# Griffiths Inequalities in the Nishimori Line

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The Griffiths inequalities for Ising spin-glass models with Gaussian randomness of non-vanishing mean are proved using the properties of the Gaussian distribution and the gauge symmetry of the system. These inequalities imply that the correlation functions are non-negative and monotonic along the Nishimori line in the phase diagram. From this result, the existence of the thermodynamic limit for the correlation functions and the free energy is proved under free and fixed boundary conditions. Relations between the location of multi-critical points are also derived for different lattices.

## §1. Introduction

The Griffiths inequalities give us significant knowledge about phase transitions in ferromagnetic Ising models.[1),2)] These inequalities are composed of two statements: correlation functions are non-negative and increase monotonically with the interaction among any set of the spins. From the Griffiths inequalities, the existence of the free energy per spin and correlation functions is proved under several boundary conditions. Furthermore, relations on critical points for various lattices are derived. However, since the proof needs the conditions that all the interactions are ferromagnetic, there was no proof of similar inequalities for the spin-glass models which have both ferromagnetic and antiferromagnetic interactions with non-vanishing mean.

Recently, the first Griffiths inequality has been proved for the Sherrington-Kirkpatrick model and the Edward-Anderson model using integration by parts.[3),4)] Moreover, in 3) and 4), monotonicity of correlation functions is proved not with respect to the strength of the interaction but with the variance of the randomness. On the other hand, the gauge theory, which uses gauge symmetry of the system, is known to be useful for analytic investigations in spin-glass models, yielding various exact results on a line called the Nishimori line (NL).[5),6)] We have been able to prove both Griffiths inequalities with respect to the mean of the randomness for the Gaussian spin glass on the NL using the gauge theory and the technique of integration by parts.[7)] The resulting inequalities can be used to prove the existence of the thermodynamic limit for correlation functions and the free energy and to derive inequalities on the location of the multicritical point for various lattices. The present contribution briefly reviews these results.

In the next section, we present our results and outline their proof. Applications of these inequalities are discussed in the third section.

## §2. Inequalities

Let us consider a finite system $\Omega$ of Ising spins $S_i = \pm 1$ and denote their product by

$$S_A = \prod_{i \in A} S_i$$

for a subset $A$ of $\Omega$. The partition function is defined as

$$Z = \sum_S \exp\left(\sum_{\{A\}} \beta_A J_A S_A\right), \tag{1}$$

where $\beta_A \geq 0$ is the inverse of local temperature for subset $A$, and $J_A$ is a quenched random variable which follows the Gaussian distribution $P_A(J_A)$ with positive mean $J_{A0}$ and variance $\sigma_A^2$.

The NL is defined in terms of a parameter $x_A$ as

$$\beta_A = \frac{x_A}{\sigma_A}, \quad J_{A0} = \sigma_A x_A. \tag{2}$$

Our results are the following inequalities:

$$[\langle S_B \rangle] \geq \frac{x_B^2}{1 + x_B^2} \quad (B \in \{A\}), \qquad [\langle S_B \rangle] \geq 0 \quad (B \notin \{A\}) \tag{3}$$

$$\frac{d}{dx_B}[\langle S_C \rangle] = 2x_B \left[ (\langle S_B S_C \rangle - \langle S_B \rangle \langle S_C \rangle)^2 \right] \geq 0, \tag{4}$$

where the angular brackets denote the thermal average with local temperature $\beta_A^{-1}$ and the square brackets stand for the configurational average. Both inequalities hold for arbitrary subsets $B, C$ as long as the parameters satisfy the NL condition (2).

The first inequality (3) is proved using the Cauchy-Schwarz inequality and the gauge theory. This inequality yields a lower bound for correlation function $[\langle S_B \rangle]$. If the parameter $x_B$ for the corresponding subset $B$ tends to infinity, this bound increases toward unity. Therefore all the spins in the subset $B$ are parallel to each other. This result is natural because large $x_B$ implies that the interaction $J_B$ is almost ferromagnetic and the local temperature is nearly zero.

For the proof of the second inequality (4), we express the total derivative by the parameter $x_B$ as

$$\frac{d}{dx_B}[\langle S_C \rangle] = \frac{1}{\sigma_B} \frac{\partial}{\partial \beta_B} [\langle S_C \rangle] \bigg|_{\mathrm{NL}} + \sigma_B \frac{\partial}{\partial J_{B0}} [\langle S_C \rangle] \bigg|_{\mathrm{NL}}. \tag{5}$$

To calculate the partial derivatives with respect to $\beta_B$ and $J_B$, the following identities for an arbitrary operator $O$ are useful.

$$[J_B O] = J_{B0}[O] + \sigma_B^2 \left[ \frac{\partial O}{\partial J_B} \right] \tag{6}$$

$$\frac{\partial}{\partial J_{B0}}[O] = \left[ \frac{\partial O}{\partial J_B} \right]. \tag{7}$$

