On the Strict Inequality Between Quenched and Annealed Ising Spin Glass

by

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Abstract

Using the cluster expansion technique, we estimate, uniformly in the volume, the difference between the quenched and annealed free energy in the high temperature phase of a d-dimensional Ising spin glass model with Bernoulli couplings.

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1. Introduction.

The study of disordered models in statistical mechanics has not yet reached the stage of systematic treatment. Even physically, the systems described by these models are not yet fully understood. For some of these models it has been rigorously proved [1-4] that in the high temperature phase the quenched and the annealed free energy density coincide in the thermodynamic limit. A question which naturally emerges from this result is the following: does the coincidence quenched-annealed for high temperatures exists even in the short range d-dimensional lattice theories? The answer seems to be negative at least in the models checked up to now [5] . In this Letter, we show how to obtain this result in an elementary way from the high temperature representation in a polymer model for a d-dimensional Ising spin glass with Bernoulli random variables. We find, in a uniformly (in the volume) estimated radius, a uniform estimate of the difference between the quenched and annealed normalized logarithms in terms of the inverse of the temperature, the result of which are of the order $\tanh(\beta)^8$.

2. The model and the estimate.

The geometrical space is the d-dimensional square lattice Z^d. We will also consider the associated bond space $\mathbb{B}^d$ defined as the space of nearest-neighbour sites. If $\Lambda \subset Z^d$, $B(\Lambda)$ will be the corresponding subset of $\mathbb{B}^d$.

To each site $n \in Z^d$ we associate the spin variable $\sigma(n)$ and to each bond $(n, n') \in \mathbb{B}^d$ the coupling variable $j(n, n')$, with the obvious symbols

$$\sigma_\Lambda \equiv \{\sigma(n)\}_{n\in\Lambda} \qquad j_{B(\Lambda)} \equiv \{j(n, n')\}_{(n,n')\in B(\Lambda)}. \tag{2.1}$$

Both j and σ take the value ± 1. For them we define the probability measures by

$$\langle\sigma(n)\rangle_\sigma = 0 \qquad \langle j(n, n')\rangle_j = 0. \tag{2.2}$$

Our disordered Ising model is defined by the Hamiltonian

$$H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) = -\sum_{(n,n')\in B(\Lambda)} j(n, n')\sigma(n)\sigma(n'). \tag{2.3}$$

The choice of the boundary conditions is irrelevant for what follows. (We chose the periodic ones only to simplify some numerical constant.) We have to compare the quenched logarithm

$$F_\Lambda\left(\beta\right) = \frac{1}{|\Lambda|}\langle\log\langle\exp\left(-\beta H_\Lambda\right)\rangle_\sigma\rangle_j \tag{2.4}$$

with the annealed one

$$\tilde{F}_\Lambda\,(\beta) = \frac{1}{|\Lambda|}\log\langle\langle\exp\left(-\beta H_\Lambda\right)\rangle_\sigma\rangle_j. \tag{2.5}$$

Also of great importance, for mathematical and physical reasons, is the random variable

$$F_\Lambda\,(\beta, j_{B(\Lambda)}) = \frac{1}{|\Lambda|}\log\langle\exp\left(-\beta H_\Lambda\right)\rangle_\sigma. \tag{2.6}$$

Using standard techniques, one obtains [6] , the existence of the thermodynamic limit and the self-averaging property.

The usual Jensen inequality for the logarithm gives $F_\Lambda(\beta) \leq \tilde{F}_\Lambda(\beta)$. A trivial calculation shows that the equality holds in the thermodynamic limit for $d = 1$ at any β. For arbitrary dimensions our results are summarized in the following:

Theorem.

The F_Λ and the $\tilde{F}_\Lambda$ are analytic functions (in β) in a radius $|\beta| \leq \overline{\beta}$ with $\overline{\beta}$ independent from the volume and defined by

$$\tanh\overline{\beta} = \frac{1}{2d-1}\left(\frac{x}{1+x}e^{-x}\right)|_{x=\frac{(\sqrt{5}-1)}{2}}. \tag{2.7}$$

In this range

$$F_\Lambda(\beta) - \tilde{F}_\Lambda(\beta) = -\frac{d(d-1)}{4}\tanh(\beta)^8 + o(\tanh(\beta)^8). \tag{2.8}$$

The analyticity of the functions implies that the remainder cannot identically compensate the first perturbative term. The following section is devoted to the proof.

3. The polymer formalism for the high temperature phase

We are interested in the high temperature phase for our model. In order to study this phase, we transform our model into a polymer model for which there are general theorems on the convergence properties of the free energy density. This is done as follows. Using the elementary identity with real a and $X = \pm 1$,

$$\exp\left(aX\right) = \cosh(a)\left(1 + X\tanh(a)\right) \tag{3.1}$$

we can write

$$\langle \exp(-\beta H_\Lambda)\rangle_\sigma = \cosh(\beta)^{d|\Lambda|} \sum_{B \subseteq B(\Lambda)} \prod_{(n,n')\in B} J(n,n')\tanh(\beta) \tag{3.2}$$

where the sum is over all subsets of $B(\Lambda)$, including the empty set, which verify the constraint that every point of the lattice intersects an even number of elements of B.

