Antiferromagnetic Potts model on the Erdős-Rényi random graph*

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Abstract

We study the antiferromagnetic Potts model on the Erdős-Rényi random graph. By identifying a suitable interpolation structure and proving an extended variational principle we show that the replica symmetric solution is an upper bound for the limiting pressure which can be recovered in the framework of Derrida-Ruelle probability cascades. A comparison theorem with a mixed model made of a mean field Potts-antiferromagnet plus a Potts-Sherrington-Kirkpatrick model allows to show that the replica symmetric solution is exact at high temperatures.

Keywords: Mean field, dilute antiferromagnet, q-state Potts model, interpolation, extended variational principle, spin glass, replica symmetry breaking.

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1 Introduction and main results

In this paper we give some rigorous results on the antiferromagnetic Potts model on the Erdős-Rényi random graph.

It is well known that antiferromagnetic Potts models on graphs are related, at zero temperature, to the graph coloring problem which consists in placing colors on the graph vertices in such a way that two of them connected by an edge have different color. In recent time an algorithmic approach was developed to study their colorability [6] based on methods and ideas from disordered system, in particular on the replica symmetry breaking scheme introduced within the mean field theory of spin glasses [20].

Since the Erdős-Rényi random graph has a locally tree-like structure and large loops, the statistical mechanics model with antiferromagnetic interactions has been reported to display some spin glass behavior in the physics literature [23]. In particular it has been argued that the one-step replica symmetry breaking solution does not get improved by a higher number of steps [25].

The rigorous theory of spin glasses has, on the other hand, made important progresses in the last decade by means of the Guerra-Toninelli [16] interpolation scheme, their thermodynamic limit control, the Guerra-Talagrand [22] theorem for the free energy of the Sherrington-Kirkpatrick model and with the general scheme introduced by Aizenman-Sims-Starr [3] which includes an extended variational principle to obtain the exact solution for a large class of models.

This paper is a first attempt to derive within mathematical rigor some properties of the antiferromagnetic Potts model on the Erdős-Rényi random graph. A full treatment of the ferromagnetic Ising

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case has been given in [10] for locally tree-like random graphs and extended in [11]. The techniques used there are heavily based on the use of ferromagnetic Griffiths-Kelly-Sherman and Griffiths-Hurst-Sherman inequalities and do not apply to our case. The extension to Potts random variables is an open problem.

For the antiferromagnetic model we introduce here an interpolation scheme and prove its monotonicity. Previously, Bayati, Gamarnik and Tetali had derived a new discrete version of the Guerra-Toninelli interpolation argument to prove existence of the thermodynamic limit for this model for all $T > 0$, as well as directly for $T = 0$ [4]. This gave a positive solution of a conjecture due to Aldous [2].

We consider the actual value of the pressure. We show that the usual continuous interpolation method of Guerra-Toninelli [16] applies, as it was extended by Franz and Leone [12]. (Also see [17] and [8].) We then prove an extended variational principle for the considered model and we use it to obtain a rigorous control of free energy bounds. In particular we show that an upper bound is obtained by restricting the optimization scheme to hierarchical structures (Ruelle probability cascades) which recovers the replica symmetric solution heuristically introduced by the physicists. We moreover show that such an upper bound is exact for high enough temperatures. Our analysis applies to an arbitrary number $q \in \mathbb{N}$ of values for the spin variables.

The paper is organized in three parts. In the first part, dealing with *existence and simple bounds*, the antiferromagnetic q-states Potts model on the Erdős-Rényi random graph is defined (Section 2) and the thermodynamic limit of its pressure is proved to exist (Section 3). By an estimate using the Cauchy-Schwarz inequality, it is also shown in Section 4 that the pressure admits an elementary upper bound.

The part on existence and simple bounds is then followed by the part on the *extended variational principle*. The cavity functional is constructed in Section 5 and it is shown to lead to a variational expression for the pressure. A class of optimizers, given by the Derrida-Ruelle random probability cascades, is discussed in Section 6. In Section 7 it is shown how this class leads to an upper bound for the pressure which coincides with the replica symmetric solution (in particular the trivial replica symmetric solution, i.e., the one involving a uniform overlap probability measure, reproduce the simple estimate of Section 4). This part on the extended variational principle is comparable to the analogous results for the Sherrington-Kirkpatrick model ([14, 3]). Note that the SK model is defined on the complete graph and the spin glass behavor has its origin in the random couplings of random signs. Here frustration is produced by deterministic anti-ferromagnetic couplings, while the randomness is given by the underlying spatial structure of the Erdős-Rényi random graph.

The last part deals with the full control of the *high temperature region*. In Section 8 it is shown that for temperatures high enough the quenched pressure admits the trivial replica symmetric expression as a lower bound. This result, combined with the result of Section 4, proves that in the high temperature region the pressure is given by the trivial replica symmetric formula. Interestingly, the lower bound leads to the study of a model defined on the complete graph with an Hamiltonian which is the one of the SK model plus the antiferromagnetic Potts-Curie-Weiss model. This last model is studied in Appendix A by a suitable extension of the techniques developed in [15].

2 The model

We study the *antiferromagnetic q-states Potts model on the Erdős-Rényi random graph*. We consider a set of N vertices, each vertex has attached a spin variables which can take $q \in \mathbb{N}$ values: $\sigma_i \in \{1, 2, \dots, q\}$. The edges of the graph are constructed using a set of N^2 independent and identical random variables $\{J_{i,j}\}$ (with $i, j = 1, \dots, N$) having a Poisson distribution with

$$\mathbb{E}[J_{i,j}] = \frac{c}{2N} \ . \tag{1}$$

In the following we will denote by $\mathbb{E}$ the expectation with respect to the randomness of the graph. Because of the double counting (we allow $J_{i,j}$ to differ from $J_{j,i}$) we obtain an average vertex degree

c. Strictly speaking, in the construction of the Erdős-Rényi random graph we should use random variables distributed as Binomial with parameter $c/(2N)$. However the two choices are equivalent in the thermodynamic limit $N \to \infty$ and we find it more convenient to work with the Poisson setting.

The Hamiltonian of the model is given by

$$H_N(\sigma) \;=\; \sum_{i,j=1}^{N} J_{i,j}\delta(\sigma_i,\sigma_j) \tag{2}$$

where $\delta(x,y)$ denotes the Kronecker delta function. Note that since the $J_{i,j}$ are non-negative, the model is anti-ferromagnetic. For a given inverse temperature β, we consider the random partition function

$$Z_N(c,\beta) \;=\; \sum_{\sigma\in\{1,\dots,q\}^N} e^{-\beta H_N(\sigma)} \;, \tag{3}$$

the expectation of a spin function $f : \{1,\dots,q\}^N \mapsto \mathbb{R}$ with respect to the random Boltzmann-Gibbs state

$$\omega(f(\sigma)) \;=\; \frac{1}{Z_N(c,\beta)} \sum_{\sigma\in\{1,\dots,q\}^N} f(\sigma) e^{-\beta H_N(\sigma)} \tag{4}$$

and the quenched expectation

$$\langle f(\sigma)\rangle \;=\; \mathbb{E}[\omega(f(\sigma))] \;. \tag{5}$$

As usual in disordered systems, it will be convenient to define the quenched state for a function of the spins of an arbitrary number $n \in \mathbb{N}$ of real copies. This is obtained by considering the product Boltmann-Gibbs state and then averaging over the disorder. Namely, for a function f of n spin configurations $\sigma^{(1)},\dots,\sigma^{(n)}$ we define

$$\Omega(f(\sigma^{(1)},\dots,\sigma^{(n)})) \;=\; \frac{1}{Z_N^n(\beta)} \sum_{\sigma^{(1)},\dots,\sigma^{(n)}} f(\sigma^{(1)},\dots,\sigma^{(n)}) e^{-\beta(H_N(\sigma^{(1)})+\ldots+H_N(\sigma^{(n)}))} \tag{6}$$

and then

$$\langle f(\sigma^{(1)},\dots,\sigma^{(n)})\rangle \;=\; \mathbb{E}[\Omega(f(\sigma^{(1)},\dots,\sigma^{(n)}))] \;. \tag{7}$$

The main thermodynamic quantity we will study is the quenched pressure per particle

$$p_N(c,\beta) \;=\; \frac{1}{N}\mathbb{E}[\ln Z_N(c,\beta)] \;. \tag{8}$$

and its thermodynamic limit

$$p(c,\beta) \;=\; \lim_{N\to\infty} p_N(c,\beta) \;. \tag{9}$$

An important observable that will appear later is the sequence (for $n \geq 1$) of arrays $q_N(r_1,r_2,\dots,r_n)$ with $(r_1,r_2,\dots,r_n) \in \{1,\dots q\}^n$, which represents the generalized multi-overlap between n spin configurations $\sigma^{(1)},\dots,\sigma^{(n)}$, and is defined as

$$q_N(r_1,r_2,\dots,r_n) \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta(\sigma_i^{(1)},r_1)\delta(\sigma_i^{(2)},r_2)\cdots\delta(\sigma_i^{(n)},r_n) \;. \tag{10}$$

3 Thermodynamic limit of the pressure

Theorem 3.1 *The pressure per particle defined in (8) is a superadditive sequence, i.e. for all N, N_1, N_2 with $N_1 + N_2 = N$*

$$N p_N(\beta) \geq N_1 p_{N_1}(\beta) + N_2 p_{N_2}(\beta) \;. \tag{11}$$

Therefore the infinite volume limit of the pressure per particle defined in (9) exists and is equal to

$$p(\beta) \;=\; \sup_N p_N(\beta) \;. \tag{12}$$

Remark 3.2 *The fact that $p_N(c,\beta)$ has a finite upper bound, uniform in N, can be trivially proved by the Jensen inequality $N^{-1}\mathbb{E}(\ln Z_N) \leq N^{-1}\ln\mathbb{E}(Z_N)$, where the right hand side is the pressure of the antiferromagnetic Potts-Curie-Weiss model. (See, for example, (82).)*

Proof: The proof is obtained by interpolation. For a partition of the systems of size N into two subsystems of sizes N_1 and N_2 and for a $t \in [0,1]$ we consider the following independent Poisson random variables:

$$\begin{aligned} J'_{i,j} &\sim \text{Poisson}\left(\frac{ct}{2N}\right) \\ J''_{i,j} &\sim \text{Poisson}\left(\frac{c(1-t)}{2N_1}\right) \\ J'''_{i,j} &\sim \text{Poisson}\left(\frac{c(1-t)}{2N_2}\right) . \end{aligned} \tag{13}$$

We define the interpolating Hamiltonian

$$H_N(\sigma,t) = \sum_{i,j=1}^{N} J'_{i,j}\delta(\sigma_i,\sigma_j) + \sum_{i,j=1}^{N_1} J''_{i,j}\delta(\sigma_i,\sigma_j) + \sum_{i,j=N_1+1}^{N} J'''_{i,j}\delta(\sigma_i,\sigma_j), \tag{14}$$

which induces an interpolating random partition function

$$Z_N(c,\beta,t) = \sum_{\sigma\in\{1,\ldots,q\}^N} e^{-\beta H_N(\sigma,t)} , \tag{15}$$

the expectation with respect to an interpolating random Boltzmann-Gibbs

$$\omega_t(f(\sigma)) = \frac{1}{Z_N(c,\beta,t)} \sum_{\sigma\in\{1,\ldots,q\}^N} f(\sigma)e^{-\beta H_N(\sigma,t)} , \tag{16}$$

and an interpolating quenched pressure

$$p_N(c,\beta,t) = \frac{1}{N}\mathbb{E}\left[Z_N(c,\beta,t)\right] . \tag{17}$$

Since $p_N(\beta,1) = p_N(\beta)$ and $p_N(\beta,0) = p_{N_1}(\beta) + p_{N_2}(\beta)$ the first statement of the theorem (Eq. (11)) follows from the fundamental theorem of calculus if one can show that the interpolating pressure is monotonically non-decreasing in t. The derivative of the interpolating pressure reads

$$\begin{aligned} \frac{dp_N(c,\beta,t)}{dt} &= \frac{c}{2}\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j=1}^{N}\ln\omega_t(e^{-\beta\delta(\sigma_i,\sigma_j)}) \right. \\ &\left. -\frac{1}{NN_1}\sum_{i,j=1}^{N_1}\ln\omega_t(e^{-\beta\delta(\sigma_i,\sigma_j)}) - \frac{1}{NN_2}\sum_{i,j=N_1+1}^{N}\ln\omega_t(e^{-\beta\delta(\sigma_i,\sigma_j)})\right] , \end{aligned} \tag{18}$$

where the following identity has been used: for a vector $X = (X_1,\ldots,X_m)$ of m independent Poisson random variables X_i with parameter $\lambda_i(t)$ and a function $f : \mathbb{N}^m \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}[f(X)] = \mathbb{E}\left[\sum_{i=1}^{m}\frac{d\lambda_i(t)}{dt}\left(f(X_1,\ldots,X_i+1,\ldots,X_m) - f(X_1,\ldots,X_i,\ldots,X_m)\right)\right]. \tag{19}$$

Expression (18) can be further simplified using the identity

$$e^{-\beta\delta(\sigma_i,\sigma_j)} = 1-(1-e^{-\beta})\delta(\sigma_i,\sigma_j), \tag{20}$$

and the Taylor expansion

$$\ln(1-x) \;=\; -\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad \forall |x|<1\;. \tag{21}$$

One obtains

$$\begin{aligned}\frac{dp_N(c,\beta,t)}{dt} \;&=\; -\frac{c}{2}\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(1-e^{-\beta})^n}{n}\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\omega_t^n(\delta(\sigma_i,\sigma_j))\right.\right.\\ &\left.\left.-\frac{1}{NN_1}\sum_{i,j=1}^{N_1}\omega_t^n(\delta(\sigma_i,\sigma_j))-\frac{1}{NN_2}\sum_{i,j=N_1+1}^{N}\omega_t^n(\delta(\sigma_i,\sigma_j))\right)\right], \end{aligned}\tag{22}$$

which can be rewritten, using the definition of the sequence of generalized multi-overlap arrays in (10), as

$$\begin{aligned}\frac{dp_N(c,\beta,t)}{dt} \;&=\; -\frac{c}{2}\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(1-e^{-\beta})^n}{n}\right. \\ &\left.\sum_{r_1,\dots,r_n=1}^{q}\Omega_t\left(q_N^2(r_1,\dots,r_n)-\frac{N_1}{N}q_{N_1}^2(r_1,\dots,r_n)-\frac{N_2}{N}q_{N_2}^2(r_1,\dots,r_n)\right)\right].\end{aligned}\tag{23}$$

From this expression one sees by inspection that the derivative is non-negative, employing a standard convexity argument as in [16]. Therefore monotonicity of the interpolating pressure is established and super-additivity (Eq.(11)) follows. The second claim of the theorem, i.e., the existence of the thermodynamic limit of the pressure and its realization as a supremum (Eq.(12)) is a standard consequence of super-additivity (see [16]). □

Remark 3.3 *The same computation goes through also for the ferromagnetic model with the change $\beta \mapsto -\beta$. However $1-e^{\beta}<0$ for $\beta>0$ and therefore the series in (23) has alternating signs and monotonicity can not be derived anymore by inspection. We believe however that the interpolation is monotone also in the ferromagnetic case, though in the opposite direction. This belief is based on two facts. Firstly, pressure sub-additivity for the ferro-magnetic model on the Erdős-Rényi random graph has been checked numerically for small system sizes [1]. This is in agreement with a monotonic behavior of the interpolating pressure. Secondly, and more importantly, the numerical checks ([1]) for the Ising case ($q=2$) show that for $0\le t\le 1$ the series in (23) is dominated by the first term and therefore one would be left with the same interpolating pressure of the Curie-Weiss model which is know to be sub-additive. This is indeed rigorously shown at zero temperature in [9]. Although the ferromagnetic model has been fully solved in [10], it would be interesting to extend the monotonicity result to all temperature.*

4 “Trivial” pressure bound

As a preliminary result, we show that the quenched pressure is bounded from above by the so-called “trivial replica symmetric pressure”, which improves the Jensen bound of the previous section.

Lemma 4.1 *For all $\beta\ge 0$ and $N\ge 1$,*

$$p_N(c,\beta)\le p^{\mathrm{TRS}}(c,\beta) \tag{24}$$

with

$$p^{\mathrm{TRS}}(c,\beta)=\frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right)+\ln q. \tag{25}$$

Proof: Using (19), we have that

$$\frac{d}{dc}p_N(c,\beta) = \frac{1}{2N^2}\mathbb{E}\sum_{i,j=1}^{N}\ln\omega(e^{-\beta\delta(\sigma_i,\sigma_j)}). \tag{26}$$

By Jensen's inequality and the concavity of the logarithm, this is bounded from above by

$$\frac{1}{2}\mathbb{E}\ln\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\omega(e^{-\beta\delta(\sigma_i,\sigma_j)})\right) \leq \frac{1}{2}\ln\mathbb{E}\left(1-(1-e^{-\beta})\omega\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j)\right)\right), \tag{27}$$

where also (20) was used. Observe that

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j) = \sum_{r_1=1}^{q}\left(\frac{1}{N}\sum_{i=1}^{N}\delta(\sigma_i,r_1)\right)^2 = \sum_{r_1=1}^{q}q_N^2(r_1)\,. \tag{28}$$

Since all the $q_N(r_1)$ are nonnegative and add up to 1, we can use Hölder's inequality to get

$$1 = \sum_{r_1=1}^{q}q_N(r_1) \leq \sqrt{\sum_{r_1=1}^{q}q_N^2(r_1)}\cdot\sqrt{q}, \tag{29}$$

and thus

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j) \geq \frac{1}{q}. \tag{30}$$

Hence,

$$\frac{d}{dc}p_N(c,\beta) \leq \frac{1}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right). \tag{31}$$

The bound in the lemma is then obtained by using the fundamental theorem of calculus and observing that

$$p_N(0,\beta) = \ln q. \tag{32}$$

□

5 Extended Variational Principle

We begin with a proposition that is useful in the ultimate definition of the extended variational principle.