These equations are derived from the properties of the Gaussian distribution $P_B(J_B)$: a) it is the function of $J_B - J_{B0}$, b) its derivative is proportional to itself, and c) it decays rapidly as $|J_B| \to \infty$. Note that both equations are valid on and away from the NL. From Eqs. (6) and (7), we obtain

$$\frac{d}{dx_B}[\langle S_C \rangle] = 2x_B[\langle S_B S_C \rangle - \langle S_B \rangle \langle S_C \rangle - \langle S_B \rangle \langle S_B S_C \rangle + \langle S_B \rangle^2 \langle S_C \rangle]. \tag{8}$$

The following identities are derived by the local gauge transformations on the NL under certain boundary conditions (free, periodic and fixed)[5),6)] :

$$\begin{aligned} &[\langle S_B S_C \rangle] = [\langle S_B S_C \rangle^2] \\ &[\langle S_B \rangle \langle S_C \rangle] = [\langle S_B \rangle \langle S_B S_C \rangle] = [\langle S_B \rangle \langle S_C \rangle \langle S_B S_C \rangle] \\ &[\langle S_B \rangle^2 \langle S_C \rangle] = [\langle S_B \rangle^2 \langle S_C \rangle^2]. \end{aligned} \tag{9}$$

Therefore we obtain the second inequality (7) from Eq. (8).

The inequality (4) implies that an arbitrary $n$-point correlation function, which includes the order parameter, monotonically increases with parameter for any subset. Since a two-point correlation function is also an increasing function of $x$, the correlation length becomes larger as $x$ increases.

It is also possible to prove the monotonicity and concavity of the pressure function $P = [\log Z]$ , which corresponds to the free energy. From Eqs. (3) and (4), we obtains that the first and second derivatives of the pressure are positive as

$$\frac{dP}{dx_B} = x_B[\langle S_B \rangle] + x_B \geq 0, \tag{10}$$

$$\frac{d^2 P}{dx_B dx_C} = \begin{cases} x_B \dfrac{d}{dx_C}[\langle S_B \rangle] \geq 0 & (B \neq C) \\ [\langle S_B \rangle] + 1 + x_B \dfrac{d}{dx_B}[\langle S_B \rangle] \geq 0 & (B = C). \end{cases} \tag{11}$$

## §3. Discussions

Let us consider the case that a parameter $x_A$ is equal to zero. The interaction $J_A$ of the subset $A$ is not zero because it distributes with variance $\sigma_A$. However, since the local temperature $T_A = \beta_A^{-1}$ is infinity, the interaction of the subset $A$ is negligible. Thus, $x_A = 0$ implies that there is no interaction among the subset $A$. From the second inequality (4), addition of an interaction for any subset increases all correlation functions.

From this result, we can prove that the correlation functions have a thermodynamic limit under free boundary conditions. Let us consider two finite sets $\Omega' \subset \Omega$. Since the subset $\Omega$ is obtained from $\Omega'$ by adding interactions, the previous argument yields

$$[\langle S_B \rangle]_{\Omega'}^{(\text{free})} \leq [\langle S_B \rangle]_{\Omega}^{(\text{free})}. \tag{12}$$

Thus, correlation functions are increasing functions with respect to the system size. Since we consider Ising spins $S_i = \pm 1$, correlation functions are bounded by unity.

Therefore each of them has its unique limit as the volume of $\Omega$ tends to infinity. Similarly, we can prove the existence of the thermodynamic limit for correlation functions under another boundary condition that all the boundary spins are fixed up. In addition, Eq. (10) can be applied to prove that the thermodynamic limit of the pressure (free energy) exists.

The location of the multicritical points, which are believed at the boundary between paramagnetic and ferromagnetic phases on the NL,[5] for the various lattices in the $\pm J$ Ising models has been discussed by Kitatani.[8] The same result for the Gaussian spin glass is obtained from the second inequality (4). For instance, we consider the simple cubic (SC) and square (SQ) lattices. Since the simple cubic lattice is obtained from the square lattice by adding interactions, the magnetization of the simple cubic lattice is greater than that of the square lattice. Therefore, we obtain

$$T_{\rm c}^{\rm SC} \geq T_{\rm c}^{\rm SQ}. \tag{13}$$

Another example is the simple lattice with only nearest neighbor interactions and the simple lattice with nearest and next-nearest neighbor interactions. In this case, the multicritical temperature of the former lattice is less than that of the latter one.

## §4. Summary

Inequalities similar to the Griffiths inequalities have been proved for the Gaussian spin glass. These inequalities are valid for any lattices and any range of interactions. However, we proved them using the properties of the Gaussian distribution and the exact result from the gauge theory under the NL conditions. Therefore, it is a future problem to extend similar inequalities to other models, and away from the NL.

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