Stability of the Spin Glass Phase under Perturbations

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We introduce and prove a new stability property of the quenched equilibrium state for the spin glass phase and show that it implies the whole set of Ghirlanda-Guerra identities. The new stability deals with perturbations which reproduces both thermal and disorder fluctuations, thus generalizing the standard stochastic stability of disordered systems.

The Gibbs-Boltzmann state $\omega_{\beta,N}$ of a statistical mechanics system of N interacting spins $\sigma = (\sigma_1, ..., \sigma_N)$, with Hamiltonian $H(\sigma)$ at inverse temperature β, admits the classical probabilistic interpretation as the *deformation* of the uniform measure over spin configurations:

$$\omega_{\beta,N}(f) \;=\; \frac{\mu_N(fe^{-\beta H})}{\mu_N(e^{-\beta H})}\;, \tag{1}$$

with

$$\mu_N(f) \;=\; \frac{1}{2^N}\sum_{\sigma} f(\sigma)\;, \tag{2}$$

and f a smooth bounded function of the spin configurations. Such a deformed state $\omega_{\beta,N}$ fulfills a remarkable *stability property* with respect to further small deformations (*perturbations*): considering the Hamiltonian per particle

$$h(\sigma) = \frac{H(\sigma)}{N} \tag{3}$$

and the perturbation with parameter λ defined as

$$\omega^{(\lambda)}_{\beta,N}(f) \;=\; \frac{\omega_{\beta,N}(fe^{-\lambda h})}{\omega_{\beta,N}(e^{-\lambda h})} \tag{4}$$

the Gibbs-Boltzmann measure is stable, i.e. λ-independent, in the thermodynamic limit $N \to \infty$. In fact one can observe that the perturbation amounts to a small temperature shift:

$$\omega^{(\lambda)}_{\beta,N}(f) \;=\; \frac{\mu_N(fe^{-\beta H-\lambda h})}{\mu_N(e^{-\beta H-\lambda h})} \;=\; \omega_{\beta+\frac{\lambda}{N},N}(f) \tag{5}$$

which implies that it has a vanishing effect in the large volume limit a part on isolated singularities, possibly related to phase transitions. More precisely one can prove the stability as follows: since for all β intervals and all values of λ one has, thanks to (5),

$$\int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\lambda}d\beta \;=\; \frac{1}{N}\int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\beta}d\beta = \frac{\omega^{(\lambda)}_{\beta_1,N}(f) - \omega^{(\lambda)}_{\beta_0,N}(f)}{N} \tag{6}$$

one obtains:

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1} \frac{d\omega^{(\lambda)}_{\beta,N}(f)}{d\lambda}d\beta \;=\; 0 \quad \forall\;\lambda, \quad \forall\;[\beta_0,\beta_1]\;. \tag{7}$$

As a consequence, computing the derivative at $\lambda = 0$, one has

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1} \left[\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)\right] d\beta \;=\; 0\;. \tag{8}$$

For the special case $f = h$ the previous formula implies that the Hamiltonian per particle converges to a constant for large volumes with respect to the Gibbs measure, at least in $\beta-$integral average. Higher order derivatives with respect to λ of $\omega^{(\lambda)}_{\beta,N}(f)$ (i.e. cumulants of h) are then enforced to vanish since cumulants are homogeneous polynomials of the constant values of h with coefficients whose sum is zero.

Formula (8) has interesting consequences. It says, for instance, that the order parameter (i.e. the magnetization) for a mean field ferromagnetic Hamiltonian has a trivial distribution. In the Curie-Weiss model at zero magnetic field, for which the Hamiltonian per particle is the square magnetization, the previous identity implies that (by choosing $f = h$),

$$\omega_\beta(\sigma_1\sigma_2\sigma_3\sigma_4) \;=\; \omega_\beta(\sigma_1\sigma_2)^2 \tag{9}$$

in $\beta-$average [CGI]. One can indeed prove that (9) holds for all β using the methods developed in [EN]. The choice $f = h^n$, or equivalently higher order derivatives in λ of the perturbed state, gives the well known factorization property of the $2n$-point function as an n-th power of the 2-point function.

