SOME PROPERTIES OF MEAN FIELD SOLUTIONS FOR A DISORDERED ISING MODEL

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Summary

We study a disordered d-dimensional Ising model defined by random couplings and quenched free energy. After some general results we concentrate on mean field solutions. For these we prove that the critical value of β is the same as in the Ising case. A particular study is devoted to the mean field solutions for fully frustrated configurations: using a Schwartz inequality for frustrated cubes we find the exact critical β.

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1. Introduction.

In the last two decades disordered mean field models in statistical mechanics have been investigated and interesting proposals have been done to solve them. Looking backwards one notices that mean field models were often proposed as exactly solvable models with phase structure very close to mean field solutions of some model in physical lattices. By mean field solution we simply mean the product state best approximating the true equilibrium state according to the prescription of entropic variational principle. For instance the Kac model defined by the Hamiltonian $H_N^K = -\frac{1}{N}\sum_{i\neq j}\sigma_i\sigma_j$ has the same spontaneous magnetization behaviour that one finds in mean field solution for Ising model (ref [1]). It is very natural to ask whether the same correspondence holds also in disordered case. For instance can Sherrington-Kirkpatrick (S.K.) or some more complicated disordered mean field model be considered, in some sense, a mimic of mean field solutions for a disordered model of Ising type? This work is only a preliminary step in this direction.

The chapters are organised as follows. In chapter 1 we summarize some theorems of existence of thermodynamical limit; using convexity estimates and cluster expansion we point out a non trivial difference between S.K. model and our lattice model. The saturation of convexity inequality for the logarithm, which in the S.K. model reflect the absence of any phase transition in the high temperature regime, doesn't hold any more in our case.

In chapter 3 we consider a possible class of mean field models corresponding to our lattice model. The main result of this section is theorem 3.1., which prove that these models reduce essentially to S.K.

Chapter 4 collects some results on mean field free energy obtained by variational principle. Theorems on thermodynamical limit are similar to the exact case. We present the proof in the appendix showing how it simply follows from reflection positivity property. Using the so obtained monotonicity property for mean field free energy we find that the critical value of β is $\frac{1}{2d}$, as in usual Ising mean field.

Finally in chapter 5 we study the mean field behaviour for fully frustrated configurations. We prove for them that the value $\frac{1}{2\sqrt{d}}$, already found as upper bound for the critical β (ref [9]), is the exact critical value. This result is obtained with a Schwartz inequality for fully frustrated cubes.

2. A disordered Ising model. Some preliminary results.

Our geometrical space is the d-dimensional lattice Z^d. We will consider also the associated bond space $I\!B^d$, and the plaquettes space $I\!P^d$. If $\Lambda \subset Z^d$, $B(\Lambda)$ will be the corresponding subset of $I\!B^d$.

To each site $n \in Z^d$ we associate the spin variable $\sigma(n)$ and to each bond $(n, n') \in I\!B^d$ the coupling variable $j(n, n')$; with obvious symbols

$$\sigma_\Lambda \equiv \{\sigma(n)\}_{n\in\Lambda} \qquad j_{B(\Lambda)} \equiv \{j(n, n')\}_{(n,n')\in B(\Lambda)} \tag{2.1}$$

Both j and σ take value ± 1; for them we define different probability measures by

$$\langle\sigma(n)\rangle_\sigma = 0 \qquad \langle j(n, n')\rangle_j = J \tag{2.2}$$

Our disordered Ising model is defined by the Hamiltonian

$$H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) = -\sum_{(n,n')\in B(\Lambda)} j(n, n')\sigma(n)\sigma(n') \tag{2.3}$$

and by the quenched free energy density

$$F_{\Lambda,J}\left(\beta\right) = -\frac{1}{\beta|\Lambda|}\langle\log\langle\exp^{-\beta H_\Lambda}\rangle_\sigma\rangle_j \tag{2.4}$$

Of great importance, for mathematical and physical reasons, is also the not j-averaged free energy, i.e. the random variable

$$F_\Lambda\left(\beta, j_{B(\Lambda)}\right) = -\frac{1}{\beta|\Lambda|}\log\langle\exp^{-\beta H_\Lambda}\rangle_\sigma \tag{2.5}$$

With the above notations the following theorems hold:
Proposition 2.1.

