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The Low Activity Phase of Some Dirichlet Series

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Abstract

We show that a rigorous statistical mechanics description of some Dirichlet series is possible. Using the abstract polymer model language of statistical mechanics and the polymer expansion theory we characterize the *low activity* phase by the suitable exponential decay of the truncated correlation functions.

1 Introduction

The idea to relate number theory and equilibrium statistical mechanics or, more precisely, zeta functions and partition functions, is now already quite old. One motivation for pursuing this idea lies in the probabilistic aspects of the prime number distribution. Statistical mechanics as an intrinsically probabilistic theory is hoped to be an appropriate language for these phenomena. The book [15] by Kac nicely presents this kind of probabilistic reasoning.

More concretely, the formulation of the famous Lee-Yang theorem was influenced by a paper [23] by Pólya on the Riemann zeta function. In that paper Pólya took the asymptotics of the Fourier transformed zeta function

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and proved for its inverse Fourier transform the 'Riemann hypothesis', saying that the non-real zeroes have real part $\frac{1}{2}$.

As described by Kac in [23], the method of Pólyas proof inspired the first version of the Lee-Yang theorem (which says that the partition function of ferromagnetic Ising models has only zeroes on the unit circle of the activity plane).

This lead to the natural question whether inversely the Riemann hypothesis or simpler number-theoretical questions could be proven by some statistical mechanics method.

In recent years two approaches have been followed. In one of them the Riemann zeta function $\zeta(s)$ itself was interpreted as partition function of a system of *interacting primes* at inverse temperature s, see Julia [13, 14] and Bost and Connes [2, 3]. In the last-mentioned paper the system was shown to exhibit a phase transition at $s = 1$ with type I states (resp. type III) at low (resp. high) temperature.

In the second approach mentioned the quotient $\zeta(s-1)/\zeta(s)$ is interpreted as a partition function at the inverse temperature s, see Cvitanović [8], Knauf [17, 18, 19, 20], Contucci and Knauf [7], and Guerra and Knauf [12].

It was shown that the partition function described a spin chain with asymptotically translation-invariant long-range ferromagnetic interaction (the *number-theoretical spin chain*). The point $s = 2$ corresponds to a phase transition where magnetization jumps form 0 to 1.

Although there exist versions of the Lee-Yang theorem predicting zero-free half planes in the inverse temperature plane, unfortunately these theorems cannot be applied to the above spin chain, since its interaction includes multi-body terms.

In this paper, using the general polymer model approach of statistical mechanics, we propose a criterium to interpret a large class of Dirichlet series as grand canonical partition functions of hard-core interacting systems.

The criterium involves a finite-volume approximation and a precise notion of activity. We present two possible polymerizations: the first is based on the notion of Euler product and works for multiplicative arithmetical functions, the second covers a wider class of cases.

We show that the natural thermodynamical quantities of the polymer model, like correlation functions, carry a deep number theoretical meaning being the probability of suitable divisibility properties.

In order to control the behaviour of the correlation functions we apply the polymer expansion technique by means of the Kirkwood-Salsburg iterative equations: the low activity expansion theory enables us to prove the exponential decay of all the truncated correlation functions and provides, in general, a full analytical control of the low temperature phase.

This shows that the language of polymer models is not only formally but also analytically adequate to describe the considered class of Dirichlet series.

Our approach clarifies the statistical mechanics meaning of the absolute convergence theory for the Dirichlet series and introduces new perspectives on it; moreover it has the merit to point out the natural limits of each polymerization. The polymerizations treated in this work, like similar techniques in number theory, provide an approximation of the Dirichlet function which works well for large real part of the complex plane but it results to be too non-uniform elsewhere, especially on the critical strip.

We believe that, in order to obtain new analytical results from the number theoretical point of view, one has to search for different polymerizations, for instance the high temperature ones, or better to explore more subtle strategies like the *rearrangement* procedure for polymer models (see [4, 5]) which, in some cases, provide a good control of the asymptotic behaviour of the correlation functions in the interesting regions of the phase space.

All these ideas can be improved and tested with the study of the number theoretical spin chain: the interacting objects are there not directly related to primes and could suggest different types of polymerization based on groups of spins (see Guerra and Knauf [12]). Moreover the approximant family φ_k [17, 8] of the Euler totient function could really be seen as a systematic way to rearrange the Euler totient function φ thought as a *bare* interaction. We will return on these question elsewhere.

Notation. Sums resp. products over empty sets equal zero resp. one. We write $\mathbb{N} := \{1, 2, 3, ...\}$ for the integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{P} \equiv \{2, 3, 5, ...\}$ for the primes. If n divides m we write $n|m$ and the symbol $\sum_{n|m}$ denotes a sum over all the divisors of m; (n, m) is the greatest common divisor of m and n.

2 The Polymer Expansion

Statistical Mechanics seeks to describe the collective behaviour of a large number of similar particles. One assumes that these particles are enclosed in a finite region $\Lambda \subset S$ of space S (typically $S = \mathbb{R}^d$ or $S = \mathbb{Z}^d$) and then considers the thermodynamic limit $\Lambda \nearrow S$.

The mutual interaction between the particles in a configuration σ is encoded by their total energy $H_\Lambda(\sigma)$. At inverse temperature s the probability of that configuration is given by $\exp(-sH_\Lambda(\sigma))/Z_\Lambda(s)$,

$$Z_\Lambda(s) := \sum_\sigma \exp(-sH_\Lambda(\sigma)) \tag{1}$$

being the partition function for volume Λ.

So the basic objects of statistical mechanics are the Boltzmann factors $\exp(-sH_\Lambda(\sigma))$ of the configurations.

Whereas the above Gibbs probability measures for the finite volume Λ are real-analytic in the parameter s, in the thermodynamic limit $\Lambda \nearrow S$ nonanalyticities arise which are called phase transitions. Different asymptotic Gibbs measures may then be compatible with a given interaction and inverse temperature.

This phenomenon is typical for random fields, i.e. random functions in several variables, and is of central interest in today's theory of probability.

Thus one basic problem of statistical mechanics consists in determining regions in parameter space (e.g., in the s plane) where intensive quantities like the free energy $|\Lambda|^{-1}\ln(Z_\Lambda(s))$ stay analytic in the thermodynamic limit.

Many of the techniques employed in that context recently turned out to be related, the common ground being the abstract *polymer model* formulation (see Glimm and Jaffe [11], Simon [24] and Kotecký and Preiss [21]).

