Local Order at Arbitrary Distances in Finite-Dimensional Spin-Glass Models

Pierluigi Contucci*
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S.Donato 5, 40127 Bologna, Italy

Francesco Unguendoli†
Dipartimento di Matematica Pura ed Applicata,
Università di Modena e Reggio Emilia,
Via Campi 213/B, 41100 Modena, Italy
(Dated: August 30th, 2004)

For a finite dimensional spin-glass model we prove local order at low temperatures for both local observables and for products of observables at arbitrary mutual distance. When the Hamiltonian includes the Edwards-Anderson interaction we prove *bond* local order, when it includes the random-field interaction we prove *site* local order.

PACS numbers: 05.50.+q, 75.50.Lk

Spin-glass models in finite dimensions, in particular the nearest neighboor Edwards-Anderson and the random-field model, are among the most intensely studied models in condensed matter. Yet their low-temperature phase remains largely unknown not only from a rigourous mathematical perspective but also from a theoretical physics point of view ([1, 2] and [3]).

In this letter we show a simple rigorous proof of the local order property, i.e. the fact that the overlap distribution concentrates close to one at low temperatures and at arbitrary distances between local observables.

We consider models of Ising spin configurations arranged in the d-dimensional lattice (for instance $\mathbb{Z}^d$); each spin interacts with a random esternal field and with its nearest neighboors so that on a finite box Λ the Hamiltonian is

$$H_\Lambda(\sigma) \;=\; -\sum_{\substack{i,j\in\Lambda\\ |i-j|=1}} J_{i,j}\sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i \;, \qquad (1)$$

where the $J_{i,j}$ are i.i.d. Gaussian random variables with $\mathrm{Av}\,(J_{i,j}) = 0$ and $\mathrm{Av}\left(J^2_{i,j}\right) = \Delta^2$, the h_i are i.i.d. Gaussian random variables with $\mathrm{Av}\,(h_i) = 0$ and $\mathrm{Av}\left(h_i^2\right) = \Gamma^2$. The h and J are moreover mutually indipendent: $\mathrm{Av}\,(h_iJ_{k,l}) = 0$ for all the i, k, l. The model of Hamiltonian (1) reduces to the standard Gaussian Edwards-Anderson [4] when $\Gamma^2 = 0$ and to a pure random-field when $\Delta^2 = 0$. The (1) is an important special case of the general Gaussian spin glass model in which not only *sites* and nearest-neighboors *bonds* have direct interactions but all the *subsets* A of the lattice do have a polynomial interaction through $\sigma_A = \prod_{i\in A}\sigma_i$:

$$H_\Lambda(\sigma) \;=\; -\sum_{A\subset\Lambda} J_A\sigma_A \;, \qquad (2)$$

where all the J are indipendent, centered, traslation invariantly distributed Gaussians. The model of Hamiltonian (2) has been introduced in [5] in the spirit of the general potentials of [6] and subsequently studied in [7]. The (1) is a special case of the (2) in which only the first two types of terms (sites and bonds) give a contribution, but it also represents an approximation of it when the effect of the large sets A ($|A| \ge 3$) is simplified to a mean field one on each spin.

The local order is a concept introduced and developped within the classical models without disorder in particular the ferromagnetic ones. For the Ising model it says that at low temperatures (large β) the Boltzmann-Gibbs expectation of the product of two sets of spins $\sigma_A = \prod_{i\in A}$ and $\sigma_B = \prod_{j\in B}\sigma_j$ is close to 1:

$$\omega(\sigma_A\sigma_B) \;\ge\; 1 - \frac{c}{\beta} \;, \qquad (3)$$

no matter how far the sets A and B are taken apart. In particular since the constant c is independent of both the volume of Λ and of the distance between the two sets the (3) remains true when the thermodynamic limit is taken and successively the distance between A and B is sent to infinity.

From such a property we know for instance that the equilibrium measure is concentrated on those configurations in which two spins are alligned, no

*Electronic address: contucci@dm.unibo.it
†Electronic address: unguendoli.francesco@unimo.it

matter how distant they are, and the lower the temperature the sharper the concentration will be.