Proposition 5.1 *Suppose that $N, M \in \mathbb{N}$ are chosen and let μ_M be any measure on $\{1,\ldots,q\}^M$. Then*

$$p_N(c,\beta) \;\leq\; G_{N,M}(c,\beta,\mu_M)\,, \tag{33}$$

where

$$G_{N,M}(c,\beta,\mu_M) = G^{(1)}_{N,M}(c,\beta,\mu_M) - G^{(2)}_{N,M}(c,\beta,\mu_M)$$

with

$$G^{(1)}_{N,M}(c,\beta,\mu_M) \;=\; \frac{1}{N}\mathbb{E}\left[\ln\sum_{\tau\in\{1,\ldots,q\}^M}\mu_M(\tau)\sum_{\sigma\in\{1,\ldots,q\}^N}\exp\left(-\beta\sum_{i=1}^{N}\sum_{j=1}^{M}K_{ij}\delta(\sigma_i,\tau_j)\right)\right], \tag{34}$$

where the K_{ij}'s are i.i.d. Poisson random variables with parameter c/M, and

$$G^{(2)}_{N,M}(c,\beta,\mu_M) \;=\; \frac{1}{N}\mathbb{E}\left[\ln \sum_{\tau\in\{1,\dots,q\}^M} \mu_M(\tau)\exp\left(-\beta\sum_{i,j=1}^{M} L_{ij}\delta(\tau_i,\tau_j)\right)\right], \tag{35}$$

where the L_{ij}'s are i.i.d. Poisson random variables with parameters $cN/(2M^2)$.

Proof: For a $t\in[0,1]$ we consider the following independent Poisson random variables:

$$(\tilde{J}_{ij})_{i,j=1}^N\,,\qquad (\tilde{K}_{ij}\;:\; i=1,\dots,N\,;\; j=1,\dots,M)\,,\qquad (\tilde{L}_{ij})_{i,j=1}^M\,, \tag{36}$$

such that

$$\mathbb{E}[\tilde{J}_{ij}] \;=\; \frac{(1-t)c}{2N}\,,\quad \mathbb{E}[\tilde{K}_{ij}] \;=\; \frac{ct}{M}\,,\quad \mathbb{E}[\tilde{L}_{ij}] \;=\; \frac{(1-t)cN}{2M^2}\,, \tag{37}$$

for all appropriate indices i,j. We define

$$H_{N,M}(\sigma,\tau,t) \;=\; \sum_{i,j=1}^{N}\tilde{J}_{ij}\delta(\sigma_i,\sigma_j)+\sum_{i=1}^{N}\sum_{j=1}^{M}\tilde{K}_{ij}\delta(\sigma_i,\tau_j)+\sum_{i,j=1}^{M}\tilde{L}_{ij}\delta(\tau_i,\tau_j)\,, \tag{38}$$

and we also define

$$Z_{N,M}(c,\beta,\mu_M,t) \;=\; \sum_{\tau\in\{1,\dots,q\}^M}\mu_M(\tau)\sum_{\sigma\in\{1,\dots,q\}^N} e^{-\beta H_{N,M}(\sigma,\tau,t)}\,, \tag{39}$$

and

$$\omega_t(f(\sigma,\tau)) = \frac{1}{Z_{N,M}(c,\beta,\mu_M,t)}\sum_{\tau\in\{1,\dots,q\}^M}\mu_M(\tau)\sum_{\sigma\in\{1,\dots,q\}^N} f(\sigma,\tau)e^{-\beta H_{N,M}(\sigma,\tau,t)}. \tag{40}$$

Then

$$\frac{1}{N}\mathbb{E}[\ln Z_{N,M}(c,\beta,\mu_M,0)] \;=\; p_N(c,\beta)+G^{(2)}_{N,M}(c,\beta,\mu_M)\,, \tag{41}$$

since if $t=0$ then $H_{N,M}(\sigma,\tau,0)$ splits into a summand only depending on σ and one only depending on τ. Furthermore,

$$\frac{1}{N}\mathbb{E}[\ln Z_{N,M}(c,\beta,\mu_M,1)] \;=\; G^{(1)}_{N,M}(c,\beta,\mu_M)\,. \tag{42}$$

Moreover, as in the proof of Theorem 3.1, one can show that

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{N}\mathbb{E}[\ln Z_{N,M}(c,\beta,\mu_M,t)]\Big)\\
&\quad= -\frac{c}{2}\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j=1}^{N}\ln\omega_t(e^{-\beta\delta(\sigma_i,\sigma_j)})-\frac{2}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\ln\omega_t(e^{-\beta\delta(\sigma_i,\tau_j)})+\frac{1}{M^2}\sum_{i,j=1}^{M}\ln\omega_t(e^{-\beta\delta(\tau_i,\tau_j)})\right]\\
&\quad= \frac{c}{2}\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(1-e^{-\beta})^n}{n}\left(\frac{1}{N^2}\sum_{i,j=1}^{N}\omega_t^n(\delta(\sigma_i,\sigma_j))-\frac{2}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\omega_t^n(\delta(\sigma_i,\tau_j))+\frac{1}{M^2}\sum_{i,j=1}^{M}\omega_t^n(\delta(\tau_i,\tau_j))\right)\right].
\end{aligned} \tag{43}$$

This can again be rewritten in terms of generalized multi-overlaps as

$$\begin{aligned}
&\frac{c}{2}\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(1-e^{-\beta})^n}{n}\sum_{r_1,\dots,r_n=1}^{q}\Omega_t\Big(q_N^2(r_1,\dots,r_n)-2q_N(r_1,\dots,r_n)q_M(r_1,\dots,r_n)+q_M^2(r_1,\dots,r_n)\Big)\right]\\
&\qquad= \frac{c}{2}\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(1-e^{-\beta})^n}{n}\sum_{r_1,\dots,r_n=1}^{q}\Omega_t\Big(\big(q_N(r_1,\dots,r_n)-q_M(r_1,\dots,r_n)\big)^2\Big)\right],
\end{aligned} \tag{44}$$

which is obviously nonnegative for all t. This proves that

$$\begin{aligned} p_N(c,\beta) + G^{(2)}_{N,M}(c,\beta,\mu_M) &= \frac{1}{N}\mathbb{E}[\ln Z_{N,M}(c,\beta,\mu_M,0)] \\ &\leq \frac{1}{N}\mathbb{E}[\ln Z_{N,M}(c,\beta,\mu_M,1)] \;=\; G^{(1)}_{N,M}(c,\beta,\mu_M)\,. \end{aligned} \tag{45}$$

In other words,

$$p_N(c,\beta) \;\leq\; G^{(1)}_{N,M}(c,\beta,\mu_M) - G^{(2)}_{N,M}(c,\beta,\mu_M)\,. \tag{46}$$

□

In order to accommodate a limit where M approaches ∞, we mention an equivalent version of the function $G_{N,M}(c,\beta,\mu_M)$:

Lemma 5.2 *Let $I(1), I(2), \ldots$ be i.i.d. random variables uniformly chosen from $\{1,\ldots,N\}$, and let $J(1), J(2), \ldots$ be i.i.d. random variables uniformly chosen from $\{1,\ldots,M\}$. Independently of this, let K be a Poisson random variable with parameter cN and let L be a Poisson random variable with parameter $cN/2$. Then*

$$G^{(1)}_{N,M}(c,\beta,\mu_M) \;=\; \frac{1}{N}\mathbb{E}\left[\ln \sum_{\tau\in\{1,\ldots,q\}^M} \mu_M(\tau) \sum_{\sigma\in\{1,\ldots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{J(k)})\right)\right]\,,$$

and

$$G^{(2)}_{N,M}(c,\beta,\mu_M) \;=\; \frac{1}{N}\mathbb{E}\left[\ln \sum_{\tau\in\{1,\ldots,q\}^M} \mu_M(\tau)\exp\left(-\beta\sum_{k=1}^{L}\delta(\tau_{J(2k-1)},\tau_{J(2k)})\right)\right]\,.$$

Proof: This follows from a property of Poisson random variables known as Poisson thinning (or sometimes called Bernoulli thinning). Previously we had i.i.d., Poisson random variables K_{ij} and L_{ij}. Now we have just two Poisson random variables K and L, but we have i.i.d., uniform random variables $I(k)$, $J(k)$. The Poisson thinning property refers to the fact that the families

$$\hat{K}_{ij} \;=\; \#\{k \leq K \,:\, (I(k),J(k)) = (i,j)\} \quad \text{and} \quad \hat{L}_{ij} \;=\; \#\{k\leq L \,:\, (J(2k-1),J(2k)) = (i,j)\}\,,$$

are distributed identically to K_{ij} and L_{ij}. The reader may also check this as an exercise. □

Corollary 5.3 *Suppose that μ_M is a random measure on $\{1,\ldots,q\}^M$. Then*

$$p_N(c,\beta) \;\leq\; \mathbb{E}\left[G_{N,M}(c,\beta,\mu_M)\right]\,,$$

where the symbol $\mathbb{E}$ denotes the expectation with respect to μ_M.

Proof: For each random realization of μ_M, we have

$$p_N(c,\beta) \;\leq\; G_{N,M}(c,\beta,\mu_M)\,,$$

almost surely, according to Proposition 5.1. So the corollary follows by elementary properties of the expectation. □

The following Proposition may be viewed as the $M\to\infty$ limit of Proposition 5.1.

Proposition 5.4 *Let $(\mu_\alpha)_{\alpha=1}^\infty$ be a random sequence such that each μ_α is positive, and the series converges, almost surely. Also let $(\tau_{\alpha,k}\, :\, \alpha \in \mathbb{N},\ k \in \mathbb{N})$ be a random array of elements of $\{1,\dots,q\}$. We write $\mathcal{L}$ for the measure which describes the joint distribution of*

$$((\mu_\alpha)_{\alpha\in\mathbb{N}}\,,\ (\tau_{\alpha,k}\, :\, \alpha \in \mathbb{N},\ k \in \mathbb{N}))\,.$$

We say that $\mathcal{L}$ is "exchangeable" if

$$((\mu_\alpha)_{\alpha\in\mathbb{N}}\,,\ (\tau_{\alpha,k}\, :\, \alpha \in \mathbb{N},\ k \in \mathbb{N})) \stackrel{\mathcal{D}}{=} ((\mu_\alpha)_{\alpha\in\mathbb{N}}\,,\ (\tau_{\alpha,\pi(k)}\, :\, \alpha \in \mathbb{N},\ k \in \mathbb{N}))\,,$$

for every non-random permutation π of $\mathbb{N}$ which moves only finitely many k's. Here $\stackrel{\mathcal{D}}{=}$ indicates equality in distribution.

For an exchangeable $\mathcal{L}$, we define

$$G_N(c,\beta,\mathcal{L})\ =\ G_N^{(1)}(c,\beta,\mathcal{L}) - G_N^{(2)}(c,\beta,\mathcal{L})\,, \tag{47}$$

where

$$G_N^{(1)}\ =\ \frac{1}{N}\,\mathbb{E}\left[\ln \sum_{\alpha=1}^\infty \mu_\alpha \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta \sum_{k=1}^K \delta(\sigma_{I(k)},\tau_{\alpha,k})\right)\right]\,, \tag{48}$$

and

$$G_N^{(2)}\ =\ \frac{1}{N}\,\mathbb{E}\left[\ln \sum_{\alpha=1}^\infty \mu_\alpha \exp\left(-\beta \sum_{k=1}^L \delta(\tau_{\alpha,2k-1},\tau_{\alpha,2k})\right)\right]\,, \tag{49}$$

where $\mathbb{E}$ is the expectation over $\mathcal{L}$ as well as the random variables $(I(k))_{k=1}^\infty$, K and L, where we assume that $(I(k)), K, L$ are all independent of one another and of $((\mu_\alpha),(\tau_{\alpha,k}))$, and $I(1), I(2),\dots$ are i.i.d, uniform on $\{1,\dots,N\}$, K is Poisson with mean cN and L which is Poisson with mean $cN/2$. Then we have

$$p_N(c,\beta)\ \leq\ G_N(c,\beta,\mathcal{L})\,,$$

for every exchangeable $\mathcal{L}$.

Remark 5.5 *The condition of exchangeability is not very restrictive. Given any non-exchangeable $\mathcal{L}$, we may obtain an exchangeable law by a standard symmetrization procedure. The advantage of assuming that $\mathcal{L}$ is symmetric is that it allows us to simplify the expression of G_N. Otherwise it would be more complicated.*

Proof: For any fixed M, consider a random realization of the sequence $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ and $(\tau_{\alpha,k}\, :\, \alpha \in \mathbb{N},\ k \in \{1,\dots,M\})$. Define the random measure μ_M on $\{1,\dots,q\}^M$ defined from this as

$$\mu_M(\tau)\ =\ \sum_{\alpha=1}^\infty \mu_\alpha \mathbf{1}\{(\tau_{\alpha,1},\dots,\tau_{\alpha,M}) = \tau\}\,,$$

for each $\tau \in \{1,\dots,q\}^M$, where $\mathbf{1}\{\cdots\}$ represents the indicator function of the condition given by $\{\cdots\}$. This is the empirical measure, but where we merely truncate the full sequence $(\tau_{\alpha,1},\tau_{\alpha,2},\dots)$ to the first M components of the spin. Another useful way to state the same thing is to notice that for any non-random function $f : \{1,\dots,q\}^M \to \mathbb{R}$, we have

$$\sum_{\tau\in\{1,\dots,q\}^M} f(\tau)\mu_M(\tau)\ =\ \sum_{\alpha=1}^\infty \mu_\alpha f(\tau_{\alpha,1},\dots,\tau_{\alpha,M})\,. \tag{50}$$

In fact, it is not necessary that f is non-random, merely that it is independent of $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ and $(\tau_{\alpha,k}\, :\, \alpha \in \mathbb{N},\ k \in \mathbb{N})$. Note that μ_M is a random measure, but according to Corollary 5.3 this still gives an upper bound. Specifically,

$$p_N(c,\beta)\ \leq\ \mathbb{E}\left[G_{N,M}(c,\beta,\mu_M)\right]\,,$$

where the expectation is over the law $\mathcal{L}$ for $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ and $(\tau_{\alpha,k} : \alpha \in \mathbb{N},\ k \in \mathbb{N})$, from which μ_M is derived as a measurable function.