In V.Hove sense

$$\exists \quad \lim_{\Lambda\nearrow Z^d} F_{\Lambda,J}\left(\beta\right) \tag{2.6}$$

$$\exists \quad \lim_{\Lambda \nearrow Z^d} F_\Lambda\left(\beta, j_{B(\Lambda)}\right) \quad \text{with probability one} \tag{2.7}$$

$$\lim_{\Lambda \nearrow Z^d} F_\Lambda\left(\beta, j_{B(\Lambda)}\right) = \lim_{\Lambda \nearrow Z^d} F_{\Lambda, J}(\beta) \quad \text{with probability one} \tag{2.8}$$

This means that the random free energy density is a selfaveraging function, in thermodynamical limit, almost everywhere.

The theorem (2.6) is completely analogous to the ordered case. (For (2.7) and (2.8) see [2], [3], [4]). The proofs are essentially based on estimates of boundary contributions and on strong law of large numbers for mutually independent and identically distributed random variables.

The two following theorems study the high temperature phase; as one expects there is no relevant difference with the ordered case. Indicating with $F^I_\Lambda(\beta)$ the free energy density for the Ising case it is easy to obtain with usual convexity arguments the following estimate:

Theorem 2.3.

$$\beta' F^I_\Lambda(\beta') - \frac{|B(\Lambda)|}{|\Lambda|} \log\left(\frac{\cosh(\beta)}{\cosh(\beta')}\right) \leq \beta F_{\Lambda, J}(\beta) \leq \beta'' F^I_\Lambda(\beta'') \tag{2.9}$$

with β' and β'' defined by

$$J \tanh(\beta) = \tanh(\beta') \tag{2.10}$$

and

$$J\beta = \beta'' \tag{2.11}$$

Proof:

The lower bound is a simple application of the convexity inequality for the logarithm, i.e. $\log\left(\langle\xi\rangle\right) \geq \langle\log\left(\xi\right)\rangle$; the relation (2.10) is needed in order to identificate the first term of the left hand side with the Ising free energy density.

The upper bound is obtainable as follows: indicating with $H^I_\Lambda(\sigma_\Lambda) = H_\Lambda(\sigma_\Lambda, \mathbb{1}_{B(\Lambda)})$ the energy of the Ising configuration and with $\langle\cdot\rangle^{I,\beta} = \frac{\langle \cdot \exp^{-\beta H^I_\Lambda}\rangle_\sigma}{\langle \exp^{-\beta H^I_\Lambda}\rangle_\sigma}$ the equilibrium state for the Ising configuration, one obtain

$$\frac{\langle \exp^{-\beta H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right)}\rangle_\sigma}{\langle \exp^{-\beta'' H^I_\Lambda(\sigma_\Lambda)}\rangle_\sigma} = \frac{\langle \exp^{-\beta H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) + \beta'' H^I_\Lambda(\sigma_\Lambda)} \exp^{-\beta'' H^I_\Lambda(\sigma_\Lambda)}\rangle_\sigma}{\langle \exp^{-\beta'' H^I_\Lambda(\sigma_\Lambda)}\rangle_\sigma} =$$

$$= \langle \exp^{-\beta H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) + \beta'' H^I_\Lambda(\sigma_\Lambda)}\rangle^{I,\beta''} \geq \exp\langle -\beta H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) + \beta'' H^I_\Lambda\left(\sigma_\Lambda\right)\rangle^{I,\beta''}.$$

We can now optimize the last inequality choosing $\beta'' = J\beta$ which gives

$$\langle -\beta H_\Lambda\left(\sigma_\Lambda, j_{B(\Lambda)}\right) + \beta'' H^I_\Lambda\left(\sigma_\Lambda\right)\rangle^{I,\beta''} = 0$$

from which the (2.9) upper bound immediately follows.

It is evident that for all $\Lambda \subseteq Z^d$ the estimates (2.9) are saturate, i.e. its become equalities, in the limit $\beta = 0$.