In a spin-glass the equilibrium state is described by the quenched measure i.e. the Gaussian expectation of the random correlation functions. Due to simmetry reasons quantities like $\omega(\sigma_i)$ or $\omega(\sigma_i\sigma_j)$ have zero Gaussian average so that the relevant physical quantities are the average of their even powers: $\mathrm{Av}\left(\omega^2(\sigma_i)\right)$, $\mathrm{Av}\left(\omega^2(\sigma_i\sigma_j)\right)$ etc.

Our main results are the following two theorems:

Theorem 1 *The quenched state of Hamiltonian (1) fulfills the site local order property at arbitrary distance when a random field is present* $(\Gamma > 0)$*: for all the* $m \in \Lambda$, $n \in \Lambda$ *and independently of their distance*

$$\mathrm{Av}\left(\omega^2(\sigma_m\sigma_n)\right) \;\geq\; 1 - \frac{s_1}{\beta} - \frac{s_3}{\beta^3} \;, \tag{4}$$

where for all the $\Gamma > 0$

$$s_1 = \frac{2}{\Gamma}\sqrt{\frac{2}{\pi}} \;, \qquad s_2 = \frac{1}{\sqrt{2\pi}\Gamma^3} \;. \tag{5}$$

Theorem 2 *The quenched state of Hamiltonian (1) fulfills the bond local order property at arbitrary distance when the two-body interaction is present* $(\Delta > 0)$*: for all the* $i, j \in \Lambda$, $|i-j| = 1$, $k, l \in \Lambda$, $|k-l| = 1$ *and independently of the distance between the two bonds*

$$\mathrm{Av}\left(\omega^2(\sigma_{i,j}\sigma_{k,l})\right) \;\geq\; 1 - \frac{b_1}{\beta} - \frac{b_3}{\beta^3} \;, \tag{6}$$

where for all the $\Delta > 0$

$$b_1 = \frac{2}{\Delta}\sqrt{\frac{2}{\pi}} \;, \qquad b_2 = \frac{1}{\sqrt{2\pi}\Delta^3} \;. \tag{7}$$

It was soon realised [1, 9] that all the relevant quantities in the spin glass models are suitable expectations of the site-overlap

$$q_i = \sigma_i\tau_i \;, \tag{8}$$

or the bond overlap

$$q_{i,j} = \sigma_i\sigma_j\tau_i\tau_j \;. \tag{9}$$

Denoting by Ω the random Boltzmann-Gibbs state over identical copies and defining the *quenched* measure by $<-> = \mathrm{Av}\left(\Omega(-)\right)$ we have in fact

$$\begin{aligned}\mathrm{Av}\left(\omega^2(\sigma_i)\right) \;&=\; \mathrm{Av}\left(\omega(\sigma_i)\omega(\tau_i)\right) = \\ &=\; \mathrm{Av}\left(\Omega(\sigma_i\tau_i)\right) = < q_i > \;,\end{aligned} \tag{10}$$

and analogously

$$\mathrm{Av}\left(\omega^2(\sigma_i\sigma_j)\right) = < q_{i,j} > \;. \tag{11}$$

According to this notation the (4) and (6) tell us that the two quantities $< q_m q_n >$ and $< q_{i,j}q_{k,l} >$ are close to 1 at low temperature no matter how far the two spins (bonds) are taken.

The proof of the two theorems is computationally elementary and only uses the integration by parts formula for centered Gaussian variables: let $\mathrm{Av}\left(\zeta^2\right) = V_\zeta$ and f a function of ζ, then

$$\mathrm{Av}\left(\zeta f(\zeta)\right) \;=\; V_\zeta \mathrm{Av}\left(\frac{df}{d\zeta}\right) \;. \tag{12}$$