According to Lemma 5.2, we may write

$$G_{N,M}^{(1)}(c,\beta,\mu_M) \;=\; \frac{1}{N}\mathbb{E}\left[\ln \sum_{\tau\in\{1,\dots,q\}^M} \mu_M(\tau) \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{J(k)})\right)\right],$$

where $I(1), I(2), \dots$ are i.i.d. random variables uniformly chosen from $\{1,\dots,N\}$, and $J(1), J(2), \dots$ are i.i.d random variables uniformly chosen from $\{1,\dots,M\}$, and independently of this, K is a Poisson random variable with parameter cN. Note that, according to (50), we may rewrite this as

$$\frac{1}{N}\ln \sum_{\tau\in\{1,\dots,q\}^M} \mu_M(\tau) \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{J(k)})\right)$$
$$= \frac{1}{N}\ln \sum_{\alpha=1}^{\infty}\mu_\alpha \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{\alpha,J(k)})\right).$$

But, conditional on the event that $J(1),\dots,J(K)$ are all distinct elements of $\{1,\dots,M\}$, we have equality in distribution,

$$\frac{1}{N}\ln \sum_{\alpha=1}^{\infty}\mu_\alpha \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{\alpha,J(k)})\right)$$
$$\stackrel{\mathcal{D}}{=} \frac{1}{N}\ln \sum_{\alpha=1}^{\infty}\mu_\alpha \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta\sum_{k=1}^{K}\delta(\sigma_{I(k)},\tau_{\alpha,k})\right) \tag{51}$$

where we have replaced the random indices $J(1),\dots,J(K)$ by the non-random indices $1,\dots,K$, because we assumed that $(\tau_{\alpha,1},\dots,\tau_{\alpha,M})$ are exchangeable, meaning equal in distribution under finite permutations. Note that here we use the fact that K and $J(1), J(2), \dots$ are independent of $(\mu_\alpha)_{\alpha\in\mathbb{N}}$ and $(\tau_{\alpha,k} : \alpha\in\mathbb{N},\ k\in\mathbb{N})$. Moreover, conditional on the value of K, the probability that $J(1),\dots,J(K)$ are all distinct is

$$\frac{M(M-1)\cdots(M-K+1)}{M^K}.$$

If we take a single realization of K for all M's, then we see that this conditional probability converges to 1, pointwise, almost surely. So we are justified in making the rearrangement in (51), with high probability. Moreover, conditioning on K, we see that the function on the left hand side of (51) is bounded in the interval $[\log(q)-\beta(K/N), \log(q)]$. This is summable against the distribution of K. Therefore, by the dominated convergence theorem, we have

$$\lim_{M\to\infty}\mathbb{E}\left[G_{N,M}^{(1)}(c,\beta,\mu_M)\right] \;=\; G_N^{(1)}(c,\beta,\mathcal{L}).$$

A similar argument holds for the second term $G_N^{(2)}(c,\beta,\mathcal{L})$. □

Theorem 5.6 *For any c and β, we have the "extended variational principle,"*

$$p(c,\beta) \;=\; \lim_{N\to\infty}\inf_{\mathcal{L}} G_N(c,\beta,\mathcal{L}) \tag{52}$$

Proof: We view the equality as a concatenation of an upper bound and a lower bound. The upper bound is obtained by optimizing the upper bound in Proposition 5.4 and taking the limit $N\to\infty$. The lower bound is proved combining sub-additivity with Fekete's lemma [21],

$$p(c,\beta) \;=\; \lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{N}\mathbb{E}\left[\ln\frac{Z_{N+M}(c,\beta)}{Z_M(c,\beta)}\right] \tag{53}$$

using a particular choice of $\mathcal{L}_*$ such that

$$\lim_{M\to\infty} \frac{1}{N}\mathbb{E}\left[\ln \frac{Z_{N+M}(c,\beta)}{Z_M(c,\beta)}\right] = G_N(c,\beta,\mathcal{L}_*)\ . \tag{54}$$

This is obtained as a weak limit of Boltzmann-Gibbs measures. By definition,

$$\frac{1}{N}\mathbb{E}\left[\ln \frac{Z_{M+N}(c,\beta)}{Z_M(c,\beta)}\right] = \frac{1}{N}\mathbb{E}\left[\ln \frac{\sum_{\tau\in\{1,\dots,q\}^M}\sum_{\sigma\in\{1,\dots,q\}^N} e^{-\beta H_{M,N}(\tau,\sigma)}}{\sum_{\tau\in\{1,\dots,q\}^M} e^{-\beta H_M(\tau)}}\right],$$

where, with $J_{ij}\sim \text{Poiss}\left(\frac{c}{2(M+N)}\right)$ and $K_{ij}\sim\text{Poiss}\left(\frac{c}{M+N}\right)$,

$$H_{M,N}(\tau,\sigma) = \sum_{i,j=1}^{M} J_{ij}\delta(\tau_i,\tau_j) + \sum_{j=1}^{M}\sum_{i=1}^{N} K_{ij}\delta(\tau_j,\sigma_i) + \sum_{i,j=1}^{N} J_{ij}\delta(\sigma_i,\sigma_j) \tag{55}$$

and, with $\widetilde{J}_{ij}\sim\text{Poiss}\left(\frac{c}{2M}\right)$,

$$H_M(\tau) = \sum_{i,j=1}^{M}\widetilde{J}_{ij}\delta(\tau_i,\tau_j). \tag{56}$$

We split the latter Hamiltonian as follows.

$$H_M(\tau) = \sum_{i,j=1}^{M} J_{ij}\delta(\tau_i,\tau_j) + \sum_{i,j=1}^{M} L_{ij}\delta(\tau_i,\tau_j), \tag{57}$$

where $L_{ij}\sim\text{Poiss}\left(\frac{cN}{2M(M+N)}\right)$. We are interested in the limit $M\to\infty$, so we will ignore the term $\sum_{i,j=1}^{N} J_{ij}\delta(\sigma_i,\sigma_j)$ in $H_{M,N}(\tau,\sigma)$, because this converges to 0 almost surely. Hence,

$$\frac{1}{N}\mathbb{E}\left[\ln \frac{Z_{M+N}(c,\beta)}{Z_N(c,\beta)}\right] = \frac{1}{N}\mathbb{E}\left[\ln \frac{\sum_{\tau\in\{1,\dots,q\}^M}\sum_{\sigma\in\{1,\dots,q\}^N}\mu_M^*(\tau)\exp\left(-\beta\sum_{j=1}^{M}\sum_{i=1}^{N}K_{ij}\delta(\tau_j,\sigma_i)\right)}{\sum_{\tau\in\{1,\dots,q\}^M}\mu_M^*(\tau)\exp\left(-\beta\sum_{i,j=1}^{M}L_{ij}\delta(\tau_i,\tau_j)\right)}\right],$$

where

$$\mu_M^*(\tau) \ = \ \exp\left(-\beta\sum_{i,j=1}^{M} J_{ij}\delta(\tau_i,\tau_j)\right) \tag{58}$$

Using Poisson thinning we can rewrite this as

$$\frac{1}{N}\mathbb{E}\left[\ln \frac{Z_{M+N}(c,\beta)}{Z_N(c,\beta)}\right] = \frac{1}{N}\mathbb{E}\left[\ln \frac{\sum_{\tau\in\{1,\dots,q\}^M}\sum_{\sigma\in\{1,\dots,q\}^N}\mu_M^*(\tau)\exp\left(-\beta\sum_{k=1}^{K_M}\delta(\tau_{J(k)},\sigma_{I(k)})\right)}{\sum_{\tau\in\{1,\dots,q\}^M}\mu_M^*(\tau)\exp\left(-\beta\sum_{k=1}^{L_M}\delta(\tau_{J(2k-1)},\tau_{J(2k)})\right)}\right],$$

where $K_M\sim\text{Poiss}(MN\frac{c}{M+N})$ and $L_M\sim\text{Poiss}(M^2\frac{cN}{2M(M+N)})$, and where $I(1),I(2),\dots$ and $J(1),J(2),\dots$ are i.i.d. random variables uniformly chosen on $\{1,\dots,N\}$ and $\{1,\dots,M\}$ respectively. Since

$$\lim_{M\to\infty} K_M = K\sim\text{Poiss}(cN)\ ,\qquad \lim_{M\to\infty} L_M = L\sim\text{Poiss}\left(\frac{cN}{2}\right)$$

we obtain formula (54) with $\mathcal{L}_* = \lim_{M\to\infty}\mu_M^*$. □

6 Derrida-Ruelle Random Probability Cascade

Theorem (5.6) shows that the cavity functional in eq. (47) needs to be optimized over measures $\mathcal{L}$. The optimal choice of the measure has been conjectured to be related to Derrida-Ruelle random probability cascades (RPC). In the following we will show how one can obtain increasing level of Replica Symmetry Breaking by considering RPC with an increasing number of levels. We will concentrate on the simplest realization of that which produces the Replica Symmetric solution.

6.1 One level RPC

Given $m \in (0,1)$, let Λ_m be the measure on $(0,\infty)$: $d\Lambda_m(x) = mx^{-m-1}\,dx$. Suppose that $\{\xi_1, \xi_2, \dots\}$ are a Poisson point process with intensity measure Λ_m. This has the important property that the set

$$\{\lambda_\alpha \xi_\alpha\}_{\alpha=1}^{\infty}$$

has the same distribution as the set

$$\{\mathbb{E}[\lambda_\alpha^m]^{1/m} \xi_\alpha\}_{\alpha=1}^{\infty}\,,$$

for i.i.d., positive random variables $\lambda_1, \lambda_2, \dots$. Let

$$\hat{\xi}_\alpha \;=\; \frac{\xi_\alpha}{\sum_{\alpha=1}^{\infty} \xi_\alpha}\,.$$

Then using properties of the logarithm

$$\mathbb{E}\left[\ln \sum_{\alpha=1}^{\infty} \hat{\xi}_\alpha \lambda_\alpha\right] \;=\; \frac{1}{m} \ln \mathbb{E}\left[\lambda_\alpha^m\right]\,.$$

As a preliminary step, we will calculate the upper bound from Proposition 5.4 for $\{\mu_\alpha\} = \{\hat{\xi}_\alpha\}$, with all the τ's being i.i.d., uniform on $\{1, \dots, q\}$.

We start with the formula for $G_N^{(2)}$, which is easier. The important thing is to condition on the value of L. This is because L is random, but it is not independent for the different values of α. Conditioning on L, the multipliers

$$\lambda_\alpha \;=\; \exp\left(-\beta \sum_{k=1}^{L} \delta(\tau_{\alpha,2k-1}, \tau_{\alpha,2k})\right)$$

are conditionally independent. We may calculate

$$\mathbb{E}[\lambda_\alpha^m \,|\, L] \;=\; \prod_{k=1}^{L} \mathbb{E}\left[e^{-m\beta\delta(\tau_{\alpha,2k-1},\,\tau_{\alpha,2k})}\right] \;=\; \left(1 - \frac{1}{q} + \frac{1}{q} e^{-m\beta}\right)^{L}\,.$$

So

$$\mathbb{E}\left[\ln \sum_{\alpha=1}^{\infty} \hat{\xi}_\alpha \exp\left(-\beta \sum_{k=1}^{L} \delta(\tau_{\alpha,2k-1}, \tau_{\alpha,2k})\right) \,\Big|\, L\right] \;=\; L \ln\left(1 - \frac{1 - e^{-m\beta}}{q}\right)\,.$$

Since L has mean $cN/2$, this gives

$$G_N^{(2)} \;=\; \frac{c}{2m} \ln\left(1 - \frac{1 - e^{-m\beta}}{q}\right)\,. \tag{59}$$

For $G_N^{(1)}$ we can rewrite this as

$$G_N^{(1)} \;=\; \frac{1}{N}\,\mathbb{E}\left[\ln \sum_{\alpha=1}^{\infty} \mu_\alpha \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left(-\beta \sum_{i=1}^{N} \sum_{k=1}^{K(i)} \delta(\sigma_i,\, \tau_{\alpha,i,k})\right)\right]\,,$$

where each $K(i) = |\{k \leq K \,:\, I(k) = i\}|$ is an independent Poisson random variable with parameter c. This holds because of the special property for Poisson random variables, that the sum of two independent Poisson random variables is Poisson.

Once again the random variables $K(1), \ldots, K(N)$ are the same for all α's, not independent. Therefore, we condition on $K(1), \ldots, K(N)$. In this case

$$\begin{aligned} \lambda_\alpha &= \sum_{\sigma \in \Sigma_r^N} \exp\left(-\beta \sum_{i=1}^{N} \sum_{k=1}^{K(i)} \delta(\sigma_i,\, \tau_{\alpha,i,k})\right) \\ &= \prod_{i=1}^{N} \sum_{\sigma_i=1}^{q} \exp\left(-\beta \sum_{k=1}^{K(i)} \delta(\sigma_i,\, \tau_{\alpha,i,k})\right) . \end{aligned}$$

The $\tau_{\alpha,i,k}$'s are all independent. In particular they are independent for different values of i. Therefore,

$$\mathbb{E}[\lambda_\alpha^m \,|\, K(1), \ldots, K(N)] \;=\; \prod_{i=1}^{N} \mathbb{E}^{\tau_1,\ldots,\tau_{K(i)}} \left[\left(\sum_{\sigma_i=1}^{q} e^{-\beta \sum_{k=1}^{K(i)} \delta(\sigma_i,\tau_k)}\right)^m\right] .$$

Using this, if we let κ be a Poisson random variable with mean c, then

$$G_N^{(1)} \;=\; \frac{1}{m}\, \mathbb{E}^\kappa \ln \mathbb{E}^{\tau_1,\ldots,\tau_\kappa} \left[\left(\sum_{\sigma=1}^{q} e^{-\beta \sum_{k=1}^{\kappa} \delta(\sigma,\tau_k)}\right)^m\right] . \tag{60}$$

6.2 Trivial bound

A special case arises in the limit $m \uparrow 1$. We get

$$G_N^{(2)} \;=\; \frac{c}{2}\, \ln\left(1 - \frac{1-e^{-\beta}}{q}\right) , \tag{61}$$

directly. For the more complicated term, we get

$$\begin{aligned} G_N^{(1)} &= \mathbb{E}^\kappa \ln \mathbb{E}^{\tau_1,\ldots,\tau_\kappa} \left[\sum_{\sigma=1}^{q} e^{-\beta \sum_{k=1}^{\kappa} \delta(\sigma,\tau_k)}\right] \\ &= \mathbb{E}^\kappa \ln \sum_{\sigma=1}^{q} \mathbb{E}^{\tau_1,\ldots,\tau_\kappa} \left[e^{-\beta \sum_{k=1}^{\kappa} \delta(\sigma,\tau_k)}\right] \\ &= \mathbb{E}^\kappa \ln \sum_{\sigma=1}^{q} \prod_{k=1}^{\kappa} \mathbb{E}^{\tau_k} \left[e^{-\beta \delta(\sigma,\tau_k)}\right] \\ &= \mathbb{E}^\kappa \ln \left[\sum_{\sigma=1}^{q} \left(1 - \frac{1-e^{-\beta}}{q}\right)^\kappa\right] \\ &= \mathbb{E}^\kappa \ln \left[q \left(1 - \frac{1-e^{-\beta}}{q}\right)^\kappa\right] \\ &= \ln q + c \ln\left(1 - \frac{1-e^{-\beta}}{q}\right) . \end{aligned}$$

Combining this with (61) we obtain

$$p_N(c,\beta) \;\leq\; \frac{c}{2}\, \ln\left(1 - \frac{1-e^{-\beta}}{q}\right) + \ln q \,.$$

which matches the upper bound from equation (24). This is not the output of the full replica symmetric ansatz. This is the output of the "trivial replica symmetric" ansatz: the one with overlap equal to 0, as we will see.

For the full replica symmetric ansatz, we must also allow for the possibility of a nontrivial overlap, which we parametrize by a number $p \in [0, 1]$. This is what we do next. Then the trivial RS ansatz is the one with $p = 0$.

6.3 Two level RPC

In order to derive the full replica symmetric ansatz, we need to take a two-level RPC, and then take a limiting case. For Proposition 5.4, all that is required of the μ_α's is that the index set $\{\alpha\}$ is countable. Therefore, we may consider instead of a single countable index, a pair $\alpha = (\alpha_1, \alpha_2)$.

We let $\{\xi^1_{\alpha_1}\}$ be a Poisson point process with intensity measure Λ_{m_1}. For each α_1, we let $\{\xi^2_{\alpha_2}(\alpha_1)\}$ be a Poisson point process with intensity measure Λ_{m_2}. These are independent of each other for different values of α_1. They are also independent of the first point process $\{\xi^1_{\alpha_1}\}$. Then we define

$$\xi_\alpha \, = \, \xi^1_{\alpha_1} \cdot \xi^2_{\alpha_2}(\alpha_1) \, .$$

We also define

$$\hat{\xi}_\alpha \, = \, \frac{\xi_\alpha}{\sum_{\alpha \in \mathbb{N}^2} \xi_\alpha} \, .$$

We assume that $m_1 < m_2$. The reason for this is that $\sum_{\alpha_1} \xi^1_{\alpha_1}$ has a moment only up to (and not including) m_1, and $\sum_{\alpha_2} \xi^2_{\alpha_2}(\alpha_1)$ has a moment only up to (and not including) m_2. But the distribution of

$$\left\{ \sum_{\alpha_2} \xi_{(\alpha_1, \alpha_2)} \right\}_{\alpha_1 \in \mathbb{N}}$$

is the same as the distribution of

$$\left\{ \mathbb{E}\left[\left(\sum_{\alpha_2} \xi^2_{\alpha_2}(\alpha_1) \right)^{m_1} \right]^{1/m_1} \xi_{\alpha_1} \right\}_{\alpha_1 \in \mathbb{N}} \, .$$

Since we need the sum to be finite almost surely, in order to define $\hat{\xi}_\alpha$, this means we must have $m_2 > m_1$.

In addition to the choice $\{\mu_\alpha\} = \{\hat{\xi}_\alpha\}$, we also choose an ansatz for the spins τ_α. Actually, we have $\tau_{\alpha,k}$'s, but we will assume that the distribution is i.i.d., for different values of k. So we just focus on a single value of k, and suppress that index from the notation momentarily. Let $\tau^1_{\alpha_1}$ be i.i.d., uniform on $\{1, \dots, q\}$. Let τ^2_α also be i.i.d., uniform on $\{1, \dots, q\}$, independent for every different choice of $\alpha = (\alpha_1, \alpha_2)$, even if the α_1's coincide (as long as the α_2's are different, and vice-versa). Let us also choose i.i.d., Bernoulli random variables X_α, with probability p to be 1, and probability $1 - p$ to be 0. Let

$$\tau_\alpha \, = \, X_\alpha \tau^1_{\alpha_1} + (1 - X_\alpha) \tau^2_{\alpha_2}(\alpha_1) \, .$$

In other words, the probability that τ_α is just $\tau^1_{\alpha_1}$ is p. This is a common value for the "cluster" consisting of all α's with the first part α_1.