In the abstract setting one starts with a denumerable set $P \equiv \{\gamma_1, \gamma_2, ...\}$ whose elements are called *polymers* and with an assigned reflexive symmetric relation of *incompatibility* between each two of them.

In the concrete application of a two-dimensional Ising model, the polymers may be the contours enclosing a region of constant spin direction, or the subgraphs of the nearest neighbour graph, depending on whether one is interested in small or large temperatures; the incompatibility between two of them is simply the mutual overlapping.

Thus one may associate to a *k-polymer* $X := \{\gamma_1, \ldots, \gamma_k\} \in P^k$ an undirected graph $G(X) = (V(X), E(X))$ with vertex set $V(X) := \{1, \ldots, k\}$, vertices $i \neq j$ being connected by the edge $\{\gamma_i, \gamma_j\} \in E(X)$ if γ_i and γ_j are incompatible. Accordingly the k-polymer X is called connected if $G(X)$ is path-connected and (completely) disconnected if it has no edges ($E(X) = \emptyset$).

The corresponding subsets of P^k are called C^k resp. D^k, with $D^0 := P^0 := \{\emptyset\}$ consisting of a single element. Moreover $P^\infty := \bigcup_{k=0}^\infty P^k$ with the subsets $D^\infty := \bigcup_{k=0}^\infty D^k$ and $C^\infty := \bigcup_{k=1}^\infty C^k$. We write $|X| := k$ if $X \in P^k$; indicating with $X(\gamma)$ the multiplicity of γ in X it results $k = \sum_i X(\gamma_i)$. It is useful to define the function $c(X) := \prod_i X(\gamma_i)!$. We will indicate with a hat the abelianized set: for instance $\hat{P}^\infty$ is the set of abelian words (which we also call polymer configurations) which arises if one identifies k-polymers $X = \{\gamma_1, \ldots, \gamma_k\}$, $Y = \{\delta_1, \ldots, \delta_k\} \in P^\infty$ if $\delta_{\pi(i)} = \gamma_i$ for some permutation π.

Statistical weights or activities $z : P \to \mathbb{C}$ of the polymers are multiplied to give the activities $z^X := \prod_{i=1}^k z(\gamma_i)$ of k-polymers.

The thermodynamical properties of the model are defined through the partition function

$$Z = \sum_{X \in \hat{D}^\infty} z^X. \tag{2}$$

We observe that no multiple occurence of a polymer is allowed since the incompatibility relation is reflexive; for this reason the sum is finite when P has finite cardinality which corresponds to a finite volume in the concrete cases.

It has to be stressed that the polymer models are not statistical mechanics models in the usual form (1). They are useful devices to study the true models in the *low activity* regime of the phase diagram: for this reason a given model is often mapped into different polymer models according to each different phase regime.

One is mainly interested in the $|P| \to \infty$ limit (thermodynamic limit) for the mean values of the configuration functions h:

$$< h >_z = \frac{\sum_{X \in \hat{D}^\infty} h(X) z^X}{\sum_{X \in \hat{D}^\infty} z^X}, \tag{3}$$

and especially for the correlation functions

$$\rho_z(Y) = < \chi_Y >_z \tag{4}$$

where χ_Y is the characteristic function of Y. Often the dependence of the correlation functions on the activity is studied in terms of parameters like, for instance in statistical mechanics, the inverse temperature or a magnetic field.

An important quantity to be studied in the thermodynamic limit is the free energy density, or pressure, which turns out to be (see for instance [10, 11]), up to a suitable normalization factor

$$\ln(Z) = \sum_{X \in \hat{P}^\infty} \frac{n^T(X)}{c(X)} z^X, \tag{5}$$

with $n^T(X) := n_+(X) - n_-(X)$, $n_\pm(X)$ being the number of subgraphs of $G(X)$ connecting all the vertices of $G(X)$ with an even resp. odd number of edges. The structure of the factors n^T implies that the previous sum is actually supported only on $\hat{C}^\infty$.

We notice that, although in the partition sum only compatible configurations of polymer may appear, the free energy contains contribution from all the configuration and also coincident polymers (multiplicities) are allowed.

Formula (5) is important from the conceptional as well as from the analytical point of view. It is complemented by the so-called tree estimate $|n^T(X)| \leq |\tau(G(X))|$, where $\tau(G)$ denotes the set of maximal subtrees of the connected graph G. This inequality is useful, since there exist techniques to estimate the number of subtrees. As an example, a theorem by Cayley says that the complete (all edges present) graph $K(k)$ with k vertices contains $|\tau(K(k))| = k^{k-2}$ maximal trees.

It's easy to check that the simplest example $P = \{p\}$, i.e., $Z = 1 + z$, of a polymer model reduces the formula (5) to the Taylor expansion for the logarithm $\ln(Z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$, since $n^T(K(k)) = (-1)^{k-1}(k-1)!$ (the last formula showing, by the way, that the tree estimate is non-optimal).

So even for a finite cardinality of P one needs bounds on the activities to ensure convergence of the free energy. In the statistical mechanics applications such bounds are given in terms of energy (or activity) and entropy bounds.

3 Dirichlet Series

A basic object of analytic number theory is the Dirichlet series consisting of terms of the form $e^{-s\lambda_n}$ whose exponents $\{\lambda_n\}_{n\in\mathbb{N}}$ being a real-valued sequence strictly increasing to $\lim_{n\to\infty} \lambda_n = \infty$.

A formal series of the form

$$\sum_{n=1}^{\infty} a(n)e^{-s\lambda_n} \tag{6}$$

with complex coefficients $a(n)$ and argument s is called a general Dirichlet series. In this context functions $A : \mathbb{N} \to \mathbb{C}$ are called *arithmetical functions.*

Dirichlet series have abscissae σ_a, σ_c of *absolute* resp. *conditional* convergence. For $\lambda_n := n$ eq. (6) is a power series in $x := e^{-s}$ so that σ_a and σ_c coincide.

For $\lambda_n := \ln(n)$ eq. (6) is called an ordinary Dirichlet series, and we write it in the form

$$Z_a(s) := \sum_{n=1}^{\infty} a(n)n^{-s}. \tag{7}$$

In that case $0 \le \sigma_a - \sigma_c \le 1$.

