})} \\
&= \frac{1}{\sqrt{\det\mathbf{\Delta}}}\int\frac{d\mathbf{z}}{2\pi}e^{-\frac{1}{2}(\mathbf{z},\mathbf{\Delta}^{-1}\mathbf{z})}\prod_{s\in\mathcal{A}}\cosh^{N^{(s)}}\left(\frac{\sqrt{2N}}{N^{(s)}}z^{(s)}\right) \\
&= \frac{1}{\sqrt{\det(\mathbf{\Delta})}}\int\frac{d\mathbf{z}}{2\pi}e^{-\frac{1}{2}(\mathbf{z},\mathbf{\Delta}^{-1}\mathbf{z})}e^{\sum_{s\in\mathcal{S}}N^{(s)}\log\cosh\left(\frac{\sqrt{2N}}{N^{(s)}}z^{(s)}\right)}
\end{aligned} \tag{84}$$

and, using the inequality $\log\cosh(x) \le \frac{x^2}{2}$, we obtain

$$\frac{\mathbb{E}(Z_N^2)}{\mathbb{E}^2(Z_N)} \le \frac{1}{\sqrt{\det(\mathbf{\Delta})}}\int\frac{d\mathbf{z}}{2\pi}e^{-\frac{1}{2}(\mathbf{z},\hat{\mathbf{\Delta}}\mathbf{z})}, \tag{85}$$

where we have defined

$$\hat{\mathbf{\Delta}} = \mathbf{\Delta}^{-1} - 2\boldsymbol{\alpha}^{-1} \tag{86}$$

and the diagonal matrix $\boldsymbol{\alpha} = \mathrm{diag}(\{\alpha^{(s)}\}_{s\in\mathcal{S}})$. Thus we have just proved the following

Theorem 2. *In the convex region, defined as* $\det\mathbf{\Delta} > 0$, *as soon as* $\hat{\mathbf{\Delta}}$ *is positively defined, the pressure of the model does coincide with the annealed approximation, i.e.*

$$p = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N = \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_N = p^A = \log 2 + \frac{1}{2}(\mathbf{1},\mathbf{\Delta}\mathbf{1}). \tag{87}$$

Remark 1. *Note that such a region does exist and can be viewed as an high temperature region. The two regions* $\det\mathbf{\Delta} > 0$ *and* $\hat{\mathbf{\Delta}} > 0$ *have a non-zero measure intersection, because, while the first is a condition on the relative size of the covariances, the latter is related to their absolute amplitude. Indeed once fixed* $\boldsymbol{\alpha}$ *and* $\mathbf{\Delta}$ *satisfying* $\det\mathbf{\Delta} > 0$, *we can rescale all the covariances with a parameter* β, *which play the role of the inverse temperature of the system, i.e.* $\Delta_{ss'} \to \beta\Delta_{ss'}$, $\forall s, s' \in \mathcal{S}$, *leaving the relative sizes unaltered and the condition* $\det\mathbf{\Delta} > 0$ *is still satisfied, such that* $\hat{\mathbf{\Delta}} \to \beta^{-S}\mathbf{\Delta}^{-1} - 2\boldsymbol{\alpha}^{-1}$ *is positively defined for* β *small enough* [1].

6 The Replica Symmetric bound

We know from the mathematical theory of Sherrington Kirkpatrick model that the whole information about the model is encoded in its covariance matrix. In particular, the study of the replica symmetric solution can be viewed as a comparison between the normalised covariance matrix and a trial parameter. In order to define the replica symmetric solution in the multi-species case, as outlined in Section 2, we can think the overlap as a vector in $\mathbb{R}^S$ and then the normalised covariance matrix can be viewed as quadratic form:

$$c_N = \left(\mathbf{q}_N, \mathbf{\Delta}\mathbf{q}_N\right).$$

[1] Since $\mathbf{\Delta}$ is positively defined then also $\mathbf{\Delta}^{-1}$. Defining $a = \max_s \alpha^{(s)}$ and ρ the smallest eigenvalue of $\mathbf{\Delta}^{-1}$, then, for any non-null vector z, $(z, \hat{\mathbf{\Delta}}z) \ge (\beta^{-S}\rho - a)(z,z) > 0$ if $\beta^S < \rho/a$.

The idea is then to compare the overlap vector with a trial vector,

$$\mathbf{q}_{trial} := \Big(q^{(s)}\Big)_{s\in\mathcal{S}} \in \mathbb{R}^S_+. \tag{88}$$

More explicitly, we want to construct an interpolating Hamiltonian such that the derivative of the interpolating pressure is proportional to

$$\Big((\mathbf{q}_N - \mathbf{q}_{trial}), \mathbf{\Delta}(\mathbf{q}_N - \mathbf{q}_{trial})\Big) = c_N - 2\Big(\mathbf{q}_N, \mathbf{\Delta}\mathbf{q}_{trial}\Big) + \Big(\mathbf{q}_{trial}, \mathbf{\Delta}\mathbf{q}_{trial}\Big). \tag{89}$$