To calculate $G^{(2)}_N$, the easier of the two parts of the cavity field functional, we condition on $\{\xi^1_{\alpha_1}\}$ and on $\{\tau^1_{\alpha_1}\}$. Since the τ^2_α's and X_α's are i.i.d., we do not condition on them. Then we note that

$$\mathbf{P}(\tau_{\alpha,2k-1} = \tau_{\alpha,2k} \,|\, \{\tau^1_{\alpha_1,k}\}) \, = \, \mathbf{P}(X_1 = X_2 = 1) \mathbf{1}\{\tau^1_{\alpha_1,2k-1} = \tau^1_{\alpha_1,2k}\} + [1 - \mathbf{P}(X_1 = X_2 = 1)] \cdot \frac{1}{q} \, .$$

This implies that

$$\mathbb{E}\left[e^{-m_2\beta\delta(\tau_{\alpha,2k-1},\tau_{\alpha,2k})}\,\middle|\,\{\tau^1_{\alpha_1}\}\right] \;=\; p^2 e^{-m_2\beta\delta(\tau^1_{\alpha_1,2k-1},\tau^1_{\alpha_1,2k})} + (1-p^2)\left(1-\frac{1-e^{-m_2\beta}}{q}\right)\,.$$

For

$$\lambda_\alpha \;=\; e^{-\beta\delta(\tau_{\alpha,2k-1},\tau_{\alpha,2k})}\,,$$

we have that

$$\{\lambda_\alpha\xi_\alpha\} \stackrel{d}{=} \{\lambda_0\xi_\alpha\}\,,$$

where

$$\lambda_0 \;=\; \mathbb{E}\left[\mathbb{E}\left[e^{-m_2\beta\delta(\tau_{\alpha,2k-1},\tau_{\alpha,2k})}\,\middle|\,\{\tau^1_{\alpha_1}\}\right]^{m_1/m_2}\right]^{1/m_1}\,.$$

So we get in general

$$\ln\lambda_0 \;=\; \frac{1}{m_1}\ln\left[\left(1-\frac{1}{q}\right)\left(1-\frac{(1-p^2)(1-e^{-m_2\beta})}{q}\right)^{m_1/m_2} + \frac{1}{q}\left(1-\left[p^2+\frac{1-p^2}{q}\right]\left(1-e^{-m_2\beta}\right)\right)^{m_1/m_2}\right]$$

Because the spin fields are independent for different values of k, the effect of the L is just to multiply this final answer. Therefore, taking the expectation of that, and dividing by N gives

$$G_N^{(2)} \;=\; \frac{c}{m_1}\ln\left[\left(1-\frac{1}{q}\right)\left(1-\frac{(1-p^2)(1-e^{-m_2\beta})}{q}\right)^{m_1/m_2} + \frac{1}{q}\left(1-\left[p^2+\frac{1-p^2}{q}\right]\left(1-e^{-m_2\beta}\right)\right)^{m_1/m_2}\right]$$

To get the replica symmetric ansatz, we use the 2-level RPC and take the limits $m_2\uparrow 1$ and $m_1\downarrow 0$. Taking $m_2\uparrow 1$, gives

$$\ln\lambda_0 \;=\; \frac{1}{m_1}\ln\left[\left(1-\frac{1}{q}\right)\left(1-\frac{(1-p^2)(1-e^{-\beta})}{q}\right)^{m_1} + \frac{1}{q}\left(1-\left[p^2+\frac{1-p^2}{q}\right]\left(1-e^{-\beta}\right)\right)^{m_1}\right]\,.$$

Then taking $m_1\downarrow 0$ gives

$$\ln\lambda_0 \;=\; \left(1-\frac{1}{q}\right)\ln\left(1-\frac{(1-p^2)(1-e^{-\beta})}{q}\right) + \frac{1}{q}\ln\left(1-\left[p^2+\frac{1-p^2}{q}\right]\left(1-e^{-\beta}\right)\right)\,.$$

Thus the replica symmetric value of the first term is

$$G_N^{(2)} \;=\; \frac{c}{2}\left(1-\frac{1}{q}\right)\ln\left(1-\frac{(1-p^2)(1-e^{-\beta})}{q}\right) + \frac{c}{2q}\ln\left(1-\left[p^2+\frac{1-p^2}{q}\right]\left(1-e^{-\beta}\right)\right)\,. \qquad (62)$$

The more complicated term is $G_N^{(1)}$. Conditioning on the spins at the first level $\{\tau^1_\alpha\}$, and all the $K(i)$ values, we get (using a notation which is clear from the context)

$$\mathbb{E}[\lambda_\alpha^{m_2}\,|\,\{K(i)\}_{i=1}^N,\{\tau^1_{\alpha,i,k}\}_{\alpha,i,k}] \;=\; \prod_{i=1}^N \mathbb{E}^{\{X_{i,k}\},\{\tau^2_{i,k}\}}\left[\left(\sum_{\sigma_i=1}^q\prod_{k=1}^{K(i)}\left[e^{-\beta\delta(\sigma_i,\tau_{i,k})}\right]\right)^{m_2}\right]\,,$$

where the $\tau^2_{i,k}$ are all i.i.d., uniform on $\{1,\dots,q\}$, and

$$\tau_{i,k} = X_{i,k}\tau^1_{i,k} + (1-X_{i,k})\tau^2_{i,k}\,.$$

The $X_{i,k}$'s are i.i.d., Bernoulli-p random variables. Since the formulas are identically distributed for different i's and since there are N such i's (cancelling the division by N), we get the formula

$$G_N^{(1)} \;=\; \frac{1}{m_1}\mathbb{E}^\kappa\ln\mathbb{E}^{\{\tau^1_k\}}\left[\mathbb{E}^{\{X_k\},\{\tau^2_k\}}\left[\left(\sum_{\sigma=1}^q\prod_{k=1}^{\kappa}\left[e^{-\beta\delta(\sigma,\tau_k)}\right]\right)^{m_2}\right]^{m_1/m_2}\right]\,,$$

where once again κ is a Poisson random variable with mean c, and now $\{\tau_k^1\}$ and $\{\tau_k^2\}$ are all i.i.d., uniform random variables on $\{1,\dots,q\}$, and $\{X_k\}$ are all i.i.d., Bernoulli random variables with mean p, and $\tau_k = X_k\tau_k^1 + (1-X_k)\tau_k^2$ for each k.

We now take $m_2 \uparrow 1$ and $m_1 \downarrow 0$. Taking $m_2 \uparrow 1$ gives

$$\frac{1}{m_1}\mathbb{E}^\kappa \ln \mathbb{E}^{\{\tau_k^1\}}\left[\mathbb{E}^{\{X_k\},\{\tau_k^2\}}\left[\sum_{\sigma=1}^q \prod_{k=1}^\kappa \left[e^{-\beta\delta(\sigma,\tau_k)}\right]\right]^{m_1}\right],$$

which can be rewritten

$$\frac{1}{m_1}\mathbb{E}^\kappa \ln \mathbb{E}^{\{\tau_k^1\}}\left[\left(\sum_{\sigma=1}^q \prod_{k=1}^\kappa \mathbb{E}^{X_k,\tau_k^2}\left[e^{-\beta\delta(\sigma,\tau_k)}\right]\right)^{m_1}\right].$$

But

$$\mathbb{E}^{X_k,\tau_k^2}\left[e^{-\beta\delta(\sigma,\tau_k)}\right] = pe^{-\beta\delta(\sigma,\tau_k^1)} + (1-p)\left(1-\frac{1-e^{-\beta}}{q}\right).$$

Using this and taking the limit $m_1 \downarrow 0$ gives

$$G_N^{(1)} = \mathbb{E}^{\kappa,\{\tau_k^1\}}\left[\ln\left(\sum_{\sigma=1}^q \prod_{k=1}^\kappa \left(pe^{-\beta\delta(\sigma,\tau_k^1)} + (1-p)\left(1-\frac{1-e^{-\beta}}{q}\right)\right)\right)\right].$$

Let us now rewrite this in a manner which is appropriate for taking derivatives at $p=0$. We can write $e^{-\beta\delta(\sigma,\tau_k^1)} = 1-(1-e^{-\beta})\delta(\sigma,\tau_k^1)$. Since the *average* value of $\delta(\sigma,\tau_k^1)$ is $1/q$, we may also incorporate that:

$$e^{-\beta\delta(\sigma,\tau_k^1)} = 1-\frac{1-e^{-\beta}}{q} - (1-e^{-\beta})(\delta(\sigma,\tau_k^1)-q^{-1}).$$

Then we may rewrite $\mathbb{E}^{X_k,\tau_k^2}\left[e^{-\beta\delta(\sigma,\tau_k)}\right]$ as

$$\left(1-\frac{1-e^{-\beta}}{q}\right) - p(1-e^{-\beta})(\delta(\sigma,\tau_k^1)-q^{-1}).$$

The formula for $G_N^{(1)}$ is simpler if we introduce a new variable, $x=(1-e^{-\beta})/(q-e^{-\beta})$. Therefore, we obtain

$$G_N^{(1)} = \ln q + c\ln\left(1-\frac{1-e^{-\beta}}{q}\right) + \mathbb{E}^{\kappa,\{\tau_k^1\}}\left[\ln \mathbb{E}^\sigma\left[\prod_{k=1}^\kappa \left(1-px\left(q\delta(\sigma,\tau_k^1)-1\right)\right)\right]\right]\Bigg|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}. \quad (63)$$

6.4 Local Replica Symmetric stability analysis of $p=0$ and the critical temperature

Now we want to consider this formula as a function of p perturbatively near 0. This allows us to derive some results of Zdeborová and Krząkała [25] in the present context. We say that the $p=0$ RS ansatz is "stable to RS perturbations" if it is a local minimizer of the extended variational principle in the set of RS ansatze.

Starting from the simpler term, (62), we rewrite $G_N^{(2)}$ as

$$\frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right) + \frac{c(q-1)}{2q}\ln\left(1+\frac{p^2(1-e^{-\beta})}{q-(1-e^{-\beta})}\right) + \frac{c}{2q}\ln\left(1-\frac{(q-1)(1-e^{-\beta})p^2}{q-(1-e^{-\beta})}\right).$$

Using $x=(1-e^{-\beta})/(q-e^{-\beta})$ this is simpler:

$$G_N^{(2)} = \frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right) + \frac{c}{2q}\left[(q-1)\ln(1+p^2x) + \ln(1-(q-1)p^2x)\right]\Bigg|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}. \quad (64)$$

This is an even function of p, so only even powers will appear. Taylor expansion shows that

$$(q-1)\ln(1+p^2x) + \ln(1-(q-1)p^2x) \;=\; -\frac{1}{2}\,q(q-1)p^4x^2\,.$$

Therefore,

$$G_N^{(2)} \;=\; \frac{c}{2}\,\ln\left(1-\frac{1-e^{-\beta}}{q}\right) - \frac{c(q-1)p^4x^2}{4} + O(p^6)\Big|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}\,.$$

This gives

$$\frac{d^4}{dp^4}G_N^{(2)}\bigg|_{p=0} \;=\; -6c(q-1)x^2\Big|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}\,. \tag{65}$$

Now turning to the more difficult term, let us start with (63). Let us write $f(\sigma,\tau) = (q\delta(\sigma,\tau)-1)$. Then we have

$$G_N^{(1)} \;=\; \ln q + c\ln\left(1-\frac{1-e^{-\beta}}{q}\right) + \mathbb{E}^{\kappa,\{\tau_k^1\}}\left[\ln \mathbb{E}^{\sigma}\left[\prod_{k=1}^{\kappa}\left(1-pxf(\sigma,\tau_k^1)\right)\right]\right]\Bigg|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}\,. \tag{66}$$

As usual, we may interpret the function

$$\mathbb{E}^{\sigma}\left[\prod_{k=1}^{\kappa}\left(1-pxf(\sigma,\tau_k^1)\right)\right]$$

as a cumulant generating function. But the random variable is multi-linear in p. Therefore, when expanding in p, we have to take account of these terms. Also, notice that $\mathbb{E}^{\sigma}[f(\sigma,\tau)] = \mathbb{E}^{\tau}[f(\sigma,\tau)] = 0$ as long as the expectations are with respect to the uniform measure. Because of this, various terms vanish either in the expectation over $\mathbb{E}^{\sigma}$ or in the expectation over $\mathbb{E}^{\{\tau_k^1\}}$.

For instance, using the fact that $\mathbb{E}^{\sigma}[f(\sigma,\tau)] = 0$, we see that the first derivative in p equals 0. Moreover, since each factor is linear in p, in taking multiple derivatives (of a single copy of the product) means we cannot repeat the derivative of any factor. So we obtain

$$\frac{d^2}{dp^2}\,\mathbb{E}^{\sigma}\left[\prod_{k=1}^{\kappa}\left(1-pxf(\sigma,\tau_k^1)\right)\right] \;=\; x^2\sum_{\substack{j,k=1\\ j\neq k}}^{\kappa}\mathbb{E}^{\sigma}[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)]\,.$$

But then taking the expectation over $\mathbb{E}^{\{\tau_k^1\}}$ gives 0 because since $j\neq k$, we have

$$\mathbb{E}^{\{\tau_k^1\}}\mathbb{E}^{\sigma}[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)] \;=\; \mathbb{E}^{\sigma}\left[\mathbb{E}^{\tau_j^1}[f(\sigma,\tau_j^1)]\cdot\mathbb{E}^{\tau_k^1}[f(\sigma,\tau_k^1)]\right] \;=\; 0\,.$$

Continuing, we may easily see that the third derivative is again 0 since $\mathbb{E}^{\sigma}[f(\sigma,\tau)] = 0$. Then, the next simplest term arises from

$$\begin{aligned}\frac{d^4}{dp^4}\,\mathbb{E}^{\sigma}\left[\prod_{k=1}^{\kappa}\left(1-pxf(\sigma,\tau_k^1)\right)\right] \;=\;& x^4\sum_{\substack{j,k,\ell,m=1\\ j\neq k\neq \ell\neq m}}^{\kappa}\mathbb{E}^{\sigma}[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)f(\sigma,\tau_\ell^1)f(\sigma,\tau_m^1)]\\ &-3x^4\left(\sum_{\substack{j,k=1\\ j\neq k}}^{\kappa}\mathbb{E}^{\sigma}[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)]\right)^2\,.\end{aligned}$$

We can rewrite this by expanding the square of the sum, and using replicated spin variables for products of expectations:

$$\frac{d^4}{dp^4}\mathbb{E}^\sigma\left[\prod_{k=1}^{\kappa}\left(1-pxf(\sigma,\tau_k^1)\right)\right] = x^4\sum_{\substack{j,k,\ell,m=1\\ j\neq k\neq \ell\neq m}}^{\kappa}\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)f(\sigma,\tau_\ell^1)f(\sigma,\tau_m^1)]$$
$$-3x^4\sum_{\substack{j,k=1\\ j\neq k}}^{\kappa}\sum_{\substack{\ell,m=1\\ \ell\neq m}}^{\kappa}\mathbb{E}^{\sigma,\sigma'}[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)f(\sigma',\tau_\ell^1)f(\sigma',\tau_m^1)]\,.$$

Any distinct terms for j,k,ℓ,m vanish in the expectation over $\mathbb{E}^{\{\tau_k^1\}}$. Therefore all must be paired. That means that the first summand vanishes entirely. In the second summand, we require $(\ell,m)=(j,k)$ or $(\ell,m)=(k,j)$. These two possibilities give an extra factor of 2. Hence, we obtain

$$\left.\frac{d^4}{dp^4}G_N^{(1)}\right|_{p=0} = -6x^4\,\mathbb{E}^{\kappa,\{\tau_k^1\}}\sum_{\substack{j,k=1\\ j\neq k}}^{\kappa}\left.\left(\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)]\right)^2\right|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}\,.$$

A calculation gives

$$\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)] = \begin{cases}-1 & \text{if } \tau_j^1\neq\tau_k^1,\\ (q-1) & \text{if } \tau_j^1=\tau_k^1.\end{cases}$$

Using the i.i.d., uniform distribution on $\{\tau_k^1\}$ gives $\mathbf{P}\{\tau_j^1=\tau_k^1\}=1/q$. Therefore,

$$\mathbb{E}^{\{\tau_k^1\}}\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)] = 0\,,$$

as we claimed before. But now we also have

$$\left(\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)]\right)^2 = \begin{cases}1 & \text{if } \tau_j^1\neq\tau_k^1,\\ (q-1)^2 & \text{if } \tau_j^1=\tau_k^1,\end{cases}$$

which gives

$$\mathbb{E}^{\{\tau_k^1\}}\left[\left(\mathbb{E}^\sigma[f(\sigma,\tau_j^1)f(\sigma,\tau_k^1)]\right)^2\right] = q-1\,.$$

Therefore, also using the fact that $\mathbb{E}^\kappa[\#\{(j,k)\in\{1,\dots,\kappa\}^2\,:\,j\neq k\}]$ equals $\mathbb{E}^\kappa[\kappa(\kappa-1)]=c^2$, we obtain

$$\left.\frac{d^4}{dp^4}G_N^{(1)}\right|_{p=0} = -6c^2(q-1)\left.x^4\right|_{x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}}\,. \tag{67}$$

Since the RS ansatz is $G_N^{(1)}-G_N^{(2)}$ this means that the local stability of the trivial RS ansatz is determined by the sign of the difference between the terms in (67) and (65)

$$\frac{d^4}{dp^4}\left[G_N^{(1)}-G_N^{(2)}\right] = -6c^2(q-1)x^4+6c(q-1)x^2 = 6c(q-1)x^2[1-cx^2]\,.$$

We assume $q>1$ otherwise the model is trivial (the 1-state Potts model). So the $p=0$ solution is locally stable if and only if

$$1-cx^2 > 0 \quad\text{ for }\quad x=\frac{1-e^{-\beta}}{q-(1-e^{-\beta})}\,.$$

Note that $(\beta=0)\Rightarrow(x=0)$ while $(\beta\to\infty)\Rightarrow(x\to\frac{1}{q-1})$. Therefore, we see that the $p=0$ solution is stable at all temperatures if $c<c_*$, where

$$c_* = (q-1)^2\,. \tag{68}$$

For instance, for the Ising model which has $q = 2$, this merely states that $c_* = 1$, which is the percolation threshold. For $c > c_*$ there is a giant component, and for $c < c_*$ there is not.