The simplest choice $a(n) := 1$ of coefficients leads to the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with inverse $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$, with the Möbius function μ (see Appendix). In that case $\sigma_a = 1$ and $\frac{1}{2} \le \sigma_c \le 1$, the Riemann hypothesis being $\sigma_c = \frac{1}{2}$.

Many Dirichlet series arising in number theory can be written as an Euler product

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p\in\mathbb{P}} f_p(p^{-s}). \tag{8}$$

By the fundamental theorem of arithmetic this is the case exactly if the arithmetical function $n \mapsto a(n)$ is *multiplicative*, that is, (it is not identically zero) and

$$a(mn) = a(m)a(n) \quad \text{if } \gcd(m,n) = 1.$$

Then $f_p(x) = \sum_{k=0}^{\infty} a(p^k)x^{-k}$. For example, $\zeta(s) = \prod_{p\in\mathbb{P}}(1 - p^{-s})^{-1}$.

The product

$$Z_f(s)Z_g(s) = Z_{f*g}(s). \tag{9}$$

of Dirichlet series has coefficients

$$f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d}). \tag{10}$$

which are given by the Dirichlet convolution product $f * g$ of the arithmetical functions of the factors.

With pointwise addition and Dirichlet multiplication the set of arithmetical functions becomes an associative algebra with unit I, $I(n) = \delta_{1,n}$ (a so-called monoid). It is easy to prove that when $f(1) \neq 0$ a Dirichlet inverse $f^{(-1)}$ exist.

Dirichlet series are used in number theory in order to make use of analytic tools in the theory of prime numbers. As an example, the prime number theorem states that the number $\pi(x) = |\{p \in \mathbb{P} \mid p \leq x\}|$ of primes smaller than x is asymptotic to $x/\ln x$. This can be shown by analyzing $\zeta'(s)/\zeta(s)$ for $\mathrm{Re}(s) = 1$, that is on the line containing the pole.

In order to give a statistical mechanics interpretation of (some) Dirichlet series as polymer partition functions we have to identify the sums (7) and (2). This can be done, of course, in many ways: the main point is that in the partition sum each polymer can only have simple multiplicity.

We propose two type of polymerization: the first works for multiplicative arithmetical function and is based on the notion of Euler product, the second is more general. We introduce both of them because the first admits a special treatment in the convergence theorems leading to better convergence estimates (see Appendix B).

1. Multiplicative Polymerization.

   - If we now interpret $\zeta(s) = \sum_{n=1}^{\infty} e^{-s\ln(n)}$ as a partition function for an infinite system with state space $n \in \mathbb{N}$ and energies $\ln(n)$, then $\zeta'(s)/\zeta(s)$ is minus the expectation of the internal energy. Moreover, in the notation of the previous section

     $$\zeta(s) \;=\; \prod_{p\in\mathbb{P}}(1-p^{-s})^{-1} = \sum_{X\in\hat{D}^{\infty}} z_s^X ,$$

     taking the primes as the polymers ($P := \mathbb{P}$), assuming different primes to be compatible and setting the activities $z_s(p) := 1/(p^s - 1)$; moreover

     $$\zeta'(s)/\zeta(s) = \frac{d}{ds}\ln(\zeta(s)) = -\sum_{p\in\mathbb{P}} \ln(p)\cdot z_s(p).$$

   - Alternatively one may consider the set $P := \{p^n \mid p \in \mathbb{P}, n \in \mathbb{N}\}$ of prime powers as polymers with the activities $z_s(x) := x^{-s}$ for

$x \in P$ and call $p_1^{n_1}, p_2^{n_2} \in P$ incompatible iff $p_1 = p_2$. Then, again, $\zeta(s)$ can be written as a polymer model (2) and thus its logarithm may be written using formula (5).

Clearly this kind of game can be played with any Dirichlet series having an Euler product (8). Then for the first choice $P = \mathbb{P}$ of polymers the activities are $z_s(p) := f_p(p^{-s}) - 1$, whereas $z_s(p^k) := a(p^k)p^{-sk}$ in the second case.

It is clear that when the multiplicative arithmetical function $a : \mathbb{N} \to \mathbb{C}$ is a square-free function (that is, it vanishes on integers containing squares), then both polymer model interpretations lead to the same activity $z_s(p) = a(p)e^{-s\ln p}$, $z_s(p^k) = 0$ for $k > 1$. The function a plays the role of an interaction.

2. General Polymerization.

   - Square-Free Case.

     A large class of *square-free* Dirichlet series admit the interpretation of polymer models where each prime number is considered a polymer. For instance we can consider the family of arithmetical functions $\phi = \omega f$ where f is multiplicative (and possibly positive to have a genuine probabilistic framework) and the function ω is defined as

$$\omega(n) = \begin{cases} 1, & \text{if } n = 1, p; \\ \prod_{pp'|n} g(p,p'), & \text{otherwise,} \end{cases} \tag{11}$$

     where $g(p,p')$ takes values $0, 1$ and is a symmetric function vanishing on the diagonal. We stress that the previous conditions define a *class* of matrices (of entries $g(p,p')$) and correspondingly a class of square-free arithmetical functions ϕ not necessarily multiplicative. Two primes with $g(p,p') = 0$ are called incompatible; two integers are incompatible if there are two incompatible primes in the respective decomposition.

     Some examples of incompatibility are the following: p, p' are compatible polymers when

     - $p \neq p'$;

- $|p - p'| > const$;
- $|p - p'| > \log \sqrt{pp'}$.

The first case corresponds to the square-free function $|\mu|$ (see appendix) in which the only interaction is the Fermi statistic; the relative zeta function is $Z_{|\mu|}(s) = \frac{\zeta(s)}{\zeta(2s)}$. The interest of the third case will be clear in the section on convergence.

The fundamental theorem of arithmetic on the unique decomposition of an integer into primes permits the formal identification of the function

$$Z_\phi(s) = \sum_{n \in \mathbb{N}} \phi(n) n^{-s}, \tag{12}$$

with a partition function of a polymer system in which each prime has activity $z_s(p) = f(p)e^{-s \log p}$ and the function ω play the role of the hard-core interaction.

- Non Square-Free case.