We can define the trial replica symmetric solution as

$$p_{RS}(\mathbf{q}_{trial}) := \log 2 + \sum_{s\in\mathcal{S}} \alpha^{(s)} p^{(s)}(\mathbf{q}_{trial}) + \frac{1}{2}\Big((\mathbf{1} - \mathbf{q}_{trial}), \mathbf{\Delta}(\mathbf{1} - \mathbf{q}_{trial})\Big), \tag{90}$$

where

$$p^{(s)}(\mathbf{q}_{trial}) := \int d\mu(z) \log\cosh\Big(\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\mathbf{q}_{trial}\Big)}z + h^{(s)}\Big), \tag{91}$$

and

$$z \sim \mathcal{N}(0,1).$$

The main result of this section is the following

Theorem 3. *The following sum rule holds*

$$p_N = p_{RS}(\mathbf{q}_{trial}) - \frac{1}{2}\int_0^1 \mathbb{E}\Omega_{N,t}\Big((\mathbf{q}_N - \mathbf{q}_{trial}), \mathbf{\Delta}(\mathbf{q}_N - \mathbf{q}_{trial})\Big). \tag{92}$$

Moreover, if the matrix $\mathbf{\Delta}$ *is positive semi-definite, then the following bound hold*

$$p_N \leq p_{RS}(\mathbf{q}_{trial}), \tag{93}$$

whose optimization gives

$$p_N \leq \inf_{\mathbf{q}_{trial}} p_{RS}(\mathbf{q}_{trial}). \tag{94}$$

Proof of the Theorem. Clearly the only result to be proved is (92), the rest follows straightforwardly and the strategy is to apply the interpolation scheme outlined in Section 2.1. To this task, let us consider the following interpolating Hamiltonian

$$H_N(\sigma, t) := \sqrt{t}H_N(\sigma) + \sqrt{1-t}H_N(\sigma, \mathbf{q}_{trial}) \tag{95}$$

with

$$H_N(\sigma, \mathbf{q}_{trial}) := \sum_{s\in\mathcal{S}} H_N^{(s)}(\sigma^{(s)}, \mathbf{q}_{trial}) \tag{96}$$

where $H_N(\sigma)$ is the Hamiltonian of the model defined in (38) and $H_N^{(s)}(\sigma, \mathbf{q}_{trial})$ are two independent one-body interaction Hamiltonian, defined as

$$H_N^{(s)}(\sigma^{(s)}, \mathbf{q}_{trial}) := -\sqrt{2}\sqrt{\mathcal{P}_s\Big(\mathbf{\Delta}\mathbf{q}_{trial}\Big)}\frac{1}{\sqrt{\alpha^{(s)}}}\sum_{i\in\Lambda_N^{(s)}} J_i^{(s)}\sigma_i^{(s)} \tag{97}$$

where $\mathcal{P}_s$ is the canonical projection operator in $\mathbb{R}^S$ introduced in Section 2 and the J's are *Gaussian i.i.d.* r.v., independent of the other r.v., such that for every s, i we have that

$$\mathbb{E}(J_i^{(s)}) = 0 \tag{98}$$

and

$$\mathbb{E}(J_i^{(s)} J_{i'}^{(s')}) = \delta_{ss'}\delta_{ii'}. \tag{99}$$

After simple computations, we get

$$\mathbb{E}\Big(H_N^{(s)}(\sigma^{(s)}, \mathbf{q}_{trial}) H_N^{(s')}(\tau^{(s')}, \mathbf{q}_{trial})\Big) = 2N\delta_{ss'}\mathcal{P}_s\Big(\mathbf{\Delta q}_{trial}\Big)\mathcal{P}_s\Big(\mathbf{q}_N\Big),$$

and then, by (13), the covariance matrix of the trial Hamiltonian becomes

$$\mathbb{E}\Big(H_N(\sigma, \mathbf{q}_{trial}) H_N(\tau, \mathbf{q}_{trial})\Big) = 2N \sum_{s\in\mathcal{S}} \mathcal{P}_s\Big(\mathbf{\Delta q}_{trial}\Big)\mathcal{P}_s\Big(\mathbf{q}_N\Big) = 2N\Big(\mathbf{q}_N, \mathbf{\Delta q}_{trial}\Big).$$

Lastly we define the interpolating pressure as

$$p_N(t) := \frac{1}{N}\mathbb{E}\log\sum_\sigma a_N(\sigma, \mathbf{h})e^{-H_N(\sigma,t)}. \tag{100}$$

Proposition 7. *The boundary values of $p_N(t)$ are*

$$\begin{aligned} p_N(1) &= p_N, & (101)\\ p_N(0) &= \log 2 + \sum_{s\in\mathcal{S}} \alpha^{(s)} p^{(s)}(\mathbf{q}_{trial}), & (102)\end{aligned}$$

Proof of the proposition. The boundary value at $t = 1$ it's obvious, on other hand at $t = 0$ the sum over the σ's and the disorder average factorize in the space of species, then we can compute separately each contribution to the interpolating pressure.□

The key result is the following

Proposition 8. *The t-derivative of $p_N(t)$ is*

$$\frac{\partial}{\partial t} p_N(t) = \frac{1}{2}\Big((\mathbf{1} - 2\mathbf{q}_{trial}), \mathbf{\Delta 1}\Big) - \frac{1}{2}\mathbb{E}\Omega_{N,t}\Big(c_N - 2\Big(\mathbf{q}_N, \mathbf{\Delta q}_{trial}\Big)\Big). \tag{103}$$

Proof of the proposition. The proof is a simple application of Proposition 11, with the identifications $i \to \sigma$, $a_i \to a_N(\sigma, \mathbf{h})$, $U_i \to H_N(\sigma)$, $\widetilde{U}_i \to H_N(\sigma, \mathbf{q}_{trial})$.
The conclusion is a straightforward computation.□

