An analyticity bound for two-dimensional Ising Model at low temperature.

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Abstract

We study the coexistence phase in two-dimensional Ising model. Optimizing the cluster expansion technique we are able to prove the phase separation phenomenon, with the Onsager value for the surface tension, in a range $\beta > \overline{\beta}$ where $\overline{\beta}$ estimates from above the critical β to within 19% and essentially coincides with the entropic bound.

Key words: cluster expansion, polymer systems.

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1. Introduction. Motivations and results.

The knowledge of the two-dimensional Ising model can be considered one of the most satisfactory in the entire field of statistical mechanics. Nevertheless there are some natural questions not yet solved. For instance there is not any proof that in the "full" coexistence phase range (i.e. [1] for $\beta \geq \beta_c$ and h=0) it is possible to control the series expansion for the surface tension and resum it to the well known Onsager value [2]. One can find in Ref. [3] that convergence and analyticity hold for $\beta \geq \widetilde{\beta}$ where $\widetilde{\beta}$ is some hundred of time β_c. The quoted work was intended to rigorously establish the coexistence phenomenon in two-dimensional Ising model for large enough beta, no matter how large; the cluster expansion technique is the main tool used there in the proof of the various theorems and lemmas. A problem emerges naturally: since the radius of convergence of cluster expansion is a priori bounded in the complex plane only by the critical point, is it possible to optimize this technique obtaining a full functional control of the theory in a range so wide to produce estimates for the critical parameters at least of the right order of magnitude? This work is a first analysis of this question.

As one expects from general considerations the best one can hope without renormalization group ideas is to obtain the entropic bound for the critical beta.

Our result is $\overline{\beta} = 0.52$ which gives an estimate from above of the critical beta ($\simeq 0.44$) to within 19% and of the entropic beta ($\simeq 0.48$) to within 9%. It is clear that in order to go below the entropic bound one has to use renormalization group ideas and, possibly, finite-size conditions; our result is, in some sense, a refined treatment of the "scale one".

We use an optimization procedure first proposed in Ref. [4]: the Banach structure on which the recursive equations live can be defined with a family of equivalent norms. This degree of freedom can be optimized and produce, in the general context of polymer models, the *golden ratio* bound (see also Refs. [5,6]). Our main idea is to observe that, when the polymers have a minimum size (volume) greater than 1 (for instance it is 4 for the self avoiding contours on square lattices) the same "tuning" of the norm produce a considerable improvement of the bound since it eliminates the first terms in a geometric series. We obtain (section 1) in this way a wider bound of analyticity for the cluster expansion series; in order to prove the well known properties for the surface tension in the two dimensional Ising model inside this wider range of temperature we give (section 2) a refined proof of the subadditivity property.

2. An optimized norm for cluster expansion.

Cluster expansion techniques have been used in many different contexts. In each of them they appear with different and special ad hoc features. A natural process of generalization has been developed in the last decade [7,5] and a kind of axiomatic structure has been singled out. One consider a general polymer model defined by a partition function in a volume V:

$$Z(V)=\sum_{\substack{\text{polymer}\\ \text{families}}}\prod_{\gamma\in\text{family}} z(\gamma) \tag{2.1}$$

where the function $z(\gamma)$ is the activity of the polymer γ and characterize the model together with the space on which one performs the sum; this last (which we call $\widehat{P}$) is the space of the words X (finite, non-ordered families) in the polymer alphabet $P \equiv (\gamma_1, \gamma_2, ...)$. The fundamental notion for polymer models is the notion of incompatibility. One can define it by a reflexive and symmetric relation represented by a characteristic function:

$$f(\gamma,\gamma')=\begin{cases}1, & \text{if } \gamma,\gamma' \text{ compatible;}\\ 0, & \text{otherwise.}\end{cases} \tag{2.2}$$

Only families composed by compatible polymers appear in the partition sum. The incompatibility relation introduce the usual graph theoretical notion for the polymer configurations: the polymers (which are the vertices) are connected by an edge if they are incompatible. It is well known that the classical Mayer expansion deals only with this abstract graphological structure of the polymer families. The cluster expansion technique is a formalism developed in order to obtain a relation between the partition function and its logarithm and to control the sums in the limiting cases. A nice way to handle the non trivial combinatoric inside is to use the algebraic method [8]; the basic object is the truncated function i.e. the formal logarithm with respect to the convolutory product between functions of polymer configuration:

$$(\phi_1 * \phi_2)(X)=\sum_{X_1+X_2=X}\phi_1(X_1)\phi_2(X_2). \tag{2.3}$$

The reason for introducing the convolution product is the property:

$$\sum_{X\in\widehat{P}}(\phi_1 * \phi_2)(X)z^X=(\sum_{X\in\widehat{P}}\phi_1(X)z^X)(\sum_{X\in\widehat{P}}\phi_2(X)z^X) \tag{2.4}$$

which implies

$$Z=\sum_{X\in\widehat{P}}\phi(X)=\exp(\sum_{X\in\widehat{P}}\phi^T(X)). \tag{2.5}$$