If $c > c_*$ then the critical temperature for stability is given by the condition

$$1 - cx^2 = 0 \quad \Leftrightarrow \quad x = \frac{1}{\sqrt{c}} \quad \Leftrightarrow \quad \frac{1}{x} = \sqrt{c}\,.$$

Since

$$x = \frac{1 - e^{-\beta}}{q - (1 - e^{-\beta})} \quad \Leftrightarrow \quad \frac{1}{x} = \frac{q}{1 - e^{-\beta}} - 1\,,$$

this means that the critical value of β satisfies

$$\frac{q}{1 - e^{-\beta}} - 1 = \sqrt{c} \quad \Leftrightarrow \quad 1 - e^{-\beta} = \frac{q}{1 + \sqrt{c}} \quad \Leftrightarrow \quad e^{-\beta} = 1 - \frac{q}{1 + \sqrt{c}}\,.$$

So

$$\beta_* = -\ln\left(1 - \frac{q}{1 + \sqrt{c}}\right)\,. \tag{69}$$

7 Replica Symmetric Solution

We would now like to compare to Dembo and Montanari's results [10], as well as those of [11]. So far we have imposed the condition that all the τ_k^1's and τ_k^2's are i.i.d., uniform on $\{1, \dots, q\}$, and

$$\tau_k = \begin{cases} \tau_k^1 & \text{if } X_k = 1, \\ \tau_k^2 & \text{if } X_k = 0. \end{cases}$$

This is a very simple special case of the replica symmetric ansatz. The more general case yields the following result.

Theorem 7.1 *The quenched pressure is bounded from above by the Replica Symmetric pressure. Namely,*

$$p(c, \beta) \le \mathbb{E} \ln \left[\sum_{s=1}^{q} \prod_{k=1}^{\kappa} \left(1 - (1 - e^{-\beta}) P_k(s)\right)\right] - \frac{c}{2} \mathbb{E} \ln \left[1 - (1 - e^{-\beta}) \sum_{s=1}^{q} P_1(s) P_2(s)\right]\,, \tag{70}$$

where κ is a Poisson random variables with mean c, the $P_k = (P_k(s))_{s=1}^q$ are i.i.d. random probability vectors, i.e. $P_k(s) \ge 0$ for each s and $\sum_{s=1}^q P_k(s) = 1$ a.s., satisfying the following equality in distribution

$$(P_1(s))_{s=1}^q \stackrel{d}{=} \left(\frac{\prod_{k=1}^{\kappa} \left(1 - (1 - e^{-\beta}) P_k(s)\right)}{\sum_{s=1}^{q} \prod_{k=1}^{\kappa} \left(1 - (1 - e^{-\beta}) P_k(s)\right)}\right)_{s=1}^{q}\,. \tag{71}$$

Proof: Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Suppose that $\mathcal{F}_1$ is a sub-σ-algebra such that $(\xi^1_{\alpha_1})_{\alpha_1=1}^\infty$ are $\mathcal{F}_1$-measurable, and have the appropriate Poisson point process distribution with intensity $\xi^{m_1} d\xi$, as stated before. Also, suppose that $(\xi^2_{\alpha_1,\alpha_2})_{\alpha_2=1}^\infty$ are independent of $\mathcal{F}_1$, and independent of each other for different α_1, and distributed according to the Poisson point process distribution with intensity $\xi^{m_2} d\xi$, for $0 < m_1 < m_2 < 1$. Finally, suppose that the conditional distribution $\{\tau_{(\alpha_1,\alpha_2),k}\}_{\alpha_2=1}^\infty$, conditioned on $\mathcal{F}_1$ are independent:

$$\mathbb{P}(\tau_{(\alpha_1,1),k} \in A_1,\, \tau_{(\alpha_1,2),k} \in A_2,\, \cdots \,|\, \mathcal{F}_1) = \prod_{\alpha_2=1}^{\infty} \mathbb{P}(\tau_{(\alpha_1,\alpha_2),k} \in A_{\alpha_2} \,|\, \mathcal{F}_1)\,,$$

assuming that only finitely many A_n's are different than $\{1, \dots, q\}$. Also, assume that the $\tau_{(\alpha_1,\alpha_2),k}$'s are independent for different values of α_1.

Finally, assume that K and L are independent of all the ξ_α's and $\tau_{\alpha,k}$'s as before. For notational simplicity, suppose that there is a $\mathcal{F}_0$ such that K and L are $\mathcal{F}_0$-measurable and such that all the ξ_α's and τ_{α_k}'s are independent of $\mathcal{F}_0$.

Then the extended variational principle works as before with

$$G_N^{(1)} \;=\; \frac{1}{m_1}\mathbb{E}\ln\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{s=1}^{q} e^{-\sum_{k=1}^{\kappa}\beta\delta(s,\tau_k)}\right)^{m_2}\,\Big|\,\mathcal{F}_1\right]\,\Big|\,\mathcal{F}_0\right]^{m_1/m_2},$$

where we write τ_k for a generic $\tau_{\alpha,k}$, and κ is a Poisson-c random variable, measurable with respect to $\mathcal{F}_0$. Similarly,

$$G_N^{(2)} \;=\; \frac{c}{2m_1}\mathbb{E}\ln\mathbb{E}\left[\mathbb{E}\left[e^{-m_2\beta\delta(\tau_1,\tau_2)}\,\big|\,\mathcal{F}_1\right]^{m_1/m_2}\,\Big|\,\mathcal{F}_0\right].$$

Taking the limit $m_1\downarrow 0$ and $m_2\uparrow 1$ gives

$$G_N^{(1)} \;=\; \mathbb{E}\ln\mathbb{E}\left[\sum_{s=1}^{q} e^{-\sum_{k=1}^{\kappa}\beta\delta(s,\tau_k)}\,\Big|\,\mathcal{F}_1\right],$$

and

$$G_N^{(2)} \;=\; \frac{c}{2}\mathbb{E}\ln\mathbb{E}\left[e^{-\beta\delta(\tau_1,\tau_2)}\,\big|\,\mathcal{F}_1\right].$$

The most general choice is to have $\mathcal{F}_1$-measurable i.i.d., random vectors $P_k = (P_k(s))_{s=1}^q$ chosen according to a distribution with support in the set such that $P_k(s)\geq 0$ for each s, and $\sum_{s=1}^q P_k(s)=1$, a.s. Then each τ_k is P_k-distributed. In this case,

$$G_N^{(2)} \;=\; \frac{c}{2}\mathbb{E}\ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_1(s)P_2(s)\right],$$

and

$$G_N^{(1)} \;=\; \mathbb{E}\ln\left[\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\left(1-(1-e^{-\beta})P_k(s)\right)\right],$$

Assuming an underlying distribution $d\rho(P)$,

$$G_N^{(2)} \;=\; \frac{c}{2}\int\ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_1(s)P_2(s)\right]\,d\rho(P_1)\,d\rho(P_2)\,,$$

and

$$G_N^{(1)} \;=\; \mathbb{E}^{\kappa}\int\ln\left[\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\left(1-(1-e^{-\beta})P_k(s)\right)\right]\,\prod_{k=1}^{\kappa}d\rho(P_k)\,.$$

Taking a Frechet derivative of ρ in the direction of $\delta\rho$ gives a condition for criticality/extremality:

$$\begin{aligned} c\int\ln&\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_1(s)P_2(s)\right]\,d\delta\rho(P_1)\,d\rho(P_2)\\ &= \mathbb{E}^{\kappa}\left[\kappa\int\ln\left[\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\left(1-(1-e^{-\beta})P_k(s)\right)\right]\,d\delta\rho(P_1)\,\prod_{k=2}^{\kappa}d\rho(P_k)\right]. \end{aligned}$$

Here we have used the product rule (Liebniz rule) and the symmetry of the formulas with respect to permutations of the indices on the P_k's. In the first integral, both $\rho(dP_1)$ and $\rho(dP_2)$ must be differentiated which gives a factor of 2. In the second one, we get an extra factor of κ. Note that

$\mathbb{E}[\kappa f(\kappa)] = c\mathbb{E}[f(\kappa+1)]$. Note that because of the differentiation, P_1 had distribution $\delta\rho$, and it is only the $\kappa-1$ last P_k's that had distribution ρ. Therefore, rewriting P_1 as P_0 and shifting, we have

$$\int \ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_0(s)P_1(s)\right] d\delta\rho(P_0)\, d\rho(P_1)$$
$$= \mathbb{E}^{\kappa}\left[\int \ln\left[\sum_{s=1}^{q}\big(1-(1-e^{-\beta})P_0(s)\big)\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)\right] d\delta\rho(P_0)\prod_{k=1}^{\kappa} d\rho(P_k)\right].$$

Note that, in order for $\rho+\delta\rho$ to be a probability measure, given that ρ is already a probability measure, it is necessary that $\delta\rho$ is a signed measure, with total measure 0. Therefore, if we integrate any function against $\delta\rho(dP_0)$, which is constant with respect to P_0, then the integral is zero. Using this, and properties of the logarithm, we see that

$$\int \ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_0(s)P_1(s)\right] d\delta\rho(P_0)\, d\rho(P_1)$$
$$= \mathbb{E}^{\kappa}\left[\int \ln\left[\sum_{s=1}^{q}\big(1-(1-e^{-\beta})P_0(s)\big)\frac{\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}{\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}\right] d\delta\rho(P_0)\prod_{k=1}^{\kappa} d\rho(P_k)\right].$$

We have normalized the multiplier, so that summed over s it just gives 0. Therefore, we may rewrite this by distributing the two terms in the factor $(1-(1-e^{-\beta})P_0(s))$, to get

$$\int \ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q} P_0(s)P_1(s)\right] d\delta\rho(P_0)\, d\rho(P_1)$$
$$= \mathbb{E}^{\kappa}\left[\int \ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q}P_0(s)\frac{\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}{\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}\right] d\delta\rho(P_0)\prod_{k=1}^{\kappa} d\rho(P_k)\right].$$

Since $\delta\rho$ is general, this implies that the condition for extremality is

$$(P_1(s))_{s=1}^{q} \stackrel{d}{=} \left(\frac{\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}{\sum_{s=1}^{q}\prod_{k=1}^{\kappa}\big(1-(1-e^{-\beta})P_k(s)\big)}\right)_{s=1}^{q}.$$

□

8 High temperature region

A guess at high temperature would be that $\omega_N(\delta(\sigma_i,\sigma_j))$ is about $1/q$ generically (for $i\neq j$) since that is the exact value at infinite temperature. A calculation, using (25) and (26), shows that

$$p_N(c,\beta)-p^{TRS}(c,\beta) = \frac{c}{2}\int_0^1 \mathbb{E}\left[\ln\left(1-a_q(\beta)\omega_{N,ct,\beta}\left(\delta(\sigma_i,\sigma_j)-q^{-1}\right)\right)\right] dt\,,$$

where $\omega_{N,c,\beta}$ is the Boltzmann-Gibbs state at inverse temperature β, associated to the Hamiltonian $H_N(c,\sigma)$ (here we write explicitly the dependence on the parameter c) and

$$a_q(\beta) = \frac{1-e^{-\beta}}{1-(1-e^{-\beta})q^{-1}}.$$

An important observation is that

$$a_q(\beta)[1-q^{-1}] = 1-\frac{e^{-\beta}}{1-(1-e^{-\beta})q^{-1}} \in [0,1) \quad \text{for } q\geq 1 \text{ and } \beta\geq 0.$$

By Taylor's theorem, we may write

$$\ln(1-ax) \;=\; -\sum_{k=1}^{n} \frac{a^k x^k}{k} + R_{k+1}(a,x)\,,$$

where $R_{k+1}(a,x)$ is the remainder term, of order $(ax)^{k+1}$:

$$R_{k+1}(a,x) \;=\; -(ax)^{k+1} \int_0^1 \frac{(1-t)^k}{(1-ax)^{k+1}}\,dx\,.$$

We want to consider this expansion with $a = a_q(\beta)$ and $x = \delta(\sigma_i,\sigma_j) - q^{-1}$. In particular, we notice that for $ax \le \mu < 1$, we can bound

$$R_{k+1}(a,x) \;\ge\; -\frac{1}{k+1}\left(\frac{ax}{1-\mu}\right)^{k+1}\,.$$

If also k is odd, then we know that $R_{k+1}(a,x) \ge 0$. This leads to the family of bounds: for any $c,\beta \ge 0$, and any integer $\ell \in \{1,2,\dots\}$,

$$\begin{aligned} 0 \;\ge\; p_N(c,\beta) - p^{TRS}(c,\beta) \;\ge\; & -\frac{c}{2N^2}\sum_{i,j=1}^{N}\sum_{k=1}^{2\ell-1}\frac{a_q^k(\beta)}{k}\int_0^1 \mathbb{E}\left[\omega_{N,ct,\beta}^{k}\left(\delta(\sigma_i,\sigma_j)-q^{-1}\right)\right]\,dt \\ & -\frac{cb_{2\ell}(q,\beta)}{2N^2}\sum_{i,j=1}^{N}\int_0^1 \mathbb{E}\left[\omega_{N,ct,\beta}^{2\ell}\left(\delta(\sigma_i,\sigma_j)-q^{-1}\right)\right]\,dt\,, \end{aligned}$$

where

$$b_{2\ell}(q,\beta) \;=\; a_q^{2\ell}(\beta)\int_0^1 \frac{(1-t)^{2\ell-1}}{(1-a_q(\beta)[1-q^{-1}]t)^{2\ell}}\,dt\,.$$

We will do a second order approximation, so we take $\ell = 1$, and note that

$$b_2(q,\beta) \;=\; a_q^2(\beta)\int_0^1 \frac{(1-t)}{(1-a_q(\beta)[1-q^{-1}]t)^2}\,dt\,.$$

In terms of doing the second order approximation, since the mean-field antiferromagnet has the correct first order behavior and the SK mean-field spin glass has the correct second order behavior, we introduce this as an intermediate step. Let us define

$$H_N^{(2)}(\sigma) \;=\; \frac{1}{2N}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j)\,,$$

which is the mean-field antiferromagnetic term, and

$$H_N^{(3)}(\sigma) \;=\; -\frac{1}{\sqrt{2N}}\sum_{i,j=1}^{N} Y_{ij}\,\delta(\sigma_i,\sigma_j)\,,$$

where the Y_{ij}'s are i.i.d., standard, normal random variables. All the random variables are supposed to be independent. We will henceforth write the original model with a superscript,

$$H_N^{(1)}(c,\sigma) \;=\; H_N(c,\sigma)\,,$$

in order to look like the other two terms.

Let us define

$$\tilde{Z}_N \,=\, \tilde{Z}_N(c,\lambda,\kappa) \,=\, \sum_{\sigma\in\{1,\dots,q\}^N} \exp\left[-\beta H_N^{(1)}(c,\sigma) - \lambda H_N^{(2)}(\sigma) - \sqrt{\kappa}\, H_N^{(3)}(\sigma)\right] .$$

The important parameter β will be implicit in this notation, since it will stay constant. The parameters c, λ and κ will vary. We write $\tilde{\omega}_N$ for the expectation with respect to the probability measure on $\{1,\dots,q\}^N$ with weights

$$\frac{\exp\left[-\beta H_N^{(1)}(c,\sigma) - \lambda H_N^{(2)}(\sigma) - \sqrt{\kappa}\, H_N^{(3)}(\sigma)\right]}{\tilde{Z}_N} . \tag{72}$$

Then the easiest calculation is

$$\frac{\partial}{\partial\lambda}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, -\frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N(\delta(\sigma_i,\sigma_j))\right] . \tag{73}$$

By Gaussian integration by parts, we have

$$\frac{\partial}{\partial\kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, \frac{1}{4N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N\left(\delta^2(\sigma_i,\sigma_j)\right) - \tilde{\omega}_N^2\left(\delta(\sigma_i,\sigma_j)\right)\right] .$$

But, of course, $\delta^2(\sigma_i,\sigma_j) = \delta(\sigma_i,\sigma_j)$. So this gives

$$\frac{\partial}{\partial\kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, \frac{1}{4N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N\left(\delta(\sigma_i,\sigma_j)\right)\right] - \frac{1}{4N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N^2\left(\delta(\sigma_i,\sigma_j)\right)\right] . \tag{74}$$

For later reference, we note that this implies

$$\frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N(\delta(\sigma_i,\sigma_j))\right] \,=\, -\frac{\partial}{\partial\lambda}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] , \tag{75}$$

and

$$\begin{aligned}\frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N^2\left(\delta(\sigma_i,\sigma_j)\right)\right] &= -2\,\frac{\partial}{\partial\kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] + \frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N\left(\delta(\sigma_i,\sigma_j)\right)\right] \\ &= -2\,\frac{\partial}{\partial\kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] - \frac{\partial}{\partial\lambda}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] .\end{aligned} \tag{76}$$

Finally, by the Poisson differentiation formula (cfr (26)),

$$\frac{\partial}{\partial c}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, \frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\ln\left(1 - [1-e^{-\beta}]\tilde{\omega}_N(\delta(\sigma_i,\sigma_j))\right)\right] .$$

But the power series and the bounds for the logarithm are still valid:

$$\begin{aligned}\frac{\partial}{\partial c}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,\geq\, \frac{1}{2}\,\ln\left(1 - \frac{1-e^{-\beta}}{q}\right) &- \frac{a_q(\beta)}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N(\delta(\sigma_i,\sigma_j) - q^{-1})\right] \\ &- \frac{b_2(q,\beta)}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N^2(\delta(\sigma_i,\sigma_j) - q^{-1})\right] .\end{aligned}$$