Finally, combining Proposition 7 and 8 and keeping in mind (89) we complete the proof of the Theorem.□
The optimization of (94) on $\mathbf{q}_{trial}$, gives a system of S coupled self consistent equations, i.e. $\forall p \in \mathcal{S}$

$$\sum_{s\in\mathcal{S}} \mathbf{\Delta}_{ps}\left[\int d\mu(z)\tanh^2\Big(\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s(\mathbf{\Delta q}_{trial})}z\Big) - q^{(s)}\right] = 0, \tag{104}$$

This system admits a unique solution as soon as $\det(\mathbf{\Delta}) \neq 0$, thus whenever $\det(\mathbf{\Delta}) > 0$, $p_{RS}(\mathbf{q}_{trial})$ has a minimum in $\mathbf{q}_{trial} = \bar{\mathbf{q}}$ satisfying $\forall s \in \mathcal{S}$

$$\bar{q}^{(s)} = \int d\mu(z) \tanh^2 \Big(\sqrt{\frac{2}{\alpha^{(s)}} \mathcal{P}_s(\mathbf{\Delta}\bar{\mathbf{q}})} z \Big) = \langle \mathcal{P}_s(\mathbf{q}) \rangle_{t=0} \tag{105}$$

The last equalities can be easily checked thanks to the factorizability of the one-body problem at $t = 0$. In other words, the value of $\mathbf{q}_{trial}$ minimizing the overlap' s fluctuations of the original model (at $t = 1$) is just the overlap's mean of the interpolating one-body trial at $t = 0$.

Let show now how the replica symmetric bound violate the entropy positivity at low temperatures. Mirroring the historical scenario for mono-partite spin-glasses, we can easily check that the replica symmetric expression for the pressure (94) is not the exact solution of the model in the low temperature region by studying the behavior of the entropy. We can define it as the *non-negative* quantity

$$s(\mathbf{\Delta}) = \lim_{N\to\infty} s_N(\mathbf{\Delta}) = -\frac{1}{N} \mathbb{E} \sum_{\sigma} G_N(\sigma, \mathbf{\Delta}) \log(G_N(\sigma, \mathbf{\Delta})), \tag{106}$$

where $G_N(\sigma, \mathbf{\Delta}) = Z_N^{-1}(\mathbf{\Delta}) e^{-H_N(\sigma,\mathbf{\Delta})}$ is the Boltzmann measure. Notice that, unlike before, we have write explicitly the dependance on the matrix $\mathbf{\Delta}$. Since $s_N(\mathbf{\Delta}) = p_N(\mathbf{\Delta}) - \frac{1}{N}\langle H(\sigma)\rangle_N$, we can write

$$s(\mathbf{\Delta}) = p(\mathbf{\Delta}) - \frac{d}{d\lambda} p(\lambda\mathbf{\Delta})|_{\lambda=1}. \tag{107}$$

Now we can define $s_{RS}(\mathbf{\Delta}) = p_{RS} - \frac{d}{d\lambda} p_{RS}(\lambda\mathbf{\Delta})|_{\lambda=1}$. We can easily show that if the amplitude of the covariances is large enough, $s_{RS}(\mathbf{\Delta})$ is strictly negative. Indeed, we have the following

Proposition 9. *In the regime of large covariances (low temperatures), the RS-entropy is strictly negative, i.e.*

$$\lim_{\beta\to+\infty} s_{RS}(\beta\mathbf{\Delta}) < 0,$$

for any choice of $\mathbf{\Delta}$ *with* $(\det(\mathbf{\Delta}) > 0)$ *and* $\boldsymbol{\alpha}$, *where* $\beta \in \mathbb{R}^+$ *play the role of the inverse temperature.*

Proof of the proposition: Using its definition

$$s_{RS}(\beta\mathbf{\Delta}) = p_{RS}(\beta\mathbf{\Delta}, \bar{\mathbf{q}}) - \frac{\partial}{\partial\lambda} p_{RS}(\lambda\beta\mathbf{\Delta}, \bar{\mathbf{q}})|_{\lambda=1}.$$

We note that, using (105), in the limit $\beta \to +\infty$, the optimized order parameters $\bar{\mathbf{q}} \to \mathbf{1}$. Explicating the derivative it is easy to see that

$$\lim_{\beta\to+\infty} s_{RS}(\beta\mathbf{\Delta}) = \lim_{\beta\to+\infty} -\frac{\beta^2}{2} \left((\mathbf{1} - \bar{\mathbf{q}}), \mathbf{\Delta}(\mathbf{1} - \bar{\mathbf{q}})\right) \leq 0$$

Finally we can state that the limit is strictly negative, using again (105) and noting that

$$\begin{aligned}\lim_{\beta\to+\infty}\beta(1-\bar{q}_{(s)}) &= \lim_{\beta\to+\infty}\beta\int d\mu(z)\Big(1-\tanh^2\Big(\beta\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\bar{\mathbf{q}}\Big)}z\Big)\Big)\\ &= \lim_{\beta\to+\infty}\frac{1}{\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\bar{\mathbf{q}}\Big)}}\int d\mu(z)z\tanh\Big(\beta\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\bar{\mathbf{q}}\Big)}z\Big)\\ &= \frac{\int d\mu(z)|z|}{\sqrt{\frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\mathbf{1}\Big)}}>0\quad\Box.\end{aligned}$$

The existence of a negative RS-entropy regime is a clear signal that the model is not always replica symmetric (certainly it is RS inside the annealed region defined in Theorem 2) but there exists a region in which the pressure $p(\mathbf{\Delta})$ is strictly lower than its RS bound $p_{RS}(\mathbf{\Delta})$.