We map our problem into a polymer model, decomposing each B in disjoint (as subsets of Z^d) components; each of these components will be a polymer. Defining $\hat{Z}_\Lambda(j_\Lambda,\beta) = \langle \exp(-\beta H_\Lambda)\rangle_\sigma \cosh(\beta)^{-d|\Lambda|}$ we obtain

$$\hat{Z}_\Lambda(j_\Lambda,\beta) = \sum_{(\gamma_1,\ldots,\gamma_n)}{}' \prod_i z(J_\Lambda,\beta,\gamma_i) \tag{3.3}$$

where the summation $\sum'$ is over all the families, including the empty one, of disjoint polymers, and where $z(J_\Lambda,\beta,\gamma_i) = \prod_{(n,n')\in\gamma_i} J(n,n')\tanh(\beta)$, with the obvious definition $z(J_\Lambda,\beta,\emptyset) = 1$ and with $z(\beta,\gamma_i) = z(\mathbb{1},\beta,\gamma_i)$ as the usual high temperature Ising activities.

For general theorems (see for instance [7],) one can prove that for $|\beta| \le \overline{\beta}$

$$\log \hat{Z}_\Lambda(J_\Lambda,\beta) = \sum_{(\gamma_1,\ldots,\gamma_n)} \alpha(\gamma_1,\ldots,\gamma_n) \prod_i z(J_\Lambda,\beta,\gamma_i) \tag{3.4}$$

where now the sum is over all the (not empty) possible families of polymers which form a connected cluster, even permitting coincident polymers.

The α 's are combinatorial factors depending only on the mutual overlapping properties of the polymers forming a given cluster. The quoted theorems state that the limit for the normalized logarithm exists in the given perturbative form and that it is an analytic function of β in $|\beta| \le \overline{\beta}$. A good estimate for this radius with explicit dimensional dependence is (ref. [8], [9])

$$\tanh\overline{\beta} = \frac{1}{\mu_d}\frac{x}{1+x}e^{-x}\Big|_{x=\frac{(\sqrt{5}-1)}{2}}, \tag{3.5}$$

where μ_d is the connectivity constant for Z^d, here trivially estimated as $2d-1$ (see [10] for a better estimate). It is possible to improve this radius in any dimension but without explicit dimensional dependence (ref. [9]). The extension of the previous theorems to our disordered case is trivial due to the fact that the random variables are bounded. The extension of this formalism to unbounded random variables could be the subject of successive work.

By direct computation, one trivially finds

$$\langle \hat{Z}_\Lambda(J_\Lambda,\beta)\rangle_j = 1 \tag{3.6}$$

since only the term corresponding to the empty family has mean different from zero. The annealed logarithm is then zero. But the quenched logarithm is not zero, due to the different structure of the sum: from the logarithm sum allowing overlapping and coincident polymers, it is clear that families with an even multiplicity for each bond will give a contribution different from zero. If we denote with $\sum^*$ the summation over these families, we obtain

$$\langle \log \hat{Z}_\Lambda(J_\Lambda, \beta)\rangle_j = \sum_{(\gamma_1,\ldots,\gamma_n)}^{*} \alpha\,(\gamma_1,\ldots,\gamma_n) \prod_i z(\beta,\gamma_i). \tag{3.7}$$

In this way we have expressed the quenched logarithm in such a way that the disorder acts only through the structure of the sum. The lowest order contribution is due to a couple of plaquettes coinciding in the lattice (pictorially $\Box^2$). We find

$$\langle \log \hat{Z}_\Lambda(J_\Lambda, \beta)\rangle_j = \sum_{(\Box^2)} \alpha\left(\Box^2\right) \tanh(\beta)^8 + o(\tanh(\beta)^8). \tag{3.8}$$

Since $\alpha(\Box^2) = -\frac{1}{2}$ (ref [7]), and the embedding index of the plaquette in Z^d is $\frac{d(d-1)}{2}$ we obtain:

$$\langle \log \hat{Z}_\Lambda(J_\Lambda, \beta)\rangle_j = -\frac{d(d-1)}{4}|\Lambda| \tanh(\beta)^8 + o(\tanh(\beta)^8) \tag{3.9}$$

from which our theorem immediately follows.

4. Conclusions

We have found that in our model with $d \geq 2$ there is no a value of β, different from zero, under which the annealed and quenched logarithm coincide; the explicit estimate of the difference, obtained uniformly in the volume, has the leading term proportional to $\tanh(\beta)^8$. One can also see that this result is stable under perturbation of the mean value for the couplings variables (ref. [11]). Our result is complementary to that obtained in [5]. We give an explicit estimate in terms of temperature and dimensions, but our method applies, at least at a first analysis, only to bounded random couplings while in [5] one can find the strict inequality without explicit estimate for short range d-dimensional models with centered gaussian random variable.

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