Expanding out the terms just involving q^{-1}, we see that

$$\begin{aligned}\frac{\partial}{\partial c}\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N\right] \;\geq\; & \frac{1}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right)+\frac{a_q(\beta)}{2q}-\frac{b_2(q,\beta)}{2q^2} \\ & -\frac{a_q(\beta)-2b_2(q,\beta)q^{-1}}{2N^2}\sum_{i,j=1}^{N}\mathbb{E}\left[\tilde{\omega}_N(\delta(\sigma_i,\sigma_j))\right] \\ & -\frac{b_2(q,\beta)}{2N^2}\sum_{i,j=1}^{N}\mathbb{E}\left[\tilde{\omega}_N^2(\delta(\sigma_i,\sigma_j))\right]\,.\end{aligned} \tag{77}$$

Now we can use the formulas (75) and (76) to obtain

$$\begin{aligned}\frac{\partial}{\partial c}\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N\right] \;\geq\; & \frac{1}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right)+\frac{a(\beta)}{2q}-\frac{b(\beta)}{2q^2} \\ & +[a(\beta)-2b(\beta)q^{-1}+b(\beta)]\,\frac{\partial}{\partial\lambda}\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N\right] \\ & +2b(\beta)\,\frac{\partial}{\partial\kappa}\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N\right]\,.\end{aligned}$$

Therefore, for a fixed β, we take a curve in (c,λ,κ) space:

$$c(t)\;=\;c_0-t\,,\quad \lambda(t)\;=\;[a(\beta)-2b(\beta)q^{-1}+b(\beta)]t\quad\text{and}\quad\kappa(t)\;=\;2b(\beta)t\,.$$

Then the differential inequality above and the fundamental theorem of calculus gives the bound

$$\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N\right]\Bigg|_{t=0}^{t=c_0}\;\leq\;-\frac{c_0}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right)-\frac{c_0a(\beta)}{2q}+\frac{c_0b(\beta)}{2q^2}\,.$$

(This is reminiscent of Gronwall's inequality, in that one integrates a differential inequality to get a typical bound.) In other words,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N(c,0,0)\right]\;\geq\;&\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N(0,[a(\beta)-2b(\beta)q^{-1}+b(\beta)]c,2b(\beta)c)\right] \\ & +\frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right)+c\left(\frac{a(\beta)}{2q}-\frac{b(\beta)}{2q^2}\right)\,.\end{aligned}$$

The previous inequality tell us that to study the high temperature of the antiferromagnetic model on the Erdős-Rényi we are led to consider $Z_N(0,\lambda,\kappa)$ for sufficiently small λ and κ. Interestingly, this is a model we have not seen studied before: it is a mean-field antiferromagnet in conjunction with a mean-field spin glass. The two models are conducive to study together since they both satisfy inequalities going in the same direction. However, as the mean-field antiferromagnet never has a phase transition, it is not surprising to learn that it does not make any significant change to the behavior of the spin glass at high temperature.

Lemma 8.1 *For λ and κ in some open neighborhood of the origin, we have*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\ln\tilde{Z}_N(0,\lambda,\kappa)\right]\;=\;\ln q-\frac{\lambda}{2q}+\frac{\kappa(q-1)}{4q^2}\,.$$

We prove this result in the appendix. For now, we mention how it results in the high temperature region for the diluted antiferromagnet.

Corollary 8.2 *There exist $\beta^* > 0$ such that for all $\beta < \beta^*$ we have the asymptotic lower bound,*

$$p(c,\beta) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N(c,0,0)\right] \geq \ln q + \frac{c}{2}\ln\left(1 - \frac{1-e^{-\beta}}{q}\right) = p^{TRS}(c,\beta)\,.$$

Combined with Lemma 4.1 this implies, for all $\beta < \beta^*$,

$$p(c,\beta) = p^{TRS}(c,\beta)\,. \tag{78}$$

Proof: (Proof of the corollary.) We already know

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N(c,0,0)\right] \;\geq\; & \mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N(0,[a(\beta)-2b(\beta)q^{-1}+b(\beta)]c, 2b(\beta)c)\right] \\ & + \frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right) + c\left(\frac{a(\beta)}{2q} - \frac{b(\beta)}{2q^2}\right)\,.\end{aligned} \tag{79}$$

Setting

$$\lambda \;=\; [a(\beta) - 2b(\beta)q^{-1} + b(\beta)]c\,,$$

and

$$\kappa \;=\; 2b(\beta)c\,,$$

we see that

$$\tilde{Z}_N(0,[a(\beta)-2b(\beta)q^{-1}+b(\beta)]c, 2b(\beta)c) \;=\; \tilde{Z}_N(0,\lambda,\kappa)\,.$$

But by the lemma, we know that

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\left[\ln \tilde{Z}_N(0,\lambda,\kappa)\right] \;=\; \ln q - \frac{\lambda}{2q} + \frac{\kappa(q-1)}{4q^2}\,.$$

In this case we see that

$$-\frac{\lambda}{2q} + \frac{\kappa(q-1)}{4q^2} \;=\; -\frac{[a(\beta)-2b(\beta)q^{-1}+b(\beta)]c}{2q} + \frac{2b(\beta)c(q-1)}{4q^2}\,.$$

Simplifying, this is

$$-\frac{a(\beta)c}{2q} + \frac{b(\beta)c}{2q^2}\,,$$

thus we see that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N(0,[a(\beta)-2b(\beta)q^{-1}+b(\beta)]c, 2b(\beta)c)\right] \;=\; \ln q - c\left(\frac{a(\beta)}{2q} - \frac{b(\beta)}{2q^2}\right)\,.$$

Putting this together with (79) gives the corollary. □

Remark 8.3 *It should be mentioned that the second moment method does not work to control the high temeperature region, because Z_N does not concentrate around its mean. To show this, note that for a Poisson random variable J with mean λ, we have*

$$\mathbb{E}e^{-\beta J} = \sum_{k=1}^{\infty} e^{-\beta k} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda e^{-\beta}} = e^{-\lambda(1-e^{-\beta})}. \tag{80}$$

Using this, a simple computation shows that the second moment of Z_N equals

$$\begin{aligned}\mathbb{E}[Z_N^2] \;=\; & \textstyle\sum_{\sigma,\sigma'\in\{1,\dots,q\}^N} \quad \exp\Bigg\{-\frac{cN}{2}(1-e^{-\beta})\sum_{r_1=1}^{q}\left(q_N^2(r_1) + q_N'^2(r_1)\right) \\ & + \frac{cN}{2}(1-e^{-\beta})^2 \sum_{r_1,r_2=1}^{q} q_N^2(r_1,r_2)\Bigg\}\,,\end{aligned} \tag{81}$$

Compared to

$$\mathbb{E}[Z_N]^2 = \sum_{\sigma,\sigma' \in \{1,\dots,q\}^N} \exp\left\{ -\frac{cN}{2}(1-e^{-\beta}) \sum_{r_1=1}^{q} \left(q_N^2(r_1) + q_N'^2(r_1) \right) \right\}, \tag{82}$$

this grows exponentially faster.

A Proof of Lemma 8.1

In this appendix, we will carry out the high temperature analysis to prove Lemma 8.1. Our approach is reminiscent of the one carried out for the Sherrington-Kirkpatrick model in [15].

A.1 Spin glass monotonicity principle

Suppose that we have two Hamiltonians: a non-random Hamiltonian $H_0 : \Omega_N \to \mathbb{R}$, and a Gaussian spin glass Hamiltonian, which we write as

$$H(\sigma, \mathcal{J}) \,=\, \sum_{k=1}^{M_N} J_k h_k(\sigma)\,,$$

where each $h_k : \Omega_N \to \mathbb{R}$ is a non-random interaction, and $\mathcal{J} = (J_1, \dots, J_{M_N})$ are i.i.d., $\mathcal{N}(0,1)$ random variables. What is important is that

$$\mathbb{E}[H(\sigma, \mathcal{J})] \,=\, 0\,,$$

for each $\sigma \in \Omega_N$, and there is a covariance which we may write explicitly in this case as

$$\mathbb{E}[H(\sigma, \mathcal{J}) H(\sigma', \mathcal{J})] \,=\, \sum_{k=1}^{M_N} h_k(\sigma) h_k(\sigma')\,.$$

For each $t \geq 0$, we define an interpolating Hamiltonian:

$$H_N(\mathcal{J}, \sigma, t) \,=\, H_N^{(0)}(\sigma) + \sqrt{t}\, H_N(\sigma, \mathcal{J}) + \frac{t}{2}\, \mathbb{E}\left[H_N^2(\sigma, \mathcal{J}) \right]\,. \tag{83}$$

We define the random partition function

$$Z_N(\mathcal{J}, t) \,=\, \sum_{\sigma \in \Omega_N} e^{-H_N(\mathcal{J}, \sigma, t)}\,,$$

and the "quenched" pressure

$$p_N(t) \,=\, \frac{1}{N}\, \mathbb{E}\left[\ln Z_N(\mathcal{J}, t) \right]\,. \tag{84}$$

We also define the random Boltzmann-Gibbs measure

$$\langle f(\sigma) \rangle_{\mathcal{J}, N, t} \,=\, \sum_{\sigma \in \Omega_N} f(\sigma)\, \frac{e^{-H_N(\mathcal{J}, \sigma, t)}}{Z_N(\mathcal{J}, t)}\,.$$

Lemma A.1 *For $t \geq 0$,*

$$\frac{dp_N}{dt} \,=\, -\frac{1}{2N}\, \mathbb{E}\left[\langle H_N(\mathcal{J}', \sigma) \rangle^2_{\mathcal{J}, N, t} \right]\,, \tag{85}$$

where $\mathcal{J}$ and $\mathcal{J}'$ are independent realizations of the i.i.d., $\mathcal{N}(0,1)$ noise. In particular $p_N(t)$ is monotone non-increasing.

Proof: Suppose that we have a general Hamiltonian

$$H_N(\mathcal{J},\sigma,t,u) \;=\; H_N^{(0)}(\sigma) + tH_N^{(1)}(\sigma) + \sqrt{u}\, H_N(\mathcal{J},\sigma)\,,$$

where $H_N^{(1)}(\sigma)$ is another non-random Hamiltonian. Then

$$\frac{\partial}{\partial t}\,\mathbb{E}\left[\frac{1}{N}\,\ln\sum_{\sigma\in\Omega_N} e^{-H_N(\mathcal{J},\sigma,t,u)}\right] \;=\; -\frac{1}{N}\,\mathbb{E}\left[\left\langle H_N^{(1)}(\sigma)\right\rangle_{\mathcal{J},N,t,u}\right]\,, \tag{86}$$

where

$$\langle f(\sigma)\rangle_{\mathcal{J},N,t,u} \;=\; \sum_{s\in\Omega_N} f(\sigma)\,\frac{e^{-H_N(\mathcal{J},\sigma,t,u)}}{Z_N(\mathcal{J},t,u)} \quad \text{ and } \quad Z_N(\mathcal{J},t,u) \;=\; \sum_{\sigma\in\Omega_N} e^{-H_N(\mathcal{J},\sigma,t,u)}\,.$$

Similarly,

$$\frac{\partial}{\partial u}\,\mathbb{E}\left[\frac{1}{N}\,\ln\sum_{\sigma\in\Omega_N} e^{-H_N(\mathcal{J},\sigma,t,u)}\right] \;=\; -\frac{1}{2N\sqrt{u}}\,\mathbb{E}\left[\langle H_N(\mathcal{J},\sigma)\rangle_{N,t,u}\right]\,.$$

But now we use Gaussian-integration-by parts to determine that

$$\begin{aligned}
-\frac{1}{2N\sqrt{u}}\,\mathbb{E}\left[\langle H_N(\mathcal{J},\sigma)\rangle_{N,t,u}\right] \;&=\; -\frac{1}{2N\sqrt{u}}\sum_{k=1}^{M_N}\mathbb{E}\left[J_k\sum_{\sigma\in\Omega_N} h_k(\sigma)\,\frac{e^{-H_N(\mathcal{J},\sigma,t,u)}}{\sum_{\sigma'\in\Omega_N} e^{-H_N(\mathcal{J},\sigma',t,u)}}\right]\\
&=\; -\frac{1}{2N\sqrt{u}}\sum_{k=1}^{M_N}\sum_{\sigma\in\Omega_N} h_k(\sigma)\mathbb{E}\left[\frac{\partial}{\partial J_k}\left(\frac{e^{-H_N(\mathcal{J},\sigma,t,u)}}{\sum_{\sigma'\in\Omega_N} e^{-H_N(\mathcal{J},\sigma',t,u)}}\right)\right]\\
=\; \frac{1}{2N}\sum_{k=1}^{M_N}\sum_{\sigma\in\Omega_N} h_k(\sigma)\mathbb{E}&\left[\frac{h_k(\sigma)e^{-H_N(\mathcal{J},\sigma,t,u)}}{\sum_{\sigma'\in\Omega_N} e^{-H_N(\mathcal{J},\sigma',t,u)}} - \sum_{\sigma'\in\Omega_N}\frac{h_k(\sigma')e^{-H_N(\mathcal{J},\sigma,t,u)-H_N(\mathcal{J},\sigma',t,u)}}{Z_N^2(\mathcal{J},t,u)}\right]\,.
\end{aligned}$$

This can be written more concisely as

$$\frac{\partial}{\partial u}\,\mathbb{E}\left[\frac{1}{N}\,\ln\sum_{\sigma\in\Omega_N} e^{-H_N(\mathcal{J},\sigma,t,u)}\right] \;=\; \frac{1}{2N}\,\mathbb{E}\left[\left\langle\mathbb{E}\left[H_N^2(\mathcal{J},\sigma)\right]\right\rangle_{\mathcal{J},N,t,u} - \langle\mathbb{E}\left[H_N(\mathcal{J},\sigma)H_N(\mathcal{J},\sigma')\right]\rangle_{\mathcal{J},N,t,u}\right]\,,$$

where we write

$$\langle f(\sigma,\sigma')\rangle_{\mathcal{J},N,t,u} \;=\; \sum_{\sigma,\sigma'\in\Omega_N} f(\sigma,\sigma')\,\frac{e^{-H_N(\mathcal{J},\sigma,t,u)-H_N(\mathcal{J},\sigma',t,u)}}{Z_N^2(\mathcal{J},t,u)}\,.$$

Putting this together with (86) and specializing to

$$H_N^{(1)}(\sigma) \;=\; \mathbb{E}\left[H_N^2(\mathcal{J},\sigma)\right]\,,$$

and setting $u = t$, we have

$$\frac{\partial}{\partial t}\,\mathbb{E}\left[\frac{1}{N}\,\ln\sum_{\sigma\in\Omega_N} e^{-H_N(\mathcal{J},\sigma,t,t)}\right] \;=\; -\frac{1}{2N}\,\mathbb{E}\left[\langle\mathbb{E}\left[H_N(\mathcal{J},\sigma)H_N(\mathcal{J},\sigma')\right]\rangle_{\mathcal{J},N,t,u}\right]\,.$$

Finally, we can make the inner expectation and outer expectations independent by using independent realizations of the noise $\mathcal{J}$. Doing this, and using the fact that conditional on $\mathcal{J}$, the probability measure associated to $\langle f(\sigma,\sigma')\rangle_{\mathcal{J},N,t,u}$ makes σ and σ' independent and identically distributed, we obtain the desired result. $\square$

A.2 Upper and lower bounds for the spin glass plus mean-field interactions

We will need to introduce various new terms to the Hamiltonian. Therefore, we take the opportunity to redefine some terms and introduce some new ones. In completing the proof we will relate these back to the original definitions from the last section. Here are the new terms in the Hamiltonian.

- The mean-field Potts-type Hamiltonian,

$$H_{MF}^{Potts}(\sigma) \;=\; \frac{1}{2N} \sum_{i,j=1}^{N} \left[\delta(\sigma_i, \sigma_j) - q^{-1}\right] . \tag{87}$$

- The mean-field Potts spin-glass. This has Hamiltonian

$$H_{Potts}^{SG}(\sigma) \;=\; -\sum_{i,j=1}^{N} \frac{J_{ij}}{\sqrt{2N}} \left[\delta(\sigma_i, \sigma_j) - q^{-1}\right] , \tag{88}$$

  for random coupling constants J_{ij}, which are i.i.d., standard, normal random variables, meaning $\mathbb{E}[J_{ij}] = 0$, $\mathbb{E}[J_{ij}^2] = 1$.

- A spin-spin overlap term. This has configuration space $\Omega_q^N \times \Omega_q^N$. We denote the pair as (σ, τ) for $\sigma, \tau \in \{1, \dots, q\}^N$. The Hamiltonian is

$$H_{over}^{(2)}(\sigma, \tau) \;=\; \frac{1}{2N} \sum_{i,j=1}^{N} \left[\delta(\sigma_i, \sigma_j) - q^{-1}\right] \left[\delta(\tau_i, \tau_j) - q^{-1}\right] . \tag{89}$$

We observe some simple calculations in the following lemma.