7 The Broken Replica Symmetry bound

In the mono-partite spin-glass model, as explained for instance in [12], the RSB interpolation is defined trough a non-decreasing, piecewise constant function $x(q):[0,1]\longrightarrow[0,1]$ which represents the order parameter of the model. A smart use of Proposition 3, lead to the proof that the Parisi's solution is an upper bound for pressure of the Sherrington-Kirkpatrick model. One of the key points of the proof of this remarkable achievement, is that the function $x(q)$ intrinsically defines a non decreasing sequence $(m_l)_{l=0,\dots,K+1}$, thus enabling the control of the sign of the *r.h.s.* of (26). In the multipartite case, keeping in mind Proposition 4, the core idea is to define the order parameter as a piecewise constant, right continuous, function

$$x(\mathbf{u}):[0,1]^S\to[0,1], \tag{108}$$

such that the corresponding sequence $(m_l)_{l=1,\dots,K}$ is non decreasing.
The explicit construction of the order parameter is the following. We choose the sequence Γ, which leads to Proposition 4, as

$$\Gamma=(\mathbf{q}_l)_{l=0,\dots,K}:=\Big(q_l^{(s)}\Big)_{s\in\mathcal{S},l=0,\dots,K}\in\mathbb{R}_+^S \tag{109}$$

such that for each $s\in\mathcal{S}$, we have

$$0=q_0^{(s)}\le q_1^{(s)}\le\cdots\le q_{K-1}^{(s)}\le q_K^{(s)}=1.$$

Roughly speaking Γ defines a path with K steps in $[0,1]^S$ which is non decreasing in each direction.
Clearly we take $(m_0,\dots,m_{K+1})$ such that $0=m_0\le m_1\le\cdots\le m_K\le m_{K+1}=1$.
If we denote by $\theta(\cdot)$ the right continuous Heaviside function, we define the functional order parameter as

$$x(\mathbf{u}):=\sum_{l=0}^{K}(m_{l+1}-m_l)\prod_{s\in\mathcal{S}}\theta(u^{(s)}-q_l^{(s)}) \tag{110}$$

where $\mathbf{u} = (u^{(s)})_{s\in\mathcal{S}}$ is vector in $[0,1]^S$.
The function x defines an S-dimensional shape, that in the case of $S=2$, looks like a *ziggurat*[2]: this can be easily understood trough the following relations, for each $p \in \mathcal{S}$ we have that:

$$x(\mathbf{u})\Big|_{u^{(p)}=q_{l'}^{(p)}} = \sum_{l=0}^{l'}(m_{l+1}-m_l)\prod_{s\neq p}\theta(u^{(s)}-q_l^{(s)}).$$

In particular we can compute, for each $p \in \mathcal{S}$, the marginal values of x

$$x(u^{(p)}) := x(\mathbf{u})\Big|_{u^{(s)}=1, s\neq p} = \sum_{l=0}^{K}(m_{l+1}-m_l)\theta(u^{(p)}-q_l^{(p)}).$$

In the RS solution, as explained in Section 4, a crucial role is played by the action of the matrix $\mathbf{\Delta}$ on the trial vector parameter. For this reason, it is useful to introduce, for each $s \in \mathcal{S}$ and $l = 0,\dots,K$, the following quantity:

$$Q_l^{(s)} = \frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}\mathbf{q}_l\Big) \tag{111}$$

which is a non decreasing sequences in l, in other words for each $l \geq l'$ we have that

$$Q_l^{(s)} - Q_{l'}^{(s)} = \frac{2}{\alpha^{(s)}}\mathcal{P}_s\Big(\mathbf{\Delta}(\mathbf{q}_l-\mathbf{q}_{l'})\Big) \geq 0. \tag{112}$$

To complete the picture we need to introduce a transformed order parameter, roughly speaking the transformation (111) on the order parameter (110), namely

$$x_\Delta(\mathbf{u}) := \sum_{l=0}^{K}(m_{l+1}-m_l)\prod_{s\in\mathcal{S}}\theta(u^{(s)}-Q_l^{(s)}) \tag{113}$$

defined for $\mathbf{u} \in \times_{s\in\mathcal{S}}[0, Q_K^{(s)}]$.

We define the trial RSB pressure as

$$p_{RSB}(x) := \log 2 + \sum_{s\in\mathcal{S}}\alpha(s)f^{(s)}(0,h^{(s)}) - \frac{1}{2}\int_{\widetilde{\Gamma}} x(\mathbf{u}) \;\; \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta}\mathbf{u}\Big)\cdot d\mathbf{u} \tag{114}$$

where, for each $s \in \mathcal{S}$, $f^{(s)}(u^{(s)},y)$ is the solution of the following Parisi's PDE

$$\frac{\partial f^{(s)}}{\partial u^{(s)}} + \frac{1}{2}\frac{\partial^2 f^{(s)}}{\partial y^2} + \frac{1}{2}x_{\mathbf{\Delta}}(u^{(s)})\Big(\frac{\partial f^{(s)}}{\partial y}\Big)^2 = 0 \tag{115}$$

where $x_{\mathbf{\Delta}}(u^{(s)})$ is the marginal value of the transformed order parameter and the boundary condition is

$$f^{(s)}(Q_K^{(s)},y) = \log\cosh(y). \tag{116}$$

The integral in (114) is a line integral on an arbitrary path $\widetilde{\Gamma}$ in the plan $\mathbf{u}$, starting from $\mathbf{0}$ and ending in $\mathbf{1}$, such that all the points $(\mathbf{q}_l)_{l=0,\dots,K}$ belong to Γ, in other words $\Gamma \in \widetilde{\Gamma}$.