In a disordered system defined by a centered Gaussian Hamiltonian $\mathcal{H}(\sigma)$ of covariance (generalized overlap)

$$\mathrm{Av}\left(\mathcal{H}(\sigma)\mathcal{H}(\tau)\right) \;=\; N c_N(\sigma,\tau) \tag{10}$$

the equilibrium measure is the quenched average of the random Boltzmann-Gibbs state $\omega_{\beta,N}$: for a bounded random function f it is defined by

$$\langle f\rangle_{\beta,N} \;=\; \mathrm{Av}\left(\omega_{\beta,N}(f)\right)\ . \tag{11}$$

The thermodynamic properties of the system are expressed in terms of a set of random variables $\{c_{i,j}\}$ related to the quenched expectation of the covariance entries. Namely, considering the Boltzmann-Gibbs product state $\Omega_{\beta,N} = \omega_{\beta,N}\times\omega_{\beta,N}$, one defines the random variables $c_{i,j}$ and their joint distribution by:

$$\mathbb{E}_{\beta,N}(c_{i,j}) = \mathrm{Av}\left(\Omega_{\beta,N}(c(\sigma^{(i)},\sigma^{(j)}))\right)\ . \tag{12}$$

In [AC] it was identified a *stochastic stability* property of the quenched state, i.e. an invariance with respect to the stochastic perturbation:

$$\langle f\rangle^{(\lambda)}_{\beta,N} \;=\; \mathrm{Av}\left(\frac{\omega_{\beta,N}(f e^{\sqrt{\lambda}\mathcal{K}})}{\omega_{\beta,N}(e^{\sqrt{\lambda}\mathcal{K}})}\right)\ , \tag{13}$$

where $\mathcal{K}(\sigma)$ is a random field (independent from the Hamiltonian) whose covariance is $c_N(\sigma,\tau)$, and it was shown that the stochastic perturbation is equivalent to a temperature shift

$$\langle f\rangle^{(\lambda)}_{\beta,N} \;=\; \langle f\rangle_{\sqrt{\beta^2+\frac{\lambda}{N}},N}\ , \tag{14}$$

from which stability follows [CG1]

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1}\frac{d\langle f\rangle^{(\lambda)}_{\beta,N}}{d\lambda}d\beta \;=\; 0 \quad \forall\ \lambda, \quad \forall\ [\beta_0,\beta_1]\ . \tag{15}$$

By consequence at $\lambda = 0$ one obtains:

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1}\mathrm{Av}\left(\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)\right)d\beta \;=\; 0\ . \tag{16}$$

The previous formula implies (taking $f = h$ and integrating by parts)

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1}\mathbb{E}_{\beta,N}(c_{1,2}^2 - 4c_{1,2}c_{2,3} + 3c_{1,2}c_{3,4})\,d\beta \;=\; 0\ . \tag{17}$$

We stress the fact that the previous identity holds for a general Gaussian Hamiltonian, both mean field or short range, in terms of its own covariance. For $f = h^n$ one can see [CG1] that the identities that can be derived from (16) are, like the (17), zero average polynomials in the $c_{i,j}$ with respect to the quenched measure. See also [Ba] for an alternative derivation.

In [Gu] it was introduced a method, based on bounds for the energy fluctuations, which leads to the set of Ghirlanda-Guerra identities [GG, CG2, Bo, T1]; the lowest order are for instance:

$$\mathbb{E}_{\beta,N}(c_{1,2}c_{2,3}) \;=\; \frac{1}{2}\,\mathbb{E}_{\beta,N}(c_{1,2}^2) + \frac{1}{2}\,\mathbb{E}_{\beta,N}(c_{1,2})^2 \tag{18}$$

$$\mathbb{E}_{\beta,N}({}_{1,2}c_{3,4}) \;=\; \frac{1}{3}\,\mathbb{E}_{\beta,N}(c_{1,2}^2) + \frac{2}{3}\,\mathbb{E}_{\beta,N}(c_{1,2})^2\;. \tag{19}$$

Unlike the set of identities that can be derived from stochastic stability, these also include non-linear terms of the overlap expectations. In recent times it was shown that an invariance under reshuffling introduced in the framework of competing particle systems [ArAi] implies the whole set of Ghirlanda-Guerra identities [A].

To this purpose we introduce and prove here a novel stability property for the spin glass quenched state. We show that from such a stability property the whole set of Ghirlanda Guerra relations can be derived.

We define the perturbation of quenched state as

$$\langle\langle f\rangle\rangle^{(\lambda)}_{\beta,N} \;=\; \frac{\mathrm{Av}\left(\omega_{\beta,N}(fe^{-\lambda h})\right)}{\mathrm{Av}\left(\omega_{\beta,N}(e^{-\lambda h})\right)}\;. \tag{20}$$

We observe that this new perturbation is the analog, for the quenched measure of a random Hamiltonian, of the standard perturbation (4) introduced for deterministic systems with respect to the Boltzmann-Gibbs measure. On the other side we notice that while the stochastic stability perturbation (13), as much as the standard perturbation for deterministic system, amounts to a small temperature shift, the newly introduced perturbation cannot be reduced to just a small temperature change but it also involves a small change in the disorder. More precisely the explicit expression of (20) reads

$$\langle\langle f\rangle\rangle^{(\lambda)}_{\beta,N} \;=\; \frac{\mathrm{Av}\left(\frac{\sum_\sigma f(\sigma)e^{-(\beta+\lambda/N)H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}}\right)}{\mathrm{Av}\left(\frac{\sum_\sigma e^{-(\beta+\lambda/N)H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}}\right)} \tag{21}$$

where it clearly appears that only the numerator of the random Boltzmann-Gibbs state is affected by the change.