It is easy to prove that if ϕ is a *hard-core* interaction $\phi(X)=z^X := \prod_\gamma z(\gamma)$, the truncated function ϕ^T admits the explicit formula [3]:

$$\phi^T(X)=\frac{n(X)}{X!}z^X \tag{2.6}$$

where $n(X)=n_+(X)-n_-(X)$ with $n_{\pm}(X)$ is the number of subgraphs of X which contain an even (resp. odd) number of lines and $X! = \prod_{\gamma\in X} m(\gamma,X)!$ where $m(\gamma,X)$

counts the multiplicity of the polymer γ in the configuration X. The convergence of the polymer expansion can be controlled with the help of the iterative equations of Kirkwood-Salzburg type [3] and optimized following the (Ref. [4]). The main idea is that the functional equations live in a Banach space where the norm is not a priori fixed; this degree of freedom permits to perform an optimization procedure.

The convergence properties are based on two bounds:
A1 (Entropy Estimate). There exists a constant μ such that

$$N(\gamma, x) \leq |\gamma| \mu^x \quad \forall \quad \gamma \in P \tag{2.7}$$

where $|\gamma|$ is the volume of the polymer γ.
A2 (Energy Estimate). There exists a constant $\lambda < 1$ such that

$$|z(\gamma)| \leq \lambda^{|\gamma|} \quad \forall \quad \gamma \in P \tag{2.8}$$

where $N(\gamma, x)$ is the number of γ-incompatible polymers in P which have a volume between x and $x+1$. The quantity to control is the correlation function $\varrho(X)$ which is the probability that the polymers in X are present:

$$\varrho(X) = \frac{\sum_{Y \in \widehat{P}} \phi(X+Y)}{\sum_{Y \in \widehat{P}} \phi(Y)} = \sum_{Y \in \widehat{P}} (\phi^{-1} * D_X \phi)(Y) = \sum_{Y \in \widehat{P}} \Delta_X(Y) \tag{2.9}$$

where

$$D_X \phi(Y) = \phi(X+Y) \frac{(X+Y)!}{Y!} \tag{2.10}$$

and

$$\Delta_X(Y) = (\phi^{-1} * D_X \phi)(Y) = \sum_{Y_1 + Y_2 = Y} \phi^{-1}(Y_1) \phi(X + Y_2). \tag{2.11}$$

Indicating with $\sum^{\gamma}$ a sum over all γ-incompatible families, the recursive equation for $\Delta_X(Y)$ is [3]:

$$\begin{aligned} \Delta_{\gamma+X}(Y) &= z(\gamma) \sum_{S \subseteq Y}{}^{\gamma} (-1)^{N(S)} \sum_{Y_1 + Y_3 = Y - S} \phi^{-1}(Y_1) \phi(X + S + Y_3) \\ &= z(\gamma) \sum_{S \subseteq Y}{}^{\gamma} (-1)^{N(S)} \Delta_{X+S}(Y - S). \end{aligned} \tag{2.12}$$

Following Ref. [4] we define the Banach structure by the m-norm depending on an optimization parameter x:

$$I_m(x) = \sup_{\substack{\gamma_1, \ldots, \gamma_n \\ m \geq n \geq 1}} \sum_{\substack{Y \\ N(Y) = m - n}} |\Delta_{\gamma_1, \ldots, \gamma_n}(Y)| (\lambda^{-1} e^{-x})^{\sum |\gamma_i|} \tag{2.13}$$

With simple manipulations one deduces from (2.12) and from A1, A2:

$$
\begin{aligned}
&\sum_{\substack{Y\\N(Y)+N(X)=m}} |\Delta_{\gamma+X}(Y)|(\lambda^{-1}e^{-x})^{|\gamma|+|X|} \leq \\
&\leq \sum_{\substack{Y\\N(Y)+N(X)=m}} \sum_{S\subseteq Y}{}^{\gamma} |\Delta_{X+S}(Y-S)|(\lambda^{-1}e^{-x})^{|\gamma|+|X|}\lambda^{|\gamma|} \leq \\
&\leq \sum_{S}{}^{\gamma} \sum_{\substack{Y\\N(Y)=m-N(X)}} |\Delta_{X+S}(Y-S)|(\lambda^{-1}e^{-x})^{|X|+|S|}e^{-x|\gamma|}(\lambda e^{x})^{|S|} \leq \\
&\leq I_m(x)e^{-x|\gamma|}\sum_{S}{}^{\gamma}(\lambda e^{x})^{|S|}.
\end{aligned}
\tag{2.14}
$$