The main result of this section is the following

[2]Ziggurats are pyramid-like structures found in the ancient Mesopotamian valley and western Iranian plateau.

Theorem 4. *The following sum rule holds*

$$p_N = p_{RSB}(x) - \frac{1}{2}\sum_{l=0}^{K}(m_{l+1} - m_l)\int_0^1 dt\Big\langle(\mathbf{q}_N - \mathbf{q}_l), \boldsymbol{\Delta}(\mathbf{q}_N - \mathbf{q}_l)\Big\rangle_{N,l,t}. \tag{117}$$

Moreover if the matrix $\boldsymbol{\Delta}$ *is positive semi-definite we have the following bound*

$$p_N \le p_{RSB}(x),$$

and the optimization gives

$$p_N \le \inf_x p_{RSB}(x).$$

Proof of the Theorem. It is enough to show that (117) holds, then we have straightforward conclusions. The strategy is to apply the RSB interpolation scheme introduced in Section 2. We define the interpolating Hamiltonian as

$$H_N(\sigma, t) := \sqrt{t}H_N(\sigma) + \sqrt{1-t}\sum_{l=1}^{K} H_N^l(\sigma, \mathbf{q}_l), \tag{118}$$

with

$$H_N^l(\sigma, \mathbf{q}_l) := \sum_{s\in\mathcal{S}} H_N^{l,(s)}(\sigma^{(s)}, \mathbf{q}_l), \tag{119}$$

where $H_N(\sigma)$ is the original Hamiltonian and, for each l, $H_N^{l,(s)}(\sigma^{(s)}, \mathbf{q}_l)$ are two independent one-body interaction Hamiltonian, defined as

$$H_N^{l,(s)}(\sigma^{(s)}, \mathbf{q}_l) := -\sqrt{2}\sqrt{\mathcal{P}_s\Big(\boldsymbol{\Delta}(\mathbf{q}_l - \mathbf{q}_{l-1})\Big)}\frac{1}{\sqrt{\alpha^{(s)}}}\sum_{i\in\Lambda_N^{(s)}} J_i^{l,(s)}\sigma_i^{(s)} \tag{120}$$

where the J's are *Gaussian i.i.d.* r.v., independent of the other r.v., such that for every l, s and i we have that

$$\mathbb{E}(J_i^{l,(s)}) = 0 \tag{121}$$

and

$$\mathbb{E}(J_i^{l,(s)}J_{i'}^{l',(s')}) = \delta_{ll'}\delta_{ss'}\delta_{ii'}. \tag{122}$$

After simple computations, we get

$$\mathbb{E}\Big(H_N^{l,(s)}(\sigma^{(s)}, \mathbf{q}_l)H_N^{l',(s')}(\tau^{(s')}, \mathbf{q}_l)\Big) = \delta_{ll'}\delta_{ss'}2N\mathcal{P}_s\Big(\boldsymbol{\Delta}(\mathbf{q}_l - \mathbf{q}_{l-1})\Big)\mathcal{P}_s\Big(\mathbf{q}_N\Big)$$

and then by (13) the covariance matrix of the trial Hamiltonian becomes

$$\mathbb{E}\Big(H_N^l(\sigma, \mathbf{q}_l)H_N^{l'}(\tau, \mathbf{q}_{l'})\Big) = \delta_{ll'}2N\sum_{s\in\mathcal{S}}\mathcal{P}_s\Big(\boldsymbol{\Delta}(\mathbf{q}_l - \mathbf{q}_{l-1})\Big)\mathcal{P}_s\Big(\mathbf{q}_N\Big) = \delta_{ll'}2N\Big((\mathbf{q}_l - \mathbf{q}_{l-1}), \boldsymbol{\Delta}\mathbf{q}_N\Big).$$

We can introduce the RSB interpolation scheme with the following identifications $i \to \sigma$, $a_i \to a_N(\sigma, \mathbf{h})$, $U_i \to H_N(\sigma)$, $B_i^{l,(s)} \to H_N^{l,(s)}(\sigma^{(s)}, \mathbf{q}_l)$,
in order to define the interpolating pressure as in Proposition 4

$$p_N(t) := \frac{1}{N}\mathbb{E}\log Z_{0,N}(t). \tag{123}$$

Then we have the following

Proposition 10. *The boundary values of $p_N(t)$ are*

$$
\begin{aligned}
p_N(1) &= p_N, &(124)\\
p_N(0) &= \log 2+\sum_{s\in\mathcal{S}}\alpha^{(s)}f^{(s)}(0,h^{(s)}), &(125)
\end{aligned}
$$

where $f^{(s)}(u^{(s)},h^{(s)})$ is the solution of the Parisi's PDE (115).