Our main result is summarized by the following

Proposition 0.1 *With the definition given above, the quenched state of a Gaussian spin glass is stable under the deformation (20), i.e.*

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1} \left.\frac{d\langle\langle f\rangle\rangle^{(\lambda)}_{\beta,N}}{d\lambda}\right|_{\lambda=0} d\beta \;=\; 0\;. \tag{22}$$

Moreover the property (22) implies the whole set of the Ghirlanda-Guerra identities: for a bounded f function of the generalized overlaps $\{c_{i,j}\}$ (with $i,j\in\{1,...,n\}$):

$$\mathbb{E}_{\beta,N}(f\,c_{1,n+1}) = \frac{1}{n}\mathbb{E}_{\beta,N}(f)\,\mathbb{E}_{\beta,N}(c_{1,2}) + \sum_{j=2}^{n}\mathbb{E}_{\beta,N}(fc_{1,j}) \tag{23}$$

Proof: A simple calculation shows that

$$\left.\frac{d\langle\langle f\rangle\rangle^{(\lambda)}_{\beta,N}}{d\lambda}\right|_{\lambda=0} = < fh >_{\beta,N} - < f >_{\beta,N} < h >_{\beta,N}\;\;. \tag{24}$$

The right hand side can be decomposed into two terms which can be identified as the thermal and the disorder correlations:

$$\begin{aligned}\left.\frac{d\langle\langle f\rangle\rangle^{(\lambda)}_{\beta,N}}{d\lambda}\right|_{\lambda=0} &= \mathrm{Av}\left(\omega_{\beta,N}(fh) - \omega_{\beta,N}(f)\omega_{\beta,N}(h)\right) + \\ &+\mathrm{Av}\left(\omega_{\beta,N}(f)\omega_{\beta,N}(h)\right) - \mathrm{Av}\left(\omega_{\beta,N}(f)\right)\mathrm{Av}\left(\omega_{\beta,N}(h)\right).\end{aligned} \tag{25}$$

In [CG2] the two previous terms were proved to converge to zero in β average and using integration Gaussian by parts it was shown how they imply formula (23). $\square$

Remark 1 *It is interesting to notice that the new stability property introduced in this paper as well as those introduced in the past admit a simple formulation in terms of cumulant generating function. Defining that function for the quenched state as*

$$\psi_{\beta,N}(\lambda) \;=\; \ln \mathrm{Av}\left(\frac{Z_{\beta+\lambda/N}}{Z_\beta}\right) \;=\; \ln\langle e^{\lambda h}\rangle_{\beta,N} \tag{26}$$

the (22) is equivalent to the property of asymptotic flatness at the origin

$$\lim_{N\to\infty}\int_{\beta_0}^{\beta_1} \left.\frac{d^2\psi_{\beta,N}(\lambda)}{d\lambda^2}\right|_{\lambda=0} d\beta \;=\; 0\,. \tag{27}$$

In particular defining the generating function of thermal fluctuations as

$$\bar{\psi}_{\beta,N}(\lambda) \;=\; \mathrm{Av}\left(\ln\omega_{\beta,N}(e^{\lambda h})\right) \tag{28}$$

and the generating function of disorder fluctuations as

$$\tilde{\psi}_{\beta,N}(\lambda) \;=\; \ln \mathrm{Av}\left(e^{\lambda\omega_{\beta,N}(h)}\right) \tag{29}$$

one has

$$\frac{d^2\psi_{\beta,N}(\lambda)}{d\lambda^2} \;=\; \frac{d^2\bar{\psi}_{\beta,N}(\lambda)}{d\lambda^2} + \frac{d^2\tilde{\psi}_{\beta,N}(\lambda)}{d\lambda^2}. \tag{30}$$

The results shown in this paper provides a straightforward method to obtain the Ghirlanda-Guerra identities of the spin glass phase by a simple computation of a derivative and a Gaussian integration by parts. This provides a new interpretation, using a stability argument, of the vanishing fluctuation property from which they were originally derived [GG].

The relevance of the stability properties and of the Ghirlanda-Guerra identities has been shown in the work [ArAi] and [Pan] where, under the hypothesis of discreteness of the overlap distribution it was proved, respectively, that competing particle systems satisfying invariance under reshuffling or spin systems satisfying Ghirlanda-Guerra identities do fulfill the hierarchical structure (ultrametricity) originally introduced in the Parisi work for the mean field spin glass [MPV].

The present work provides a further bridge between those two approaches, whose mutual relation has still to be fully clarified [T2], suggesting that the invariance under reshuffling is well represented by our newly introduced stability under perturbation.

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