An important observation is that to treat the case of non square-free Dirichlet series we have simply to change the polymer identification: the polymers are now the prime powers $P := \{p^n \mid p \in \mathbb{P}, n \in \mathbb{N}\} \equiv \{2, 3, 4, 5, 7, 8, 9, ...\}$ with the activities $z_s(x) := f(x)x^{-s}$ for $x \in P$. The class of arithmetical function treated in this way is defined by $\phi = f\omega$ where f is multiplicative and ω

$$\omega(n) = \begin{cases} 1, & \text{if } n = 1, p^k; \\ \prod_{x,x' \in P, xx'|n} g(x, x'), & \text{otherwise}, \end{cases} \tag{13}$$

where $g(x, x')$ takes values $0, 1$ and is a symmetric function vanishing on all the couples $(x, x') = (p^k, p^{k'})$. The previous conditions define a *class* of matrices and correspondingly a class of arithmetical functions ϕ in general not square-free nor multiplicative.

Also in this case there is plenty of examples; for instance the second and third example of the previous polymerization can be rephrased exactly in this one. The simplest example is just the Riemann zeta function: it corresponds to the element of the previous class in which $f(n) = 1$ and $g(x, x') = 1$ if $(x, x') \neq (p^k, p^{k'})$ which says that two polymers are incompatible when they are power of the same prime and they are compatible otherwise.

This means that the Riemann zeta function admits the interpretation of the partition function of an hard-core interacting polymer system.

4 The Hard-Core Models

Let consider, for simplicity, the square-free case with $\phi = \omega$. We introduce now a family of approximating functions ω_k depending on a integer k; the meaning of this approximation is just the finite volume approximation in statistical mechanics which manifest itself with a finite number of polymers. The corresponding partition function becomes a finite series for each k and the problem to control the thermodynamical limit for the correlation functions concerns the possibility to obtain bounds which are uniform in k.

We first define the k-th set of square-free integers $\mathbb{N}_k$ as the integers of the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \text{ where } \alpha_i = 0 \text{ or } 1, \tag{14}$$

and $p_1, ..., p_k$ are the first k prime numbers. Then, for instance, $\mathbb{N}_0 = \{1\}$, $\mathbb{N}_1 = \{1,2\}$, $\mathbb{N}_2 = \{1,2,3,6\}$, $\mathbb{N}_3 = \{1,2,3,5,6,10,15,30\}$, etc, and $|\mathbb{N}_k| = 2^k$. Now we define:

$$\omega_k(n) = \begin{cases} \omega(n), & \text{if } n \in \mathbb{N}_k; \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

It is easy to prove that:

$$\omega_k(n) = \omega(n), \text{ for } n \le p_k, \tag{16}$$

and

$$\omega_k(n) = 0, \text{ for } n > p_1 \cdots p_k. \tag{17}$$

The origin of this approximation is quite simple: we consider the natural numbers progressively generated by prime numbers; the nature of the function ω implies that for each generation only a finite quantity of integers gives a contribution.

Remark: The above mechanism induces in general a one-to-one correspondence between the functions of the variable $(\alpha_1, ..., \alpha_k)$ and the k-th approximation of a square-free arithmetical functions.

It clearly turns out that the approximating zeta function admits the interpretation of a grand canonical partition function for a system of k particles interacting via a hard-core two-body potential:

$$Z_{\omega_k}(s) = \sum_{n\in\mathbb{N}} \omega_k(n) n^{-s} = \sum_{\alpha}\prod_{i} p_i^{-s\alpha_i} \prod_{i<j}[1+\alpha_i\alpha_j(g(p_i,p_j)-1)]. \qquad (18)$$

where $\alpha = (\alpha_1, ..., \alpha_k)$. A mean values is:

$$< f >_k (s) = \frac{\sum_\alpha f(\alpha)\prod_i p_i^{-s\alpha_i}\prod_{i<j}[1+\alpha_i\alpha_j(g(p_i,p_j)-1)]}{Z_{\omega_k}(s)}. \qquad (19)$$

The basic objects of our model are the r-points ($r \leq k$) correlation functions:

$$< \alpha_{i_1} \cdots \alpha_{i_r} >_k (s), \qquad (20)$$

with $i_1 < i_2 < \cdots < i_r$; it is interesting to notice that, by comparing with (20), they represent the probability of divisibility by the integer $p_{i_1} \cdots p_{i_r}$. Without loss we assume $i_1, \ldots, i_r \leq k$ because otherwise (20) vanishes.

One of our main problems is to study the limit $k \to \infty$ of the correlation functions, and to prove that they describe indeed, the equilibrium state of a system of interacting polymers defined by the partition function (18).

There are various approaches in the study of the properties of the polymer models. One of them, the one we consider here, is based on the use of the Kirkwood-Salsburg type iterative equations to control the analytical behaviour of the correlation functions and related quantities. We will follow the ideas of [10] (see also [9, 16]) with a different proof of the convergence theorems according to the number theoretical framework which requires a slightly different identification of interaction and activity function.

5 The Iterative Equation for the Correlations

The analytic control of a polymer model is based on two interrelated axioms carrying a deep statistical mechanics meaning.
Activity Bound: There exists a constant $a < 1$ and a positive function $v(\gamma)$ (the volume) such that

$$|z(\gamma)| \leq a^{v(\gamma)}. \qquad (21)$$

Defining $N(\gamma, x)$ as the number of γ-incompatible polymers for which the function v stay inside the interval $[x, x+1)$ we impose the
Entropy Bound: There exist a constant c such that

$$N(\gamma, x) \leq v(\gamma) c^x. \tag{22}$$

From an analytic point of view the two requirements are simply saying that the terms we sum have to be not to large and not too many. In our context the activity bound can be naturally fulfilled with the choice $a = e^{-s}$ and $v(p) = \log(p)$ and the entropy bound defines the class of function we are treating. It is easy to see that the three concrete examples of the general polymerization fulfill the entropy axiom for any $c > 1$; for each of them one could actually improve the general convergence strategy we are going to present.

Our first goal is to express the correlation function at the temperature s as a zeta function:

$$< \alpha_{i_1} \cdots \alpha_{i_r} >_k (s) = \sum_l < n > (l) l^{-s}, \tag{23}$$

where $n = p_{i_1} \cdots p_{i_r}$. The algebraic properties of the Dirichlet convolution imply that eq. (23) can be solved in the arithmetical function $< n >$ and the solution is:

$$< n >= n^{-s} (\omega_k^{(-1)} * D_n \omega_k), \tag{24}$$

where we have introduced the operation D_n as

$$D_n f(k) = f(nk). \tag{25}$$

One immediately realizes that the arithmetic function corresponding to the correlations of a square-free model *is not* square-free. This is because the Dirichlet inverse operation does not conserve the square-free property and it is the main motivation to introduce a formalism able to handle generic polymer configurations with the suitable convolution. Moreover it also implies that, even for finite k, the correlations zeta function is no more a finite series; we want to show how it is possible to control its properties in the limit $k \to \infty$ using the statistical mechanics method of the iterative equations. This will provide a statistical mechanics meaning to the limiting correlations and a new point of view in the study of some number theoretical quantities.