Proof of the proposition. The boundary value at $t=1$ is obvious; on other hand at $t=0$ the sum over the σ's and the disorder average factorize over the species and we can compute separately each contribution to the interpolating pressure, obtaining

$$p_N(0)=\sum_{s\in\mathcal{A}}p_0^{(s)}(\mathbf{q}_l)$$

where $p_0^{(s)}(\mathbf{q}_l)$ are defined recursively as in Proposition 4 with

$$p_K^{(s)}(\mathbf{q}_l)=2^{\alpha^{(s)}}\cosh^{\alpha^{(s)}}\Big[\sum_{l=1}^{K}\frac{\sqrt{2}}{\sqrt{\alpha^{(s)}}}\sqrt{\mathcal{P}_s\Big(\mathbf{\Delta}(\mathbf{q}_l-\mathbf{q}_{l-1})\Big)}z_l+h^{(s)}\Big]$$

and for each l

$$z_l\sim\mathcal{N}(0,1).$$

It is immediate to check that $p_0^{(s)}(\mathbf{q}_l)=\alpha^{(s)}(\log 2+f^{(s)}(0,h^{(s)})$ where $f^{(s)}(u^{(s)},h^{(s)})$ is defined in(115). $\Box$

In order to apply the interpolation argument we have to compute the t-derivative of the interpolating pressure, which leads to the following

Proposition 11. *The t-derivative of $p_N(t)$ is*

$$\frac{\partial}{\partial t}p_N(t)=-\frac{1}{2}\Big(\mathbf{1},\mathbf{\Delta 1}\Big)-\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\Big\langle\Big(c_N-2\Big(\mathbf{q}_N,\Delta\mathbf{q}_l\Big)\Big)\Big\rangle_{N,l,t}. \tag{126}$$

Proof of the proposition. The proof is a simple application of Proposition 4, observing that

$$\sum_{l=1}^{K}\Big((\mathbf{q}_l-\mathbf{q}_{l-1}),\mathbf{\Delta 1}\Big)=\Big(\mathbf{1},\mathbf{\Delta 1}\Big),$$

and

$$\sum_{l'=0}^{l}\Big(\mathbf{q}_N,\mathbf{\Delta}(\mathbf{q}_{l'}-\mathbf{q}_{l'-1})\Big)=\Big(\mathbf{q}_N,\mathbf{\Delta q}_l\Big),$$

and then the conclusion is straightforward.$\Box$

In order to complete the proof of the Theorem, we still need the following equivalence

Proposition 12. *The following representation holds*

$$-\frac{1}{2}\Big(\mathbf{1},\mathbf{\Delta 1}\Big)+\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\Big(\mathbf{q}_l,\mathbf{\Delta q}_l\Big)=-\frac{1}{2}\int_{\widetilde{\Gamma}}d\mathbf{u}\;\; x(\mathbf{u})\;\;\nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big)\cdot d\mathbf{u}.$$

Proof of the proposition. We can use the explicit definition of $x(\mathbf{u})$ given in (110) to compute

$$-\frac{1}{2}\int_{\widetilde{\Gamma}} x(\mathbf{u}) \ \ \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big)\cdot d\mathbf{u} = -\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\int_{\widetilde{\Gamma}}\prod_{s\in\mathcal{S}}\theta(u^{(s)}-q_l^{(s)}) \ \ \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big)\cdot d\mathbf{u} =$$

$$= -\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\int_{\Gamma_l} \ \ \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big)\cdot d\mathbf{u}$$

where Γ_l is the result of the action of the θ's on the path $\widetilde{\Gamma}$, that is his component between the points $\mathbf{q}_l$ and $\mathbf{1}$. By the Gradient's Theorem, the result of the integral is path independent and is equal to the increment of the potential function, then finally we obtain

$$-\frac{1}{2}\int_{\widetilde{\Gamma}} d\mathbf{u} \ \ x(\mathbf{u}) \ \ \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big) = -\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\Big[\Big(\mathbf{1},\mathbf{\Delta 1}\Big)-\Big(\mathbf{q}_l,\mathbf{\Delta q}_l\Big)\Big] =$$

$$= -\frac{1}{2}\Big(\mathbf{1},\mathbf{\Delta 1}\Big)(m_{K+1}-m_0)+\frac{1}{2}\sum_{l=0}^{K}(m_{l+1}-m_l)\Big(\mathbf{q}_l,\mathbf{\Delta q}_l\Big)$$

that is the desired result. □
Finally, combining Proposition 10, 11 and 12 we obtain the proof of the Theorem.□

The RSB bound achieved by Theorem 4 and the corresponding *ziggurat*-like order parameter, includes the RS bound (Theorem 3) in the case of $K = 2$, $m_1 = 0, m_2 = 1$ and $\mathbf{q}_1 = \mathbf{q}_{trial}$ and improves it.
However it is quite natural to ask if the $ziggurat - ansatz$ we introduced contains the exact solution of the model in each point $(\mathbf{\Delta},\boldsymbol{\alpha})$ when Δ is positive semidefinite. This is an hard question on which we are not able to give a complete proof, but we can at least show that the $ziggurat - ansatz$ is able to reproduce limits under control. Let us consider for simplicity the case of two non interacting species that corresponds to a system composed of two independent SK models. From the results by Guerra and Talagrand [12, 17] we know that, since the species are decoupled, the pressure is given by the sum of two independent Parisi's solutions. We can prove (see last section), that the previous solution is contained in the *ziggurat*-RSB variational principle, and this is our claim of consistency.

We stress however that, while we recovered through our *ansatz* the known limit of decoupled spin-glasses -which is somehow trivial (at least physically)- the ziggurat prescription is able to tackle the complexity of their mutual interaction, if the latter does not exceed a threshold where the strength of the off-diagonal terms prevails on the inter-party interactions. The latter is intrinsically different because the model approaches the Hopfield model for neural network [6], on which we plan to report soon.