The last sum can be bounded as follows:

$$
\sum_{S}{}^{\gamma}(\lambda e^{x})^{|S|} = \sum_{n\geq 1}\frac{1}{n!}(\sum_{\sigma\in P}{}^{\gamma}(\lambda e^{x})^{|\sigma|})^{n} \leq \exp(|\gamma|\sum_{l\geq r}(\mu\lambda e^{x})^{l}) = \exp(|\gamma|\frac{(\mu\lambda e^{x})^{r}}{1-\mu\lambda e^{x}})
\tag{2.15}
$$

if

$$
\mu\lambda e^{x} < 1
\tag{2.16}
$$

where r is the minimum size of a polymer. Extending to infinity the previous sum in l corresponds to consider the thermodynamic limit; the resulting estimates are uniform in the volume. Taking the sup over all the X of the formula (2.14) and observing that the sup of the left hand side is an upper bound for I_{m+1} we obtain:

$$
I_{m+1}(x) \leq I_m(x)\exp\left[|\gamma|(-x+\frac{(\mu\lambda e^{x})^{r}}{1-\mu\lambda e^{x}})\right].
\tag{2.17}
$$

It's clear that for the m-norm to be a contraction and the expansion to be convergent the argument in the exponential has to be negative:

$$
-x+\frac{(\mu\lambda e^{x})^{r}}{1-\mu\lambda e^{x}} \leq 0.
\tag{2.18}
$$

If $r=1$ the optimization in x produce the *golden ratio* result [4] $x=\frac{(\sqrt{5}-1)}{2}$; the same result has been obtained with different methods in Ref. [5] and more recently in Ref. [6]; it is, so far, the best estimate obtained in a so general context.

Our main observation is that when the polymers have a minimum size greater than 1 the previous bound can be considerably improved using the same optimizing procedure. In the concrete case of the non-overlapping contours on square lattices, in fact, the minimum size of a polymer is 4 and we can optimize in x the (2.18); it is clear that this will produce good results because it eliminates the first three terms in a geometric sum. We observe that the previous formula can be used to obtain analyticity bound for every polymer model living on a given lattice; the topological structure of the lattice (in particular the connectivity) is encoded in the value of the constant μ.

The (2.18) is not explicitly solvable as in the case $r = 1$ but we can optimize the choice of x numerically using, for the low temperature 2D Ising model where $\lambda = \exp(-2\beta)$, the value $\mu = 3$; one can check by direct computation that the result, after the optimization in x has been performed, is quite insensible to the value of μ.

We have obtained (see fig. 1) $\overline{\lambda} = 0.35$ or, in terms of *beta*:

$$\beta > \overline{\beta} = 0.52 \tag{2.19}$$

which can be compared with the golden ratio bound which gives

$$\beta > 1.04$$

The $\overline{\beta}$ gives an estimate from above of the critical β ($\beta_c \simeq 0.44$) to within 19%; it is moreover interesting to compare this value with the (2.16) which give the "a priori" bound, on the best cluster expansion result, $\tilde{\beta} = 0.48$. We approximate it to within 9%.

3. The series expansion for the surface tension in $\beta > \overline{\beta}$

We finally show the absolute convergence, for $\beta \geq \overline{\beta}$, of the perturbative series for the surface tension. We follow the line of Ref. [3] where the microscopic description of the surface tension is

$$\tau = \lim_{N \to \infty} \frac{1}{N} \log \frac{Z_N^{+-}(m, \beta)}{Z_N^{++}(m^*, \beta)} \tag{3.1}$$

where m^* is the value of the spontaneous magnetization, $m = \alpha m^* + (1 - \alpha)(-m^*)$, $0 < \alpha < 1$ and the symbol $+-$ $(++)$ denote the ensemble with periodic boundary conditions in the horizontal direction and $+-$ (resp $++$) in the vertical one; one of the main result of Ref. [3] is that the limit (3.1) exist and is independent of α. The bounds which can be derived from the cluster expansion method are

$$\sum_X |\phi^T(\gamma + X)| \leq a \exp(-b\beta) \exp(-c\beta|\gamma|) \tag{3.2}$$

and

$$\sum_{\substack{X \ni p \\ X i Q}} |\phi^T(X)| \leq a' \exp(-b'\beta) \exp(-c' d(p, Q)\beta) \tag{3.3}$$

where the a,b etc. are positive constants and XiQ means that the sum has to be performed only over Q-incompatible families; they easily imply that the (3.1) can be transformed in a simpler object, and at the end, the surface tension will appear as the thermodynamic limit of a partition function of the ensemble of *big* contours λ (the ones which turn around the cilinder), each one having a weight of the form $\exp(-2\beta|\lambda| + \nu(\lambda, \beta))$ with $|\nu(\lambda, \beta)| \leq a|\lambda|e^{-b\beta}$, with a, b positive constants. The proof of the existence of the limit (3.1) and its evaluation has to be modified since we are working in a wider range of temperature; we give here new proof of the *subadditivity property* which is the essential ingredient for our result.