The idea, which is a central one in statistical mechanics, is to study the "interaction" between one particle and the remaining ones or, in number theoretical terms, to have some control on the non multiplicativity of the ω.

Defining the function

$$\Gamma_n := \omega_k^{(-1)} * D_n \omega_k, \tag{26}$$

we consider an integer of the form pn, where p is a prime compatible with n (otherwise $\Gamma_{pn} = 0$). By definition we have

$$\Gamma_{pn}(l) = \sum_{d|l} \omega_k^{(-1)}(d) \omega_k(pn\frac{l}{d}); \tag{27}$$

indicating $\sum_{k \subset n}$ a sum over all the divisors of n counted with multiplicity (see appendix), we first observe that

$$\omega_k(pn\frac{l}{d}) = \omega_k(n\frac{l}{d}) {\sum_{r \subset \frac{l}{d}}}^{p} \lambda(r), \tag{28}$$

where the $\sum^p$ means a sum over all square-free integers build on p-incompatible primes and the function λ is the Liouville function defined by $\lambda(n) = (-1)^{\Omega(n)}$ where Ω is the number of prime factors counted with multiplicity. Since $\omega(p) = 1$ the previous formula gives an evaluation of how much the interaction ω deviates from a completely multiplicative function; it can be proved, for instance, observing that the factor $G(p, h)$ defined by

$$\omega(ph) = \omega(h) G(p, h) \tag{29}$$

is

$$G(p, h) = \prod_{p' \subset h} g(p, p') = \prod_{p' \subset h} ((g(p, p') - 1) + 1) = {\sum_{r \subset h}}^{p} (-1)^{\Omega(r)}, \tag{30}$$

which is the (28) since the integer pn is supposed to be compatible. Substituting the (28) inside the (27) we have

$$\Gamma_{pn}(l) = \sum_{d|l} \omega_k^{(-1)}(d) \omega_k(p) \omega_k(n\frac{l}{d}) {\sum_{r \subset \frac{l}{d}}}^{p} \lambda(r), \tag{31}$$

and interchanging the summation order

$$\Gamma_{pn}(l) = \sum_{r\subset l}^{p} \lambda(r) \sum_{d|\frac{l}{r}} \omega_k^{(-1)}(d)\,\omega_k(\frac{nl}{d}), \tag{32}$$

which is, up to renaming the sets,

$$\Gamma_{pn}(l) = \sum_{r\subset l}^{p} \lambda(r)\Gamma_{nr}(\frac{l}{r}). \tag{33}$$

This is the iterative equation we want to consider. In order to control its solutions we observe that, by inspection, it lives naturally as equation for the *two-variable* arithmetical function Γ; moreover defining the "index" of the quantity $\Gamma_n(l)$ as $\Omega(nl)$, the equations (33) can be solved iteratively observing that they allow to compute the family of index $\Omega(nl)+1$ in terms of that whose index is $\Omega(nl)$. This fact not only makes it possible to study the iterative solutions with the initial condition $\Gamma_1(1) = 1$ but it also gives hints on the suitable Banach space structure to be introduced in order to make use of the contraction principle.

6 The Contraction Regime for the Iteration

In our number theoretical context we can introduce the seminorms for the family of the $< n > (l)$ with $\Omega(nl) = m$, depending on a parameter δ to be optimized at the end,

$$N_m(\delta) = \sup_{n, 1\leq \Omega(n) \leq m} \sum_{l, \Omega(nl)=m} | < n > (l) | l^{-s} n^{(s-\delta)}; \tag{34}$$

we claim that, for suitable values of δ, this norm is contractive for the iterative equations. The proof is along the following lines. Using the (33) we observe that it holds the bound

$$\sum_{l,\Omega(nl)=m} | < pn > (l) | l^{-s}(pn)^{(s-\delta)} \leq \sum_{l,\Omega(nl)=m} p^{-s} \sum_{r\subset l}^{p} r^{s} | < rn > (\frac{l}{r}) | l^{-s}(pn)^{(s-\delta)}, \tag{35}$$

since the Liouville function is bounded in modulus by one. It follows that

$$\sum_{l,\Omega(nl)=m} | < pn > (l) | l^{-s}(pn)^{(s-\delta)} \leq N_m(\delta) p^{-\delta} \sum_{r}^{p} r^{-(s-\delta)} \tag{36}$$

where the last sum runs over the square-free r build on p-incompatible primes. We also observe that, since r runs over square-free integers,

$$\sum_{r}^{p} e^{-(s-\delta)\log r} \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{p'}^{p} e^{-(s-\delta)\log p'} \right)^{n} . \tag{37}$$

Making use of the entropy bound it is possible to control the sum on p-incompatible primes observing that it can be written as

$$\sum_{v=1}^{\infty} \sum_{p', v \leq \log p' < v+1} e^{-(s-\delta)\log p'} = \tag{38}$$

$$\leq \log p \sum_{v=1}^{\infty} (ce^{-(s-\delta)})^{v} = \log p \frac{ce^{-(s-\delta)}}{1-ce^{-(s-\delta)}}, \tag{39}$$

provided $ce^{-s} < e^{-\delta}$; we notice that the extension of the previous sum up to infinity makes the resulting bound uniform in k. Taking the supremum norm of the (35) we obtain

$$N_{m+1}(\delta) \leq N_m(\delta) \exp\left(\log 2(-\delta + \frac{ce^{-(s-\delta)}}{1-ce^{-(s-\delta)}}) \right) ; \tag{40}$$

which means that the we are in the contraction regime when $ce^{-s} < \frac{\delta e^{-\delta}}{1+\delta}$. The right hand side can easily be optimized observing that it is, on the positive real line, a function with the only maximum reached at the *golden ratio* $\bar{\delta} = \frac{\sqrt{5}-1}{2}$ (see [6] for a discussion). The previous convergence theorem means that the limit $k \to \infty$ of the correlation functions exist in the range defined by

$$ce^{-s} < \frac{\bar{\delta} e^{-\bar{\delta}}}{1+\bar{\delta}} \approx e^{-1.58} \tag{41}$$

and describe the equilibrium state of an hard-core interacting polymer system.