Lemma A.2 *There are the following relations among these Hamiltonians:*

$$\mathbb{E}\left[H_{Potts}^{SG}(\sigma) H_{Potts}^{SG}(\tau)\right] \;=\; H_{over}^{(2)}(\sigma, \tau) \qquad \textit{and}$$

$$H_{over}^{(2)}(\sigma, \sigma) \;=\; (1 - 2q^{-1}) H_{MF}^{Potts}(\sigma) + \frac{N}{2q^2} .$$

We leave the proof of Lemma A.2 as an exercise for the reader. The most important step is the identity

$$\left[\delta(\sigma_i, \sigma_j) - q^{-1}\right]^2 \;=\; (1 - 2q^{-1}) \delta(\sigma_i, \sigma_j) + q^{-2} .$$

We are interested in the following quantities:

$$H_N(y, z, \sigma) \;=\; y H_{MF}^{Potts}(\sigma) + \sqrt{z}\, H_{Potts}^{SG}(\sigma) - \frac{Nz}{2q^2} , \tag{90}$$

$$Z_N(y, z) \;=\; \sum_{\sigma \in \Omega_q^N} e^{-H_N(y, z, \sigma)} , \tag{91}$$

$$p_N(y, z) \;=\; \frac{1}{N} \mathbb{E}\left[\ln Z_N(y, z)\right] . \tag{92}$$

Guerra and Toninelli proved that the thermodynamic limit of this pressure exists in [18]. It is remarkable that they are able to prove the existence of the thermodynamic limit even with mean-field ferromagnetic interactions, not just antiferromagnetic terms. More recently, other researchers have considered this model in certain regimes outside the high-temperature region, for the Ising case, $q = 2$, to try to establish the Ghirlanda-Guerra identities [7]. We will not need such results, since we restrict attention to a high temperature region.

Guerra and Toninelli's general spin glass monotonicity principle implies the following.

Corollary A.3 *For any $y \in \mathbb{R}$, $z \geq 0$*

$$p_N\,(y,z) \;\leq\; p_N\left(y - \frac{1-2q^{-1}}{2} z, 0\right) . \tag{93}$$

More precisely,

$$\begin{aligned} \frac{d}{dt}\, p_N\left(y + \frac{1-2q^{-1}}{2} t, z+t\right)\Big|_{t=0} \;&=\; -\frac{1}{2N}\,\mathbb{E}\left[\left\langle H^{(2)}_{over}(\sigma,\sigma')\right\rangle_{y,z}\right] \\ &=\; -\frac{1}{4N^2}\sum_{i,j=1}^{N}\mathbb{E}\left[\langle\delta(\sigma_i,\sigma_j)-q^{-1}\rangle^2_{y,z}\right] , \end{aligned} \tag{94}$$

where $\langle\cdot\rangle_{y,z}$ denotes the random Boltzmann-Gibbs average,

$$\langle f(\sigma)\rangle_{y,z} \;=\; \sum_{\sigma\in\Omega_N} f(\sigma)\,\frac{e^{-H_N(y,z,\sigma)}}{Z_N(y,z)} \qquad \textit{and} \qquad \langle f(\sigma,\sigma')\rangle_{y,z} \;=\; \sum_{\sigma,\sigma'\in\Omega_N} f(\sigma,\sigma')\,\frac{e^{-H_N(y,z,\sigma)-H_N(y,z,\sigma')}}{Z_N^2(y,z)} .$$

Proof: The proof is an application of Lemma A.1. Suppose that y and z are fixed, and, let us define the interpolating Hamiltonians for the definition of (83). Let

$$H_N(\mathcal{J},\sigma) \;=\; H^{SG}_{Potts}(\sigma) .$$

According to Lemma A.2, this has variance

$$H^{(1)}_N(\sigma) \;=\; \mathbb{E}\left[H^2_N(\mathcal{J},\sigma)\right] \;=\; (1-2q^{-1})H^{Potts}_{MF}(\sigma) + \frac{N}{2q^2} ,$$

We also define the non-random Hamiltonian

$$H^{(0)}_N(\sigma) \;=\; \left(y - \frac{1-2q^{-1}}{2}\,z\right) H^{Potts}_{MF}(\sigma) .$$

Then we see that the interpolating Hamiltonian,

$$H_N(\sigma,t) \;:=\; H^{(0)}_N(\sigma) + \sqrt{t}\,H_N(\mathcal{J},\sigma) + \frac{t}{2}\,\mathbb{E}\left[H^2_N(\mathcal{J},\sigma)\right] ,$$

may actually be written in terms of the Hamiltonian in (90) as

$$H_N(\sigma,t) \;=\; H_N(\tilde{y}(t),\tilde{z}(t),\sigma) \qquad \text{for} \qquad \tilde{y}(t) \;=\; y - \frac{1-2q^{-1}}{2}\,(z-t) ,\;\; \tilde{z}(t) \;=\; t .$$

Defining $p_N(t)$ relative to the interpolating Hamiltonian in (84), we see that it related to (92) as

$$p_N(t) \;=\; p_N(\tilde{y}(t),\tilde{z}(t)) .$$

Therefore, we apply (85), applied to $t = z$. Again using Lemma A.2 to get the covariance, this gives precisely (94). □

The inequality (93) is a simple application of Lemma A.1. But that lemma will always only lead to upper bounds. In order to get lower bounds, we use the explicit form of the right hand side of (94) and introduce a term to the Hamiltonian corresponding to this. We define

$$H^{(2)}_N(x,y,z,\sigma,\tau) \;=\; -xH^{(2)}_{over}(\sigma,\tau) + y[H^{Potts}_{MF}(\sigma)+H^{Potts}_{MF}(\tau)] + \sqrt{z}\,[H^{SG}_{Potts}(\sigma)+H^{SG}_{Potts}(\tau)] - \frac{Nz}{q^2} , \tag{95}$$

for $x, y \in \mathbb{R}$ and $z \geq 0$. The reason we take $-x$ is that we want the two spin configurations σ and τ to be ferromagnetically coupled. We will define the general partition function for all three of these terms as

$$Z_N^{(2)}(x,y,z) \;=\; \sum_{(\sigma,\tau)\in\Omega_q^N\times\Omega_q^N} e^{-H_N^{(2)}(x,y,z,\sigma,\tau)}\,, \tag{96}$$

and the quenched finite-volume approximation to the "pressure,"

$$p_N^{(2)}(x,y,z) \;=\; \frac{1}{2N}\,\mathbb{E}\left[\ln Z_N^{(2)}(x,y,z)\right]\,. \tag{97}$$

Then, by a calculation like (86), we obtain

$$\frac{\partial p_N^{(2)}}{\partial x}(x,y,z) \;=\; \frac{1}{2N}\,\mathbb{E}\left[\langle H_{over}^{(2)}(\sigma,\tau)\rangle_{x,y,z}\right]\,, \tag{98}$$

where now we write $\langle \cdots \rangle_{x,y,z}$ for the random Gibbs measure associated to the partition function $Z_N^{(2)}(x,y,z)$, which is consistent with $\langle\cdot\rangle_{y,z}$ in the special case $x=0$.

Because of this, one may rewrite (94) as

$$\frac{d}{dt}\,p_N\left(y+\frac{1-2q^{-1}}{2}t, z+t\right)\bigg|_{t=0} \;=\; -\frac{\partial p_N^{(2)}}{\partial x}(x,y,z)\bigg|_{x=0}. \tag{99}$$

Guerra and Toninelli's next key step is to replace the derivative by a finite difference approximation. A general property guarantees that $p_N(x,y,z)$ is convex in x. Therefore

$$\frac{\partial p_N^{(2)}}{\partial x}(x,y,z)\bigg|_{x=0} \;\leq\; \frac{p_N^{(2)}(x,y,z)-p_N^{(2)}(0,y,z)}{x} \;=\; \frac{1}{x}\,p_N^{(2)}(x,y,z) - \frac{1}{x}\,p_N(y,z)\,,$$

for any $x>0$. So we may obtain an inequality from (99):

$$\frac{d}{dt}\,p_N\left(y+\frac{1-2q^{-1}}{2}t, z+t\right) \;\geq\; \frac{1}{x}\,p_N\left(y+\frac{1-2q^{-1}}{2}t, z+t\right) - \frac{1}{x}\,p_N^{(2)}\left(x, y+\frac{1-2q^{-1}}{2}t, z+t\right).$$

This leads to the following:

Corollary A.4 *For any $x>0$,*

$$\frac{d}{dt}\left(e^{-t/x}p_N\left(y+\frac{1-2q^{-1}}{2}t, z+t\right)\right) \;\geq\; -\frac{e^{-t/x}}{x}\,p_N^{(2)}\left(x, y+\frac{1-2q^{-1}}{2}t, z+t\right). \tag{100}$$

This corollary is a direct consequence of the previous inequality. The key fact is that now, turning the inequality around,

$$-\frac{d}{dt}\left(e^{-t/x}p_N\left(y+\frac{1-2q^{-1}}{2}t, z+t\right)\right) \;\leq\; \frac{e^{-t/x}}{x}\,p_N^{(2)}\left(x, y+\frac{1-2q^{-1}}{2}t, z+t\right),$$

and now Guerra and Toninelli's monotonicity principle in the form of Lemma A.1 may be applied to yield and upper bound for the right hand side.

Corollary A.5 *For $x, y \in \mathbb{R}$, $z \geq 0$, and $t \in [0,z]$,*

$$p_N^{(2)}(x,y,z) \;\leq\; p_N^{(2)}\left(x+z, y-\frac{1-2q^{-1}}{2}\,z, 0\right), \tag{101}$$

This is proved similarly to Corollary A.3, except that now we use the pair, $(\sigma, \tau) \in \Omega_{2N}$, as the basic spin configuration.

Proof: Now we replace N by $2N$ and consider the pair $(\sigma, \tau) \in \Omega_{2N}$. Let

$$H_{2N}(\mathcal{J}, \sigma, \tau) \;=\; H^{SG}_{Potts}(\sigma) + H^{SG}_{Potts}(\tau)\,.$$

Using Lemma A.2, this has variance

$$H^{(1)}_{2N}(\sigma, \tau) \;=\; \mathbb{E}\left[H^2_{2N}(\mathcal{J}, \sigma, \tau)\right] \;=\; (1 - 2q^{-1})\left(H^{Potts}_{MF}(\sigma) + H^{Potts}_{MF}(\tau)\right) + 2H^{(2)}_{over}(\sigma, \tau) + \frac{N}{q^2}\,.$$

This time, we define the non-random Hamiltonian

$$H^{(0)}_{2N}(\sigma, \tau) \;=\; -(x+z)H^{(2)}_{over}(\sigma, \tau) + \left(y - \frac{1-2q^{-1}}{2}\, z\right)\left(H^{Potts}_{MF}(\sigma) + H^{Potts}_{MF}(\tau)\right)\,.$$

The interpolating Hamiltonian is

$$H_{2N}(\sigma, \tau, t) \;:=\; H^{(0)}_{2N}(\sigma, \tau) + \sqrt{t}\, H_{2N}(\mathcal{J}, \sigma, \tau) + \frac{t}{2}\, H^{(1)}_{2N}(\sigma, \tau)\,.$$

In terms of (95) this is

$$H_{2N}(\sigma, \tau, t) \;=\; H_N(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t), \sigma)\,,$$

for

$$\tilde{x}(t) \;=\; x + z - t\,, \quad \tilde{y}(t) \;=\; y - \frac{1-2q^{-1}}{2}\,(z - t)\,, \quad \tilde{z}(t) \;=\; t\,.$$

Defining $p_{2N}(t)$ relative to the interpolating Hamiltonian $H_{2N}(\sigma, \tau, t)$ as in (84), but with N replaced by $2N$, it is related to (97) as

$$p_{2N}(t) \;=\; p^{(2)}_N(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))\,.$$

Therefore, we apply (85), applied to $t = 0$ and $t = z$. This gives the inequality that was claimed. This time, we do not calculate the remainder, so we do not need to use the general covariance formula. □

A.3 Conclusion of Lemma 8.1.

We may now establish Lemma 8.1 by completing the proof. By combining (93), (100) and (101), we have

$$\begin{aligned} p_N\left(y - \frac{1-2q^{-1}}{2} z, 0\right) &\leq p_N(y, z) \\ &\leq e^{z/x} p_N\left(y - \frac{1-2q^{-1}}{2} z, 0\right) - \int_0^z p^{(2)}_N\left(x + z - t, y - \frac{1-2q^{-1}}{2} z, 0\right) \frac{e^{t/x}}{x}\, dt\,. \end{aligned} \tag{102}$$

Note that we may rewrite this as

$$\begin{aligned} p^{(2)}_N\left(0, y - \frac{1-2q^{-1}}{2} z, 0\right) &\leq p_N(y, z) \\ &\leq e^{z/x} p^{(2)}_N\left(0, y - \frac{1-2q^{-1}}{2} z, 0\right) - \int_0^z p^{(2)}_N\left(x + z - t, y - \frac{1-2q^{-1}}{2} z, 0\right) \frac{e^{t/x}}{x}\, dt\,. \end{aligned}$$

But $p^{(2)}_N(x, y, 0)$ represents the pressure from a true mean-field classical spin system. This may be calculated by large deviation techniques. Thus both the lower and upper bounds may be calculated this way. When they match, we will be guaranteed to be in the high temperature region.

Lemma A.6 *For each $u, v \in \mathbb{R}$, let us define*

$$\mathcal{L}(u,v;\rho) \;=\; \frac{u}{2}\sum_{a,b=1}^{q}\left(\rho_{ab}-q^{-2}\right)^2+\frac{v}{2}\left(\sum_{a=1}^{q}\left[\sum_{b=1}^{q}(\rho_{ab}-q^{-2})\right]^2+\sum_{b=1}^{q}\left[\sum_{a=1}^{q}(\rho_{ab}-q^{-2})\right]^2\right)-\sum_{a,b=1}^{q}\rho_{ab}\ln\rho_{ab}\,.$$

where we assume ρ is in $\mathcal{M}_q^2$, the set of all probability distributions $\rho = (\rho_{ab})_{a,b=1}^q$ on $\{1,\dots,q\}^2$. For any $x, y \in \mathbb{R}$, we have

$$\lim_{N\to\infty}\ln p_N^{(2)}(x,y,0) \;=\; \frac{1}{2}\max_{\rho\in\mathcal{M}_q^2}\mathcal{L}(x,xq^{-1}-y;\rho)\,.$$

Proof: For a pair $(\sigma,\tau)\in\Omega_q^N\times\Omega_q^N$, let us define $\rho(\sigma,t)=(\rho_{ab}(\sigma,t))_{a,b=1}^q\in\mathcal{M}_q^2$, where

$$\rho_{a,b}(\sigma,\tau) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta(\sigma_i,a)\delta(\tau_i,b)\,,$$

for $a,b=1,\dots,q$. Let us also define

$$\rho_a^{(1)}(\sigma) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta(\sigma_i,a) \quad\text{and}\quad \rho_b^{(2)}(\tau) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta(\tau_i,b)\,.$$

We note that

$$\rho_a^{(1)}(\sigma) \;=\; \sum_{b=1}^{q}\rho_{a,b}(\sigma,\tau) \quad\text{and}\quad \rho_b^{(2)}(\tau) \;=\; \sum_{a=1}^{q}\rho_{a,b}(\sigma,\tau)\,.$$

Therefore, we may write

$$\begin{aligned}
H_{over}^{(2)}(\sigma,\tau) \;&=\; \frac{1}{2N}\sum_{i,j=1}^{N}\left[\delta(\sigma_i,\sigma_j)-q^{-1}\right]\cdot\left[\delta(\tau_i,\tau_j)-q^{-1}\right]\\
&=\; \frac{1}{2N}\sum_{i,j=1}^{N}\left[\sum_{a,b=1}^{q}\delta(\sigma_i,a)\delta(\sigma_j,a)\delta(\tau_i,b)\delta(t_j,b)-q^{-1}\sum_{a=1}^{q}\delta(\sigma_i,a)\delta(\sigma_j,a)-q^{-1}\sum_{b=1}^{q}\delta(\tau_i,b)\delta(\tau_i,b)+q^{-2}\right]\\
&=\; \frac{N}{2}\left(\sum_{a,b=1}^{q}[\rho_{a,b}-q^{-2}]^2-q^{-1}\sum_{a=1}^{q}\left[\rho_a^{(1)}-q^{-1}\right]^2-q^{-1}\sum_{b=1}^{q}\left[\rho_b^{(2)}-q^{-1}\right]^2\right)\,.
\end{aligned}$$

Similarly,

$$H_{MF}^{Potts}(\sigma)+H_{MF}^{Potts}(\tau) \;=\; \frac{N}{2}\left(\sum_{a=1}^{q}\left[\rho_a^{(1)}-q^{-1}\right]^2+\sum_{b=1}^{q}\left[\rho_b^{(2)}-q^{-1}\right]^2\right)\,.$$

Therefore, using (96), we have the non-random partition function when $z=0$:

$$Z^{(2)}(x,y,0) \;=\; \sum_{\sigma,\tau\in\Omega_q^N}e^{N\mathcal{E}(x,xq^{-1}-y;\rho(\sigma,\tau))}\,,$$

where

$$\mathcal{E}(u,v;\rho) \;=\; \frac{u}{2}\sum_{a,b=1}^{q}\left[\rho_{ab}-q^{-2}\right]^2+\frac{v}{2}\left(\sum_{a=1}^{q}\left[\sum_{b=1}^{q}(\rho_{ab}-q^{-2})\right]^2+\sum_{b=1}^{q}\left[\sum_{a=1}^{q}(\rho_{ab}-q^{-2})\right]^2\right)\,.$$

But standard large deviation calculations give

$$\lim_{N\to\infty}\frac{\ln\#\{(\sigma,\tau)\in\Omega_q^N\;:\;\rho(\sigma,\tau)\in A\}}{N} \;=\; -\max_{\rho\in A}\sum_{a,b=1}^{q}\rho_{ab}\ln\rho_{ab}$$

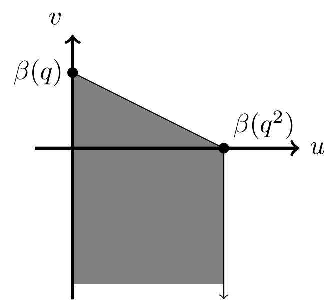

Figure 1: By Lemma A.7, we can deduce that the region in grey is a subset of the region of pairs (u, v) such that $\mathcal{L}(u, v; \rho)$ has the symmetric state $\rho = \rho^*$ as an optimizer.

for any set $A \subseteq \mathcal{M}_q^2$ which is the closure of its interior. (See for instance, [26].) Then Varadhan's lemma implies that

$$\lim_{N\to\infty} \frac{\ln Z^{(2)}(x, y, 0)}{N} = \max_{\rho\in\mathcal{M}_q^2} \left(\mathcal{E}(x, xq^{-1} - y; \rho) - \sum_{a,b=1}^{q} \rho_{ab} \ln \rho_{ab} \right) .$$

This is the desired result because $\mathcal{L}(u, v; \rho) = \mathcal{E}(u, v; \rho) - \sum_{a,b=1}^{q} \rho_{ab} \ln \rho_{ab}$. □

We define $\rho^* \in \mathcal{M}_q^2$ to be the symmetric state:

$$\rho^*_{ab} = q^{-2} \quad \text{for each } a, b = 1, \ldots, q.$$

We say that (u, v) is a "trivial point" of $\mathcal{L}$ if ρ^* is *an* optimizer of $\mathcal{L}(u, v; \rho)$. In other words, (u, v) is a trivial point of $\mathcal{L}$ if

$$\max_{\rho\in\mathcal{M}_q^2} \mathcal{L}(u, v; \rho) = 2 \ln q .$$

Note that in the case that there are several optimizers, we still say that (x, y) is a trivial point if one of the optimizers is ρ^*.