8 The decoupled RSB Ziggurat ansatz

In this section we show that the $ziggurat - ansatz$ contain the solution of the case of two species $\mathcal{S} = \{a, b\}$ with $\Delta_{ab} = 0$, that corresponds to a system composed of two independent SK models, with sizes $\alpha^{(s)}N$ and at inverse temperatures $\beta_s = \sqrt{2\alpha^{(s)}}\Delta_{ss}$, with $s \in \{a, b\}$. We know that the pressure of the model is given by

$$p(\mathbf{\Delta}, \boldsymbol{\alpha}) = \sum_{s\in\mathcal{S}} \alpha^{(s)} p^{SK}(\beta_s), \tag{127}$$

where $p^{SK}(\beta) = \inf_{x(q)} p^{Parisi}(\beta; x(q))$ is the Parisi full-RSB solution of the Sherrington-Kirkpatrick model with

$$p^{Parisi}(\beta; x(q)) = \log 2 + f(0, 0; \beta, x(q)) - \frac{\beta^2}{2}\int_0^1 dq\; q x(q) \tag{128}$$

and $f(q, y; \beta, x(q))$ satisfying the Parisi equation

$$\partial_q f + \frac{1}{2}\partial_y^2 f + \frac{1}{2}x(q)(\partial_y f)^2 = 0, \tag{129}$$

with the boundary condition $f(1, y) = \log\cosh(\beta y)$.
A basic observation is the following.
Consider for each $s \in \{a, b\}$ an integer $K^{(s)}$ and two non decreasing sequences

$$m^{(s)} := (m_l^{(s)})_{l=0,\ldots,K^{(s)}+1}$$

with $m_0^{(s)} = 0$ and $m_{K^{(s)}+1}^{(s)} = 1$, and

$$p^{(s)} := (p_l^{(s)})_{l=0,\ldots,K^{(s)}}$$

with $p_0^{(s)} = 0$ and $p_K^{(s)} = 1$. Then the following proposition holds

Proposition 13. *Given, for each $s \in \{a, b\}$, the above sequences $m^{(s)}$ and $p^{(s)}$, there exists a suitable integer K, such that we can always construct a non decreasing sequence*

$$(m_l)_{l=0,\ldots,K+1}$$

with $m_0 = 0, m_{K+1} = 1$ and a sequence of points

$$\Gamma = (\mathbf{q}_l)_{l=0,\ldots,K} := \left(q_l^{(a)}, q_l^{(b)}\right)_{l=0,\ldots,K} \in \mathbb{R}_+^2$$

such that for each $s \in \{a, b\}$, we have

$$0 = q_0^{(s)} \leq q_1^{(s)} \leq \cdots \leq q_{K-1}^{(s)} \leq q_K^{(s)} = 1,$$

and if

$$x(\mathbf{u}) = \sum_{l=0}^{K}(m_{l+1} - m_l)\theta(u^{(a)} - q_l^{(a)})\theta(u^{(b)} - q_l^{(b)}), \tag{130}$$

then the marginals are

$$x(u^{(s)}) = \sum_{l=0}^{K^{(s)}}(m_{l+1}^{(s)} - m_l^{(s)})\theta(u^{(s)} - p_l^{(s)}).$$

Proof of the Proposition: The proof works by explicit construction. We can assume without loss of generality that the sequences of $m_l^{(s)}$ and m_l are strictly increasing.
We define the following sets

$$M = \{m_1^{(a)}, m_1^{(b)}, \dots, m_{K^{(a)}}^{(a)}, m_{K^{(b)}}^{(b)}\},$$

and

$$D := \{x, y \in M : x = y\},$$

and the integer

$$K_D := \frac{1}{2}|D|,$$

namely the number of equal elements in M.
Setting $K = K^{(a)} + K^{(b)} - K_D$, we define recursively the following strictly increasing sequence

$$\begin{aligned} m_{K+1} &:= 1, && (131)\\ m_l &:= \max\Big(M \backslash \bigcup_{l'=l+1,\dots,K} \{m_{l'}\}\Big), && (132)\\ m_0 &:= \max\Big(\emptyset\Big) := 0. && (133) \end{aligned}$$

We define

$$D^{(s)} := \{l : \exists! \;\; l' : m_{l'}^{(s)} = m_l\}$$

the uniqueness follows from the hypothesis of strictly increasing sequence. Clearly we have that

$$\{1, \dots, K\} = \bigcup_{s \in \{a,b\}} D^{(s)}$$

and $|D^{(s)}| = K^{(s)}$.
We denote by

$$\mathbf{q}_l := \Big(q_l^{(s)}\Big)_{s \in \{a,b\}},$$

and we set $\mathbf{q}_K = (1,1) = \mathbf{1}$, $\mathbf{q}_0 = (0,0) = \mathbf{0}$ and for each $l \in \{1, \dots K\}$ and $s \in \{a,b\}$ we define

$$q_l^{(s)} = \begin{cases} q_{l-1}^{(s)} & l \notin D^{(s)} \\ p_{l'}^{(s)} & l \in D^{(s)} \end{cases}$$

It is easy to see by simple inspection starting from $q_0^{(s)} = 0$ that the above sequence is not decreasing:
Let us fix $s \in \{a,b\}$, then by definition we have that

$$x(u^{(s)}) = m_{K+1}\theta(u^{(s)} - q_K^{(s)}) - \sum_{l=1}^{K} m_l\Big(\theta(u^{(s)} - q_l^{(s)}) - \theta(u^{(s)} - q_{l-1}^{(s)})\Big).$$