An easy corollary to be used in the control of connected correlations functions is that, for all the s defined by the (41), one has the bound

$$N_{m+1} \leq N_m e^{-\rho}, \tag{42}$$

where, defining the positive number $\epsilon := \frac{\bar{\delta} e^{-\bar{\delta}}}{1+\bar{\delta}} - ce^{-s}$ one has $\rho = \epsilon e^{\bar{\delta}} \log 2(2+\bar{\delta})$. In particular it is possible to check that, since $N_1 \leq 1$, it results $N_m \leq e^{-m\rho}$.

The reader should compare this general result with the one for the multiplicative case (Appendix B), where one has optimal convergence estimates.

7 The Exponential Decay of the Correlations

The statistical mechanics theory of the low activity expansion gives a systematic way to obtain bounds for the free energy and for all its derivatives with respect to external parameters. It is well known that those bound are equivalent to the bounds for the truncated correlation function and are usually given in terms of the distance between the polymers.

In our context the convolutory algebra permits a natural rephrasing of all these properties: the bound we present are given in terms of the volume of each polymer, i.e. $\log p$.

The quantities we are mainly interested in are the generalization of the two point truncated correlation function

$$< \alpha_{i_1}, \alpha_{i_2} >^T (s) :=< \alpha_{i_1} \alpha_{i_2} > (s) - < \alpha_{i_1} > (s) < \alpha_{i_2} > (s). \tag{43}$$

This function represents the deviation from the independence of the two events "p_{i_1} divides an integer" and "p_{i_2} divides an integer". As for the simple correlations an easy computation shows that it is possible to express them as Dirichlet series of a suitable arithmetical function:

$$< \alpha_{i_1}, \alpha_{i_2} >^T (s) = \sum_l (p_{i_1} p_{i_2})^{-s} (\Gamma_{p_{i_1} p_{i_2}} - \Gamma_{p_{i_1}} * \Gamma_{p_{i_2}}) l^{-s}. \tag{44}$$

We recognize that the two point truncated expectation is the Dirichlet series of the arithmetical function given by the second order of the formal logarithm of the functions Γ with respect to the *lower variable*:

$$\Gamma^T := \mathrm{Log}\Gamma. \tag{45}$$

Let us clarify the geometrical meaning of the operations which naturally appear considering the simple and the truncated correlation functions. First we notice that for the operation D_n it holds (see [22]) the Leibnitz rule with respect to the circle product (see appendix):

$$D_n(f \circ g) = D_n f \circ g + f \circ D_n g. \tag{46}$$

From it one can easily prove that the operation ∂_n defined by

$$\partial_n f(k) = f(nk)\frac{c(nk)}{c(k)}, \tag{47}$$

plays the role of a *multiple* derivative with respect to the Dirichlet product since it fulfills the composition rule $\partial_{n_1}\partial_{n_2} = \partial_{n_1 n_2}$ and, when n is a prime number, the Leibnitz rule with respect to the Dirichlet multiplication. This can be seen observing that defining the operation $\mathcal{D}$ from the set of the one-variable to that of the two-variable arithmetical functions by

$$(\mathcal{D}f)(n,k) = f(nk)\frac{c(nk)}{c(n)c(k)}; \tag{48}$$

it holds for it the important property:

$$\mathcal{D}(f * g) = \mathcal{D}f * \mathcal{D}g, \tag{49}$$

where the convolution at the right hand side is the two variable Dirichlet convolution.

In particular it holds the

$$\partial_p \mathrm{Exp} f = \partial_p f * \mathrm{Exp} f, \tag{50}$$

and

$$\partial_p \mathrm{Log} g = g^{(-1)} * \partial_p g. \tag{51}$$

Since ω is a square-free function $D_n\omega = \partial_n\omega$ we have, with $\omega^T := \mathrm{Log}\omega$,

$$\partial_n \omega^T = \Gamma^T_n. \tag{52}$$

Choosing $n = p$ it holds $\partial_p\omega^T = \Gamma_p$ which is

$$\omega^T(pn) = \frac{c(n)}{c(pn)}\Gamma_p(n). \tag{53}$$

This relation enable us to obtain a bound, inside our contraction regime, on a quantity which represents the free energy density centered around the prime p:

$$\sum_{l=1,p|l}^{\infty} \omega^T(l) l^{-s}. \tag{54}$$

In fact applying the (53) and the contraction scheme for the norm one has:

$$| \sum_{l=1,p|l}^{\infty} \omega^T(l) l^{-s} | \leq \sum_{k=1}^{\infty} \sum_{l,\Omega(pl)=k} |\Gamma_p(l)| p^{-s} l^{-s} = \tag{55}$$

$$\leq p^{-(s-\bar{\delta})} \sum_{k=1}^{\infty} e^{-\rho k} = e^{-(s-\bar{\delta})\log p} \frac{e^{-\rho}}{1-e^{-\rho}}; \tag{56}$$

which is the claimed exponential decay in terms of the polymer volume (notice that $s > \bar{\delta}$ in the contraction regime). In the same way it is possible to obtain the decay for the multiple truncated correlations functions; let us show it for the two-point case.

From (44) we have

$$| < \alpha_{i_1}, \alpha_{i_2} >^T (s)| \leq (p_{i_1} p_{i_2})^{-s} \sum_{l=1}^{\infty} (|\Gamma_{p_{i_1} p_{i_2}}(l)| + |\Gamma_{p_{i_1}} * \Gamma_{p_{i+2}}(l)|) l^{-s}. \tag{57}$$

The term with the convolution product of the right hand side is bounded using the (56) and the multiplicative property of the relative Dirichlet series. For the first terms one has:

$$\sum_{l=1}^{\infty} |\Gamma_{p_{i_1} p_{i_2}}(l)| l^{-s} = (p_{i_1} p_{i_2})^{\bar{\delta}} \sum_{k=2}^{\infty} \sum_{l,\Omega(p_{i_1} p_{i_2})=k} < p_{i_1} p_{i_2} > (l) l^{-s} (p_{i_1} p_{i_2})^{(s-\bar{\delta})} = \tag{58}$$

$$\leq (p_{i_1} p_{i_2})^{\bar{\delta}-s} \sum_{k=2}^{\infty} e^{-k\rho} = e^{-(s-\delta)\log(p_{i_1} p_{i_2})} \frac{e^{-2\rho}}{1-e^{-\rho}}. \tag{59}$$

Summing the two contributions we obtain

$$| < \alpha_{i_1}, \alpha_{i_2} >^T (s)| \leq e^{-(s-\bar{\delta})\log(p_{i_1} p_{i_2})} \left(\frac{e^{-2\rho}}{1-e^{-\rho}} + \frac{e^{-2\rho}}{(1-e^{-\rho})^2} \right) \tag{60}$$

which is the desired result. In the same way one can obtain the same exponential decay for all the other truncated correlation functions.