Lemma A.7 *(i) If (u, v) is a trivial point of $\mathcal{L}$, then so is (u', v') for all pairs such that $u' \le u$ and $v' \le v$.*
(ii) If (u_1, v_1) and (u_2, v_2) are trivial points, then (u, v) is also a trivial point for any convex combination

$$u = tu_1 + (1 - t)u_2 \quad \textit{and} \quad v = tv_1 + (1 - t)v_2 , \quad \textit{for } t \in [0, 1] .$$

(iii) For $q \ge 2$, let

$$\beta(q) = \frac{2(q - 1)}{q - 2} \ln(q - 1) ,$$

which is the critical temperature for the q-state Potts ferromagnet. Then $(\beta(q^2), 0)$ and $(0, \beta(q))$ are both trivial points.

In Figure 1, we have sketched a region of pairs (u, v) which are trivial, in the sense defined above.

Proof: Suppose $u' \le u$ and $v' \ge v$. Then for any $\rho \in \mathcal{M}_q^2$, we have

$$\mathcal{L}(u, v; \rho) - \mathcal{L}(u', v'; \rho) = \frac{u - u'}{2} \sum_{a,b=1}^{q} \left(\rho_{ab} - q^{-2}\right)^2 + \frac{v - v'}{2} \left(\sum_{a=1}^{q} \left[\sum_{b=1}^{q} \left(\rho_{ab} - q^{-2}\right) \right]^2 + \sum_{b=1}^{q} \left[\sum_{a=1}^{q} \left(\rho_{ab} - q^{-2}\right) \right]^2 \right) .$$

Hence, relative to the symmetric measure ρ^* such that $\rho^*_{ab} \equiv q^{-2}$, we have

$$\mathcal{L}(u,v;\rho) - \mathcal{L}(u,v;\rho^*) \;=\; \mathcal{L}(u',v';\rho) - \mathcal{L}(u',v';\rho^*) + \frac{u-u'}{2}\sum_{a,b=1}^{q}\left(\rho_{ab}-q^{-2}\right)^2$$

$$+\,\frac{v-v'}{2}\left(\sum_{a=1}^{q}\left[\sum_{b=1}^{q}\left(\rho_{ab}-q^{-2}\right)\right]^2 + \sum_{b=1}^{q}\left[\sum_{a=1}^{q}\left(\rho_{ab}-q^{-2}\right)\right]^2\right).$$

The last two terms are nonnegative. Therefore if

$$\mathcal{L}(u',v',\rho) \;>\; \mathcal{L}(u',v';\rho^*)\,,$$

for some $\rho \in \mathcal{M}_q^2$, then it follows that

$$\mathcal{L}(u,v;\rho) \;>\; \mathcal{L}(u,v;\rho^*)\,,$$

for that same ρ. Part (i) of the lemma is the statement of the contrapositive of this result.

To prove part (ii), note that for $(u,v) = t(u_1,v_1) + (1-t)(u_2,v_2)$, we have

$$\mathcal{L}(u,v;\rho) \;=\; t\mathcal{L}(u_1,v_1;\rho) + (1-t)\mathcal{L}(u_2,v_2;\rho)\,,$$

for every $\rho \in \mathcal{M}_q^2$. Therefore, if $\mathcal{L}(u,v;\rho) > \mathcal{L}(u,v;\rho^*)$, then either

$$\mathcal{L}(u_1,v_1;\rho) \;>\; \mathcal{L}(u_1,v_1;\rho^*)\,,$$

or

$$\mathcal{L}(u_2,v_2;\rho) \;>\; \mathcal{L}(u_2,v_2;\rho^*)\,,$$

or both. So, once again, the result follows by taking the contrapositive statement.

For part (iii) first note that when $v=0$, then $\mathcal{L}(u,0;\rho)$ is precisely the large-deviation optimization problem for the mean-field r-state Potts model with $r=q^2$. This proves that $(u=\beta(q^2), v=0)$ is a trivial point. This follows from the rigorous analysis of the mean-field Potts model. See for instance, [19] or [5].

When $u=0$ the model is not equivalent to the q-state Potts model. But we may show that the optimizer is still the same. Given any $\rho \in \mathcal{M}_q^2$, let us define two mesaures in $\mathcal{M}_q$, which are measures on just the set $\{1,\dots,q\}$, as $\rho^{(1)}$ and $\rho^{(2)}$, the marginals

$$\rho_a^{(1)} \;=\; \sum_{b=1}^{q}\rho_{ab} \quad \text{and} \quad \rho_b^{(2)} \;=\; \sum_{a=1}^{q}\rho_{ab}\,.$$

Then we may define a new "product measure," $\tilde\rho \in \mathcal{M}_q^2$, such that

$$\tilde\rho_{ab} \;=\; \rho_a^{(1)}\rho_b^{(2)}\,.$$

Note that the term attached to $v/2$ in the partition function is

$$\sum_{a=1}^{q}(\rho_a^{(1)}-q^{-1})^2 + \sum_{b=1}^{q}(\rho_b^{(2)}-q^{-1})^2\,.$$

But the process of taking marginals and product measures stabilizes in the sense that the marginals of $\tilde\rho$ are: $\tilde\rho^{(1)} = \rho^{(1)}$ and $\tilde\rho^{(2)} = \rho^{(2)}$. So, if $u=0$ then the only difference between the function evaluated at ρ and $\tilde\rho$ comes from the entropy term, and we obtain:

$$\mathcal{L}(0,v;\tilde\rho) - \mathcal{L}(0,v;\rho) \;=\; -\sum_{a,b=1}^{q}\left[\tilde\rho_{ab}\ln(\tilde\rho_{ab}) - \rho_{ab}\ln(\rho_{ab})\right].$$

But a standard calculation with entropy (see for example, chapter 1 of [24]) shows that

$$-\sum_{a,b=1}^{q}[\tilde{\rho}_{ab}\ln(\tilde{\rho}_{ab})-\rho_{ab}\ln(\rho_{ab})]\;=\;\sum_{a,b=1}^{q}\rho_{ab}\ln\left(\frac{\rho_{ab}}{\tilde{\rho}_{ab}}\right)\,,$$

and this is always nonnegative. So, given any $\rho \in \mathcal{M}_q^2$, one can always replace ρ by $\tilde{\rho}$, and $\mathcal{L}(0,v;\cdot)$ will not decrease.

But also,

$$\mathcal{L}(0,v;\tilde{\rho})\;=\;\tilde{\mathcal{L}}(v;\rho^{(1)})+\tilde{\mathcal{L}}(v;\rho^{(2)})\,,$$

where for any $\rho \in \mathcal{M}_q$, we have

$$\tilde{\mathcal{L}}(\beta;\rho)\;=\;\frac{\beta}{2}\sum_{a=1}^{q}(\rho_a-q^{-1})^2-\sum_{a=1}^{q}\rho_a\ln(\rho_a)\,.$$

This is the large-deviation optimization problem for the mean-field q-state Potts model, at inverse temperature β. Also note that there is not restriction on $\rho^{(1)}$ or $\rho^{(2)}$: given any two such measures, we may have started out by choosing ρ to be the their product measure. Therefore, the optimization problem for $\mathcal{L}(0,v;\cdot)$ has the same solution as twice the optimization problem for the q-state Potts model at inverse temperature v. (If $v>0$ then saying the Potts model has a negative inverse temperature just means we consider the antiferromagnet instead of the ferromagnet, which has no phase transition because of "convexity.") In particular, this proves that the critical value of v is $v=\beta(q)$. If $v \geq -\beta(q)$ then ρ^* is still an optimizer for $\mathcal{L}(0,v;\cdot)$. Again, this follows from the rigorous analysis of the mean-field q-state Potts model, as in [19] or [5]. □

Proof of Lemma 8.1: Combining (102), Lemma A.6 and Lemma A.7, we see that

$$\ln q\;\leq\;\liminf_{N\to\infty}p_N(y,z)\;\leq\;\limsup_{N\to\infty}p_N(y,z)\;\leq\;e^{z/x}\ln q-\int_0^z(\ln q)\,\frac{e^{t/x}}{x}\,dx\;=\;\ln q\,,$$

as long as (u,v) is a trivial point of $\mathcal{L}$ for

$$u\;=\;x+z\quad\text{ and }\quad v\;=\;(x+z)q^{-1}-y+\frac{1-2q^{-1}}{2}z\;=\;xq^{-1}-y+\frac{1}{2}\,z\,.$$

Here it is required that $x>0$. But the various bounds we established for the partial derivatives of p_N apply to show that it is Lipschitz. Therefore, even if (u,v) is only a trivial point of $\mathcal{L}$ for

$$u\;=\;z\quad\text{ and }\quad v\;=\;\frac{1}{2}\,z-y\,,\tag{103}$$

it is still the case that $p_N(y,z)\to\ln q$ as $N\to\infty$. From Lemma A.7 once again, (u,v) is a trivial point if it lies in the generalized triangle with vertices $(0,\beta(q))$, $(\beta(q^2),0)$ and $(0,-\infty)$. But (103) is a linear mapping. So this means the condition for (z,y) is that it lies in the generalized triangle with vertices $(0,-\beta(q))$, $(\beta(q^2),\frac{1}{2}\beta(q^2))$ and $(0,\infty)$. This is shown in Figure 2. In this region, we know that $p_N(y,z)$ has the limiting value $\ln q$. Note that this does include a neighborhood of $(0,0)$: $0\leq z\leq\epsilon$ and $-\delta<y<\delta$ for some $\epsilon,\delta>0$, as desired.

Now, combining this with (94), since $p_N(y,z)$ is constant in this region, we see that

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left[\langle\delta(\sigma_i,\sigma_j)-q^{-1}\rangle_{y,z}^2\right]$$

converges to zero, as $N\to\infty$, almost everywhere in this region. This also means that for almost every (y,z) in this region,

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle\delta(\sigma_i,\sigma_j)\rangle_{y,z}\;\xrightarrow{P}\;q^{-1}\quad\text{ as }N\to\infty,\,,$$

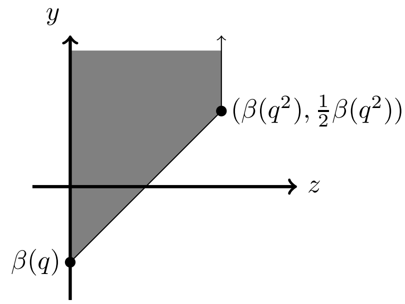

Figure 2: The region in grey is a subset of the region of pairs (y, z) (drawn with the coordinate pairs reversed as (z, y) on the ordinate and abscissa) such that $p_N(y, z)$ converges to its trivial value $\ln q$.

where $\stackrel{P}{\to}$ indicates convergence in probability. We can use this to determine the desired formula for $N^{-1}\mathbb{E}\left[\ln \tilde{Z}_N(0, \lambda, \kappa)\right]$.

When $\lambda = \kappa = 0$ we have the trivial formula

$$\frac{1}{N}\,\mathbb{E}\left[\ln \tilde{Z}_N(0,0,0)\right] \,=\, \ln q\,.$$

(Recall that we will assume $c = 0$, throughout.) That serves as a starting point. We may calculate

$$\frac{1}{N}\,\mathbb{E}\left[\ln \tilde{Z}_N(0,\lambda,\kappa)\right] \,=\, \ln q + \int_0^1 \frac{d}{dt}\,\mathbb{E}\left[\frac{1}{N}\,\ln \tilde{Z}_N(0,\tilde{\lambda}(t),\tilde{\kappa}(t))\right]\,dt\,,$$

for any differentiable path $(\tilde{\lambda}(t), \tilde{\kappa}(t))$ such that: $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(1) = \lambda$, $\tilde{\kappa}(0) = 0$, $\tilde{\kappa}(1) = \kappa$.

We can relate the partial derivatives of this to the partial derivatives of $p_N(y, z)$. Recall that in our definition of $H_N(y, z)$ we did not exactly match the definition needed to obtain $\tilde{Z}_N(0, \lambda, \kappa)$ from Lemma 8.1. We subtracted some trivial terms to obtain $p_N(y, z)$ that made the analysis proceed more easily. Specifically, we subtracted q^{-1} from $\delta(\sigma_i, \sigma_j)$ in (87) and (88), and we subtracted $Nz/(2q^2)$ in (90). But these are all constant shifts, not depending on the spin configurations. Therefore, none of these changes affects the random Boltzmann-Gibbs measures, because they cancel in the numerator and denominator in the definition of the Boltzmann-Gibbs weights. In other words, considering the definition of $\tilde{\omega}_N(\cdots)$ from (72), we do have

$$\tilde{\omega}_N(\cdots) \,=\, \langle \cdots \rangle_{y,z} \quad \text{if} \quad c = 0\,,\ \lambda = y \ \text{ and } \ \kappa = z\,.$$

The partial derivatives of $N^{-1}\mathbb{E}[\ln \tilde{Z}_N(0, \lambda, \kappa)]$ are calculated in (73) and (74):

$$\frac{\partial}{\partial \lambda}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, -\frac{1}{2N^2}\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N(\delta(\sigma_i,\sigma_j))\right] \qquad \text{and}$$

$$\frac{\partial}{\partial \kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,=\, \frac{1}{4N^2}\,\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N\left(\delta(\sigma_i,\sigma_j)\right)\right] - \frac{1}{4N^2}\,\sum_{i,j=1}^N \mathbb{E}\left[\tilde{\omega}_N^2\left(\delta(\sigma_i,\sigma_j)\right)\right]\,.$$

But since we know that

$$\frac{1}{N^2}\,\sum_{i,j=1}^N \langle \delta(\sigma_i,\sigma_j)\rangle_{y,z} \,\stackrel{P}{\to}\, q^{-1} \qquad \text{as } N \to \infty,,$$

for almost every (y, z) in the specified region, this means that

$$\frac{\partial}{\partial \lambda}\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,\stackrel{N\to\infty}{\longrightarrow}\, -\frac{1}{2q} \qquad \text{and} \qquad \frac{\partial}{\partial \kappa}\,\mathbb{E}\left[\frac{1}{N}\ln \tilde{Z}_N\right] \,\stackrel{N\to\infty}{\longrightarrow}\, \frac{1}{4q} - \frac{1}{4q^2}\,,$$

for almost every pair (λ,κ) in the same region.

Now there seems to be a small problem. We have established convergence of the derivatives for Lebesque-a.e. pair (λ,κ) in the desired region. But we have to integrate over a 1-dimensional curve, which has Lebesgue measure zero. (In fact, we could have concluded convergence a.e. with respect to a 1-dimensional Lebesgue measure, but it would have been for a curve, with a specification for the slope between y and z, which may not have matched our desired curve.) However, we also know that the derivatives in question are all uniformly bounded. That implies continuity of $\mathbb{E}\left[\frac{1}{N}\ln\tilde{Z}_N(0,\lambda,\kappa)\right]$ with respect to λ and κ. So we can average a bit over these points, to turn the contour integral into an area integral. Then we can reduce the averaging window, and use continuity to conclude that we obtain the original pressure. Therefore, using this, we do conclude that for (λ,κ) in this region we have

$$\lim_{N\to\infty}\frac{1}{N}\,\mathbb{E}\left[\ln\tilde{Z}_N(0,\lambda,\kappa)\right] \;=\; \ln q-\frac{\lambda}{2q}+\frac{(q-1)\kappa}{4q^2}\,,$$

as claimed. □

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