Following Ref. [3] one has:

$$\tau = \lim_{N\to\infty} N^{-1} \log \sum_{(\lambda)} \exp(-2\beta|\lambda| - \nu(\lambda, \beta)) \tag{3.4}$$

where the sum is over the *big* contours (up to vertical congruence) such that $|\lambda| \leq N(1 + \beta/\overline{\beta})$ and where

$$\nu(\lambda, \beta) = \sum_{X i \lambda} \phi^T(X). \tag{3.5}$$

To show that the limit (3.4) exists we will prove a weak form of the subadditivity property for the function

$$S_N(\beta) = \log \sum_{(\lambda)} \exp(-2\beta|\lambda| - \nu(\lambda, \beta)). \tag{3.6}$$

We start observing that if $\beta > \overline{\beta}$ the λ appearing in the sum have length bounded by $2N$. This gives us the possibility of finding a column C with the following properties:

(a) The strip of width one immediately to the right of C contains only one horizontal step of λ.

(b) The strip of width $2N^{1/3}$ centered at C contains a portion of λ at most $N^{1/2}$ long.

These properties can be proved as follows. Define N_s and N_m the number of columns with, respectively, intersection simple or multiple with λ. It is easily seen that a multiple column consumes at least 5 units in the length of λ; at least three steps are in fact consumed in the horizontal direction and two in the vertical direction in order to connect the horizontal ones. Hence $5N_m + N_s \leq |\lambda| \leq 2N$ which gives $N_s \geq 3/4N$. This bound, which trivially proves the (a), is essential to prove the (b). Consider now for each simple step the strip of width $2N^{1/3}$ centered at its left end, and let L be the shortest length of λ contained in any of these strips. Let M be the maximal cardinality of a family of disjoint such strips; it is easy to see that if the strips of this family are widened to $4N^{1/3}$ their union will contain all simple steps. Hence $M \cdot 4N^{1/3} \geq N_s \geq 3/4N$. Moreover, $LM \leq |\lambda| \leq 2N$, so $L \leq 2N/M \leq 11N^{1/3} < N^{1/2}$ if N is large. This proves the (b).

Using the property (a) we can now construct a mapping F which associates to any pair $(\lambda_N, \lambda_{N'})$ coming from $S_N(\beta)$ and $S_{N'}(\beta)$ a $\lambda_{N+N'}$ in $S_{N+N'}(\beta)$; we cut λ_N and $\lambda_{N'}$ at some C_N and $C_{N'}$ as described above and join them together in the given order on a cylinder with circumference $N + N'$ to a closed path $\lambda_{N+N'}$. The property (a) says that the so constructed $\lambda_{N+N'}$ is an allowed contour in $S_{N+N'}(\beta)$. Considering that the mapping F can have a $N \cdot N'$ degeneracy and using the property (b) one can prove [3] with cluster expansion techniques that $\nu(\lambda, \beta)$ is weakly additive:

if $\beta > \overline{\beta}$

$$|\nu(\lambda_{N+N'}, \beta) - \nu(\lambda_N, \beta) - \nu(\lambda_{N'}, \beta)| \leq c(N^{1/2} + {N'}^{1/2}) \tag{3.7}$$

and hence in the same range

$$S_{N+N'}(\beta) = (NN')^{-1} \exp(-c(N^{1/2} + {N'}^{1/2}))S_N(\beta)S_{N'}(\beta). \tag{3.8}$$

from which the existence and analyticity of the limit (3.4) immediately follows in the range $\beta > \overline{\beta}$.

4. Conclusions

The simple result obtained in this work is a first step toward the optimal bound that one expects for the analyticity radius of the expansion series at low temperature in the 2D Ising model.

It can be interesting to compare our result with the one obtained in Ref. [9]. One can find in that reference the elegant theorem which establishes the abstract existence of the limit (3.1) at any value of the temperature simply using duality property and Griffith's inequalities. Our result is in some sense complementary to that one since we have a complete functional control of phase equation in a slightly smaller range. More recently [10] it has been proved, with large deviations methods, that the Wulff bounds on the block spin magnetization holds up to the critical temperature; it would be very interesting to see how this nice result can be proved using cluster expansion and renormalization group methods. The problem, in this language, is to find a resummation algorithm which permit a systematic improvement of the convergence radius up the critical point. Also, it would be interesting to apply our method to different models and different definitions of surface tension [11] and also to the theory of the surface tension proposed in Ref. [12]; we hope to return on these problems in future works.

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