Let us carry out the proof for the general Hamiltonian (2) and then we will apply it to the (1). We introduce the convenient notation

$$\omega_A \;=\; \frac{\sum_\sigma \sigma_A e^{-\beta H_\Lambda(\sigma)}}{\sum_\sigma e^{-\beta H_\Lambda(\sigma)}} \tag{13}$$

and for every J_A with $\mathrm{Av}\left(J_A^2\right) = \Xi_A^2 > 0$ we apply integration by parts to the quantity $\mathrm{Av}\left(J_A\omega_A\right)$

$$\mathrm{Av}\left(J_A\omega_A\right) \;=\; \Xi_A^2\beta(1 - \mathrm{Av}\left(\omega_A^2\right)) \;. \tag{14}$$

Since an elementary estimate of the Gaussian integral gives (for $\Xi_A = \sqrt{\mathrm{Av}\left(J_A^2\right)}$)

$$\mathrm{Av}\left(J_A\omega_A\right) \;\leq\; \mathrm{Av}\left(|J_A|\right) \;=\; \Xi_A\sqrt{\frac{2}{\pi}} \;, \tag{15}$$

we deduce from (14) that

$$< q_A > = \; \mathrm{Av}\left(\omega_A^2\right) \;\geq\; 1 - \frac{1}{\Xi_A\beta}\sqrt{\frac{2}{\pi}} \;. \tag{16}$$

The previous formula says that the overlap distribuition of the set A does concentrates close to 1 at low temperatures (see [10]).

To obtain the local order at arbitrary distances we proceed as follows: let consider two subset A and B of Λ, and the notation:

$$A \cdot B = A \cup B - A \cap B \tag{17}$$

we have

$$\sigma_A\sigma_B = \sigma_{A\cdot B} \tag{18}$$

because the sites i in the intersections of A and B appear twice $\sigma_i^2 = 1$. Moreover

$$\begin{aligned}0 \;&\leq\; \mathrm{Av}\left((\omega_{A\cdot B} - \omega_A\omega_B)^2\right) = \\ &=\; \mathrm{Av}\left(\omega_{A\cdot B}^2\right) + \mathrm{Av}\left(\omega_A^2\omega_B^2\right) - 2\mathrm{Av}\left(\omega_{A\cdot B}\omega_A\omega_B\right) \;,\end{aligned} \tag{19}$$

so that

$$\mathrm{Av}\left(\omega_{A\cdot B}^2\right) \;\geq\; 2\mathrm{Av}\left(\omega_{A\cdot B}\omega_A\omega_B\right) - \mathrm{Av}\left(\omega_A^2\omega_B^2\right) \;. \tag{20}$$

We apply then twice the (12) to the positive quantity

$$0 \;\le\; \mathrm{Av}\left(J_A^2(1-\omega_B^2)\right) = \tag{21}$$
$$= \;\Xi_A^2 \mathrm{Av}\left(1-\omega_B^2\right) - 2\beta^2\Xi_A^4 \mathrm{Av}\left((\omega_{A\cdot B}-\omega_A\omega_B)^2\right)$$
$$-\;\; 4\beta^2\Xi_A^4 \mathrm{Av}\left(\omega_A^2\omega_B^2\right) + 4\beta^2\Xi_A^4 \mathrm{Av}\left(\omega_{A\cdot B}\omega_A\omega_B\right)\;,$$

from which

$$2\mathrm{Av}\left(\omega_{A\cdot B}\omega_A\omega_B\right) - \mathrm{Av}\left(\omega_A^2\omega_B^2\right) \ge \tag{22}$$
$$\mathrm{Av}\left(\omega_A^2\omega_B^2\right) + \mathrm{Av}\left((\omega_{A\cdot B}-\omega_A\omega_B)^2\right) +$$
$$-\frac{1}{2\Xi_A^2\beta^2}\mathrm{Av}\left(1-\omega_B^2\right) \ge$$
$$\mathrm{Av}\left(\omega_A^2\omega_B^2\right) - \frac{1}{2\Xi_A^2\beta^2}\mathrm{Av}\left(1-\omega_B^2\right) \ge$$
$$\mathrm{Av}\left(\omega_A^2\omega_B^2\right) - \frac{1}{\sqrt{2\pi}\Xi_A^2\Xi_B\beta^3}\;,$$

where in the second inequality we have eliminated a positive term and in the third we applyed the (16). Concatenating the (20) to the (22) we get

$$\mathrm{Av}\left(\omega_{A\cdot B}^2\right) \;\ge\; \mathrm{Av}\left(\omega_A^2\omega_B^2\right) - \frac{1}{\sqrt{2\pi}\Xi_A^2\Xi_B\beta^3}\;. \tag{23}$$