A Some Arithmetical Functions

Some arithmetical functions considered on this work are: the identity for the pointwise multiplication:

$$u(n) = 1 \quad \forall n, \tag{61}$$

the identity for the Dirichlet product

$$I(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{62}$$

the identity map from $\mathbb{N}$ to $\mathbb{N}$

$$N(n) = n, \tag{63}$$

and the "square" function:

$$Q(n) = \begin{cases} 1, & \text{if } n \text{ is a square;} \\ 0, & \text{otherwise,} \end{cases}. \tag{64}$$

In terms of them it is easy to express other important functions: the Möbius function

$$\mu = u^{-1}, \tag{65}$$

its absolute value

$$|\mu| = u * Q^{(-1)}, \tag{66}$$

the Liouville function

$$\lambda = \mu * Q = |\mu|^{(-1)}. \tag{67}$$

It can be useful to introduce another convolution product: considering an integer as an unordered sequence of primes $n \equiv \{p_{i_1}, ..., p_{i_1}, p_{i_2}, ..., p_{i_2}, ..., p_{i_n}\}$ the natural definition of convolution is the sum over all the subsequences

$$f \circ g(n) = \sum_{d \subset n} f(d) g(\frac{n}{d}), \tag{68}$$

where, for instance, the set of subsequences of 4 is $\{1, 2, 2, 4\}$. It is easy to see that it is related to the Dirichlet one by:

$$f \circ g(n) = \sum_{d|n} f(d) g(\frac{n}{d}) c(d, \frac{n}{d}), \tag{69}$$

Function	1	2	3	4	5	6	7	8	9	10
Q	1	0	0	1	0	0	0	0	1	0
$Q^{(-1)}$	1	0	0	-1	0	0	0	0	-1	0
λ	1	-1	-1	1	-1	1	-1	-1	1	1
μ	1	-1	-1	0	-1	1	-1	0	0	1
ω_3	1	1	1	0	1	1	0	0	0	1
$\omega_3^{(-1)}$	1	-1	-1	1	-1	1	0	-1	1	1
$D_2\omega_3$	1	0	1	0	1	0	0	0	0	0
$\omega_3^{(-1)} * D_2\omega_3$	1	-1	0	1	0	0	0	-1	0	0

where $c(l,m) = \frac{c(lm)}{c(l)c(m)}$ with $c(p_{i_1}^{\alpha_{i_1}} \cdots p_{i_k}^{\alpha_{i_k}}) = \prod_{j=1}^{k} \alpha_{i_k}!$. This property is equivalent to the fact that the $\circ$-product plays the role of the convolution for the deformed zeta functions with the non-character activity $\tilde{z}(n) = \frac{n^{-s}}{c(n)}$

$$\tilde{Z}_f(s) = \sum_n f(n)\tilde{z}(n), \tag{70}$$

i.e. it holds

$$\tilde{Z}_f(s)\tilde{Z}_g(s) = \tilde{Z}_{f\circ g}(s). \tag{71}$$

For both the convolution products it is possible to define the powers of a function and, in some cases, power series like exponential and logarith: Defining the sets of arithmetical functions $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively by the conditions $f(1) = 0$ and $f(1) = 1$ it is possible to construct well defined power series in the convolution products; in particular the arithmetical function corresponding to the exponential for $f \in \mathcal{A}_0$ is

$$\mathrm{Exp} f = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} \tag{72}$$

and the logarithm, for $h \in \mathcal{A}_1$ which is, defining $h = I + \tilde{h}$,

$$\mathrm{Log} h = \sum_{k=0}^{\infty} (-1)^k \frac{\tilde{h}^{(k)}}{k}. \tag{73}$$

It is easy to see that the operation Exp: $\mathcal{A}_0 \to \mathcal{A}_1$ and Log: $\mathcal{A}_1 \to \mathcal{A}_0$ are mutually inverse.

From a combinatorial point of view the main advantage to consider the circle product is that it permits to define the exponential of a function as the sum over the *partitions*. Example: the $\circ$-exponential of f in $12 = 2^2 \cdot 3$ is in fact

$$\mathrm{Exp}(f)(12) = f(12) + 2f(2)f(6) + f(3)f(4) + f(2)f(2)f(3); \tag{74}$$

On the other hand the Dirichlet exponential implies the important *formal* property:

$$Z_f(s) = exp(Z_{\mathrm{Log}f}(s)), \tag{75}$$

which permits to obtain the *free energy* series expansion starting from the partition function series expansion on a Dirichlet series.

B Convergence in the Multiplicative Case

First keeping within the context of general polymer models, we set $\psi(X) := \phi(X)z^X$ so that the partition function equals $Z = \sum_{X \in \hat{P}^\infty} \psi(X)$. Then the probability that the k-polymer X is present is defined by

$$\rho(X) := \frac{\sum_{Y \in \hat{P}^\infty} \psi(Y \cdot X)}{\sum_{Y \in \hat{P}^\infty} \psi(Y)} = \sum_{Y \in \hat{P}^\infty} \Delta_X(Y) \tag{76}$$

with $\Delta_X(Y) = (\psi^{-1} * D_X \psi)(Y)$.

The terms $\Delta_X(Y)$ meet the following recursive equation w.r.t. addition of a polymer $\gamma \in P$ to X:

$$\Delta_{\gamma \cdot X}(Y) = z(\gamma) \sum_{S \subset Y}{}^{\gamma} (-1)^{|S|} \Delta_{X \cdot S}(Y/S). \tag{77}$$

Here the superscript γ means that summation is restricted to multi-polymers S of Y which are incompatible with γ.

We now express the correlation function

$$\rho_k(X) =< \alpha_{i_1} \cdots \alpha_{i_r} >_k (s) = \sum_{Y \in \mathbb{N}} \Delta^k_X(Y)$$

at the inverse temperature s as a series in the activities. By definition $\Delta^k_X(Y) = (\omega_k^{-1} * D_X \omega_k)(Y) \cdot z^X z^Y$, with D_n defined in (25).