It has been proved ([5], [6]) that the equivalent lower bound for the S.K. model, i.e. the estimate obtained with the convexity property for the logarithm is saturate in the thermodynamical limit when β is sufficiently small.

We can prove that this property never holds in our case unless $\beta = 0$. In fact with classical cluster expansion techniques ([7]), at high temperature regime and with periodic boundary conditions, we easily obtain the following:

Theorem 2.3.

$$\beta F_{\Lambda,J}(\beta) - \beta' F_{\Lambda}(\beta') = -d \log\left(\frac{\cosh(\beta)}{\cosh(\beta')}\right) - \frac{d(d-1)}{4}(2J^6+1)\left(\tanh(\beta)\right)^8 + o\left(\left(\tanh(\beta)\right)^8\right) \tag{2.12}$$

We can conclude that the hight temperature phase for our lattice model is not characterised by the saturation of convexity inequality for the logarithm.

3. A class of disordered mean field models. The reduction to S.K.

As explained in the introduction one of the principal motivation to study mean field models is that they are generally exactly solvable and its phase structure reproduce the mean field solution of some model in physical lattice. A mean field model can be thought as a group of N spin arranged in a infinite dimensional space in such a way that each of them sees the remanent ones at the same distance; the interaction between two spin decrease with the inverse of N. The Kac model is one of the simplest examples of mean field model. In the case of disordered systems a possible class of mean field models can be defined by an Hamiltonian sum of two parts representing respectively a disordered version of Kac Hamiltonian (D.K.) and the S.K. Hamiltonian.

$$H_N = H_N^{D.K.} + H_N^{S.K.} = -\sum_{i \neq j}\left(\frac{J_{ij}}{N} + \frac{\tilde{J}_{ij}}{\sqrt{N}}\right)\sigma_i\sigma_j. \tag{3.1}$$

where the sum runs to all the couples of spin. The variables σ, J, $\tilde{J}$ take value ± 1 and J, $\tilde{J}$ are the random variables with

$$\langle J_{ij} \rangle_J = J \qquad \langle \tilde{J}_{ij} \rangle_J = 0 \tag{3.2}$$

The suggested Hamiltonian can be thought as a series expansion in $\frac{1}{N}$ truncated at the second order with random coupling variables able to control the relative normalization factors.

The main result of this section is that the D.K. part is exactly solvable and that the disorder act on it as a simple temperature rescaling of the ordered case. We have obtained this conclusion simply rephrasing the theorem 2.2 for the Hamiltonian $H_N^{D.K.}$ which give the

Theorem 3.1

indicating with $F^{DK}(\beta)$ and $F^K(\beta)$ the free energy density respectively for the disordered Kac model and the usual one we have

$$\beta' F_N^K(\beta') - \frac{N(N-1)}{2} \log\left(\frac{\cosh(\frac{\beta}{N})}{\cosh(\frac{\beta'}{N})}\right) \leq \beta F_N^{D.K.}(\beta) \leq \beta'' F_N^K(\beta'') \tag{3.3}$$

with β' and β'' defined by

$$J \tanh(\frac{\beta}{N}) = \tanh(\frac{\beta'}{N}) \tag{3.4}$$

and

$$J\beta = \beta'' \tag{3.5}$$

Now in the limit $N \to \infty$ one obtain $\beta'' = \beta' = J\beta$ and $\frac{N(N-1)}{2} \log\left(\frac{\cosh(\frac{\beta}{N})}{\cosh(\frac{\beta'}{N})}\right) \to 0$ so that the upper and the lower bound coincide in the thermodynamical limit:

Theorem 3.2

$$F^{D.K.}(\beta) = JF^K(\beta J) \tag{3.6}$$

This means that the disordered behaviour for H_N is entirely due to the S.K. part.