Let us fix an $l_0 \notin D^{(s)}$ then by definition

$$q_{l_0}^{(s)} = q_{l_0-1}^{(s)}.$$

Then

$$x(u^{(s)}) = m_{K+1}\theta(u^{(s)} - q_K^{(s)}) - \sum_{l=1, l\neq l_0}^{K} m_l \Big(\theta(u^{(s)} - q_l^{(s)}) - \theta(u^{(s)} - q_{(l-1)_0}^{(s)})\Big),$$

where $(l-1)_0$ denote previous element of l excluding l_0. After a repeated application of the previous argument we get

$$x(u^{(s)}) = m_{K+1}\theta(u^{(s)} - q_K^{(s)}) - \sum_{l\in D^{(s)}}^{K} m_l \Big(\theta(u^{(s)} - q_l^{(s)}) - \theta(u^{(s)} - q_{(l-1)_D}^{(s)})\Big),$$

where $(l-1)_D$ denote the previous element of l in $D^{(s)}$.
Clearly we have that if $l \in D^{(s)}$, then by definition

$$q_l^{(s)} = p_{l'}^{(s)}, \quad q_{(l-1)_D}^{(s)} = p_{l'-1}^{(s)}, \quad m_l = m_{l'}^{(s)},$$

hence

$$x(u^{(s)}) = m_{K+1}\theta(u^{(s)} - q_K^{(s)}) - \sum_{l'=1}^{K^{(a)}} m_{l'}^{(s)} \Big(\theta(u^{(s)} - p_{l'}^{(s)}) - \theta(u^{(s)} - p_{l'-1}^{(s)})\Big).$$

Since $m_{K+1}^{(s)} = m_{K+1}$, $m_{K+1}^{(s)} = m_{K+1}$, $q_0^{(s)} = 0\, q_K^{(s)} = 1$ then we get the thesis $\square$.
The meaning of the previous Proposition is that the set of all possible *ziggurat* order parameters is surely able to reproduce the situation of two arbitrary decoupled one dimensional order parameters which is precisely the case of $\Delta_{ab} = 0$. However the fact that we can recover two decoupled Parisi's solutions is not immediate. We show briefly how this happens. Consider for example the species a, in the case $\Delta_{ab} = 0$, the relation (111) becomes

$$Q_l^{(a)} = \beta_a^2 q_l^{(a)}$$

The Parisi's PDE for the party a is

$$\frac{\partial f^{(a)}}{\partial u^{(a)}} + \frac{1}{2}\frac{\partial^2 f^{(a)}}{\partial y^2} + \frac{1}{2} x_{\mathbf{\Delta}}(u^{(a)})\Big(\frac{\partial f^{(a)}}{\partial y}\Big)^2 = 0 \tag{134}$$

with boundary condition

$$f^{(s)}(Q_K^{(s)}, y) = \log\cosh(y) \tag{135}$$

and can be rewritten in the standard Parisi's form by setting

$$u^{(a)} = \beta_a^2 q \quad y = \beta_a y'$$

and then by Proposition 13 we know that there exists a choice of the *ziggurat* such that

$$x_{\mathbf{\Delta}}(u^{(a)}) = x_a(q) = \sum_{l=0}^{K^{(a)}} (m_{l+1}^{(a)} - m_l^{(a)})\theta(q - p_l^{(a)}).$$

The same argument holds for the partite b, hence we have proved that the *ziggurat ansatz* can describe two decoupled solutions of the Parisis's PDE. It remains two show that also the integral

over the path is decoupled in the same way.
For this purpose we observe that, in the case of $\Delta_{ab}=0$, we have

$$-\frac{1}{2}\int_{\widetilde{\Gamma}} d\mathbf{u}\ \ x(\mathbf{u})\ \ \nabla_{\mathbf{u}}\Big(\mathbf{u},\mathbf{\Delta u}\Big)=\sum_{s\in\mathcal{A}}\Big[-\frac{\alpha^{(s)}\beta_s^2}{4}(m_{K+1}-m_0)+\frac{\alpha^{(s)}\beta_s^2}{4}\sum_{l=0}^{K}(m_{l+1}-m_l)(q_l^{(s)})^2\Big]$$

Consider for example the party a, then the following holds

$$\sum_{l=0}^{K}(m_{l+1}-m_l)(q_l^{(a)})^2=m_{K+1}-\sum_{l=1}^{K}m_l\Big[(q_l^{(a)})^2-(q_{l-1}^{(a)})^2\Big]$$

The *r.h.s.* of the previous relation can be rewritten, by the same argument which leads to the proof of Proposition 13, as

$$\sum_{l=0}^{K^{(a)}}(m_{l+1}^{(a)}-m_l^{(a)})(p_l^{(a)})^2$$

then

$$-\frac{\alpha^{(a)}\beta_a^2}{4}\Big[(m_{K+1}-m_0)-\sum_{l=0}^{K}(m_{l+1}-m_l)(q_l^{(a)})^2\Big]=-\frac{\alpha^{(a)}\beta_a^2}{2}\int_0^1 dq\ q\ \ x_a(q).$$

Clearly the same argument hold for the species b.

Acknowledgements

Authors are grateful to MiUR trough the FIRB grants number $RBFR08EKEV$ and $RBFR10N90W$ and PRIN grant number $2010HXAW77$ and to Sapienza University of Rome and Alma Mater Studiorum, Bologna University. AB is partially funded by GNFM (Gruppo Nazionale per la Fisica Matematica) which is also acknowledged.

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