Now by (77) for a prime $p \in \mathbb{P}_k$

$$\begin{aligned}\Delta^k_{p\cdot X}(Y) &= z_s(p){\sum_{S|Y}}^{p}(-1)^{|S|}\Delta^k_{S\cdot X}(Y/S) \\ &= z_s(p)\sum_{S:p|S|Y}\lambda(S)\Delta^k_{S\cdot X}(Y/S), \qquad (78)\end{aligned}$$

since $S \in \mathbb{N}$ is incompatible with $p \in \mathbb{N}$ iff $p|S$. Furthermore for an integer S of the form $S = \prod_i p_i^{\alpha_i}$ by definition $|S| = \sum_i \alpha_i = \Omega(S)$. Moreover, the Liouville function λ is defined by $\lambda(S) = (-1)^{\Omega(S)}$, showing (78).

Eqs. (78) are the iterative equations we want to consider. Defining the "index" of the quantity $\Delta^k_X(Y)$ as $\Omega(XY)$, the equations (78) can be solved iteratively observing that give the the family of index $\Omega(XY) + 1$ in terms of that whose index is $\Omega(XY)$. This fact not only makes it possible to study the iterative solutions with the initial condition $\Delta^k_1(1) = 1$ but it also gives hints on the suitable Banach space structure to be introduced in order to make use of the contraction principle.

In our number-theoretical context the seminorms N^δ_m have the form

$$\begin{aligned}N^\delta_m(\rho_k) &= \sup_{X\in\mathbb{N}}\sum_{Y\in\mathbb{N},\Omega(XY)=m}|\Delta^k_X(Y)|e^{-(\ln(a)+\delta)v(X)} \\ &= \sup_{X\in\mathbb{N}}\sum_{Y\in\mathbb{N},\Omega(XY)=m}|\Delta^k_X(Y)|X^{(A'-\delta')} \qquad (79)\end{aligned}$$

with $A' = -\ln(a)/\ln 2$, $\delta' = \delta/\ln 2$ and $v(X) = \ln(X)/\ln(2)$.

In the multiplicative case $\omega = |\mu|$ we can improve the convergence estimate to the optimal value.

$$\begin{aligned}N^\delta_m(\rho_k) &= \sup_{X\in\mathbb{N}}\sum_{Y\in\mathbb{N},\Omega(XY)=m}|\omega_k^{-1} * D_X\omega_k(Y)||z^X z^Y|e^{-(\ln(a)+\delta)v(X)} \\ &= \sup_{X\in\hat{P}_k^\infty}\sum_{Y\in\hat{P}_k^\infty,\Omega(XY)=m}|\omega^{-1} * D_X\omega(Y)||z^X z^Y|e^{-(\ln(a)+\delta)v(X)} (80)\end{aligned}$$

Remember that $\mathbb{P}_k$ consists of of the first k primes. So

$$\hat{P}_k^\infty = \{n \in \mathbb{N} \mid p \in \mathbb{P} \text{ and } p|n \Rightarrow p \in \mathbb{P}_k\}.$$

Since we have assumed $\omega = |\mu|$, $\omega^{-1} = \lambda$. So

$$\omega^{-1} * D_X\omega(Y) = \sum_{d|Y}\lambda(Y/d)|\mu|(Xd)$$

$$= \sum_{d|Y,(d,X)=1} \lambda(Y/d) = \begin{cases} \lambda(Y) & ,p|Y \Rightarrow p|X (p \in \mathbb{P}) \\ 0 & ,\text{otherwise.} \end{cases}$$

Now if there is a m-independent bound b on the minimal number of prime factors of an X which attains the supremum in (79), then $m \mapsto N_m^\delta(\rho_k)$ converges exponentially fast to zero, since then there are only $\mathcal{O}(b^m)$ terms in the sum (79). Then we are done.

So we can assume w.l.o.g. that the maximal number of prime factors of the X grows with m. Now since $m+1 \geq 2$, the $\tilde{X} \in \mathbb{N}$ which attain the supremum in $N_{m+1}^\delta(\rho_k)$ are unequal 1 so that we can write them in the form $\tilde{X}_0 = pX$ and assume that $p \in \mathbb{P}$ is the largest prime factor. Then we use the recursion relation (77):

$$\begin{aligned} N_{m+1}^\delta(\rho_k) &= \sup_{\tilde{X}\in\mathbb{N}} \sum_{Y\in\mathbb{N},\Omega(\tilde{X}Y)=m+1} |\Delta_{\tilde{X}}^k(Y)|\tilde{X}^{(A'-\delta')} \\ &= \sum_{Y\in\mathbb{N},\Omega(XY)=m} |\Delta_{pX}^k(Y)|(pX)^{(A'-\delta')} \\ &= \sum_{Y\in\mathbb{N},\Omega(XY)=m} |z_s(p) \sum_{S:p|S|Y} \lambda(S)\Delta_{XS}^k(Y/S)|(pX)^{(A'-\delta')} \\ &\leq p^{-\delta'} \sum_{Y\in\mathbb{N},\Omega(XY)=m} \sum_{S:p|S|Y} |\Delta_{XS}^k(Y/S)|(SX)^{(A'-\delta')}S^{(\delta'-A')} \\ &= p^{-\delta'} \sum_{S:p|S} S^{(\delta'-A')} \sum_{Y:S|Y,\Omega(XY)=m} |\Delta_{XS}^k(Y/S)|(SX)^{(A'-\delta')} \\ &\leq p^{-\delta'} \sum_{S:p|S} S^{(\delta'-A')} N_m^\delta(\rho_k) \\ &= p^{-A'}\zeta(A'-\delta')N_m^\delta(\rho_k) \end{aligned} \tag{81}$$

Now we assume that $A' > 1$ and $\delta' = \frac{1}{2}(A'-1)$. Then as m and thus p become large, the constant $c := p^{-A'}\zeta(A'-\delta')$ in

$$N_{m+1}^\delta(\rho) \leq cN_m^\delta(\rho)$$

coming from (81) is getting strictly smaller than one, implying convergence. In other words, we have absolute convergence if $|z(p)| \leq p^{-1-\varepsilon}$ for some $\varepsilon > 0$. This is clearly optimal.

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