We may then use the elementary inequality which states that for all $a \le 1$, $b \le 1$

$$(1-a)(1-b) \;=\; ab - a - b + 1 \;\ge\; 0\;; \tag{24}$$

it implies

$$\mathrm{Av}\left(\omega_A^2\omega_B^2\right) \;\ge\; \mathrm{Av}\left(\omega_A^2\right) + \mathrm{Av}\left(\omega_B^2\right) - 1\;, \tag{25}$$

which together with (16) gives

$$\mathrm{Av}\left(\omega_A^2\omega_B^2\right) \;\ge\; 1 - \left[\frac{1}{\Xi_A} + \frac{1}{\Xi_B}\right]\frac{1}{\beta}\sqrt{\frac{2}{\pi}}\;. \tag{26}$$

Putting together the (23) with (26) we obtain

$$\mathrm{Av}\left(\omega_{A\cdot B}^2\right) \;\ge \tag{27}$$
$$1 - \left[\frac{1}{\Xi_A} + \frac{1}{\Xi_B}\right]\frac{1}{\beta}\sqrt{\frac{2}{\pi}} - \frac{1}{\sqrt{2\pi}\Xi_A^2\Xi_B}\frac{1}{\beta^3}.$$

The previous formula gives immediately Theorems 1 and 2 when applied to sites or to bonds.

It is interesting to observe that while for the ferromagnetic Ising model with zero magnetic field the property (3) is related to the phase transition of the model and is called long range order (see [6] for a general definition and [8] for its proof in the two-dimensional Ising model), in the case of spin glasses our result is not directly related to a phase transitions. A possible way to detect the existence of a phase transition in a spin glass model would be in fact to bound from below the quantity $\mathrm{Av}\left(\omega^2(\sigma_m\sigma_n)\right)$ indipendently of Γ, which our theorem 1 fails to achieve. The method of integration by parts which we exploit here can of course be iterated to two spins at distance k but an easy computation show that it leads to a bound $\mathrm{Av}\left(\omega^2(\sigma_1\sigma_k)\right) \ge 1 - \frac{c_k}{\beta}$ where the quantity c_k diverges for large k.

Acknowledgments

We thank M.Aizenman, A. Bovier, A. van Enter, C. Giardinà, S. Graffi, C. Newman, H. Nishimori, D. Stein and F.L.Toninelli for interesting discussions.

[1] M.Mezard, G.Parisi, M.A.Virasoro, Spin Glass theory and beyond, World Scien. (1987)
[2] D.S.Fisher and D.H.Huse, Phys. Rev. Lett., 56, 1601 (1986)
[3] C. M. Newmnan and D. L. Stein http://arxiv.org/abs/cond-mat/0301403
[4] S.F.Edwards and P.W.Anderson, J.Phys.F. Vol. 5, May (1975)
[5] B.Zegarlinski, Comm. Math. Phys, Vol 139, 305-339 (1991)
[6] D. Ruelle, Statistical Mechanics, Rigorous Results, W.A.Benjamin, New York 1969
[7] P.Contucci, S.Graffi, Jou. Stat. Phys., Vol. 115, Nos. 1/2, 581-589, (2004)
[8] T.D.Schultz, D.C.Mattis and E.H.Lieb, Rev.Mod.Phys., Vol. 36, 856-871 (1964)
[9] F.Guerra, Int. Jou. Mod. Phys. B, Vol. 10, 1675-1684, (1997)
[10] C.M.Newmnan D.L.Stein, Phys.Rev. B, Vol 46, 973-982, (1992)