4. Study of mean field solutions.

In the following we will indicate the product state by $\omega_\Lambda = \underset{n\in\Lambda}{\otimes} \omega_n$ with $\omega_n(\sigma(n)) = m(n)$ and $M_\Lambda \equiv \{m(n)\}_{n\in\Lambda}$
It is trivial to give an upper bound of free energy with a functional of product states; with standard convexity estimates we find, for every configuration j

$$F_\Lambda(\beta, j_{B(\Lambda)}) \leq U_\Lambda(M_\Lambda, j_{B(\Lambda)}) - \beta^{-1} S(M_\Lambda) \tag{4.1}$$

where U and S are respectively the internal energy density and entropy density in a general product state i.e.

$$|\Lambda|U_\Lambda\left(M_\Lambda,\beta\right)=\omega_\Lambda\left(H_\Lambda\left(\sigma_\Lambda,j_{B(\Lambda)}\right)\right)=-\sum_{(n,n')\in B(\Lambda)}j(n,n')m(n)m(n')$$

and

$$|\Lambda|S\left(M_\Lambda\right)=\sum_{n\in\Lambda}S_n \tag{4.2}$$

with

$$S_n=-\frac{1+m(n)}{2}\log\left(1+m(n)\right)-\frac{1-m(n)}{2}\log\left(1-m(n)\right)$$

Defining the right hand side of (4.1) as $\tilde{F}_\Lambda\left(M_\Lambda,j_{B(\Lambda)},\beta\right)$ we define the mean field free energy density by

$$F^{M.F.}_{\Lambda,J}(\beta)=\langle F^{M.F.}_\Lambda\left(j_{B(\Lambda)},\beta\right)\rangle_j \tag{4.3}$$

and

$$F^{M.F.}_\Lambda\left(j_{B(\Lambda)},\beta\right)=\inf_{M_\Lambda}\tilde{F}_\Lambda\left(M_\Lambda,j_{B(\Lambda)},\beta\right)$$

The thermodynamical limit for the mean field case is similar to the exact one:

Theorem 4.1.

$$F^{M.F.}_{\Lambda,J}(\beta)\le F^{M.F.}_{\tilde{\Lambda},J}(\beta)\quad\text{monotonicity property} \tag{4.4}$$

for those Λ which admit a rational disjoint decomposition

$$\Lambda=\cup_{i=1}^{N_\Lambda}\tilde{\Lambda}_i$$

where $\tilde{\Lambda}_i$ are congruents to an arbitrary $\tilde{\Lambda}$.
Being free energy density lower bounded one obtains that, in Van Hove sense:

$$\exists\quad F^{M.F.}_J(\beta)=\lim_{\Lambda\nearrow Z^d}F^{M.F.}_{\Lambda,J}(\beta)=\liminf_{\Lambda\nearrow Z^d}F^{M.F.}_{\Lambda,J}(\beta) \tag{4.5}$$

$$\exists\lim_{\Lambda\nearrow Z^d}F^{M.F.}_\Lambda\left(j_{B(\Lambda)},\beta\right)=F^{M.F.}_J(\beta)\quad\text{with probability one}\quad. \tag{4.6}$$

For the proof see Appendix A.

We look now for those M which minimize the functional $\tilde{F}_\Lambda$; imposing the stationarity condition one obtains the well known mean field consistency equation

$$m(n)=\tanh(\beta\sum_{n'\in N(n)}j(n,n')m(n')) \tag{4.7}$$

where N(n) is the set of nearest-neighbours of n.

The positivity of second order variation has to be added to exclude maximum and saddle points.

It's clear that, for a given j, the (4.7) is a fixed point equation for an operator acting in the space of M_Λ equipped with some metric. This suggests the possibility that the property of stationary points of $\tilde{F}_\Lambda\,(M_\Lambda, j_{B(\Lambda)}, \beta)$ can be translated in terms of ergodicity property of the abstract dynamical system associated to (4.7). We will use for our purpose the degree of freedom connected to this association in the choice of operator and metric. The *sup* metric seems to be the only candidate able to survive in the infinite volume. We choose the operator acting on the metric space in such a way that $\tilde{F}$ shall be a Ljapounov function for it; in this way the set of fixed stable points will correspond to the points of relative minima for our functional.

The operator can be obtained as follows: defining

$$(K_{even}M)(n) = \begin{cases} \tanh\left(\beta \sum_{n' \in N(n)} j(n,n')m(n')\right) & \text{if n is even} \\ M(n) & \text{otherwise} \end{cases} \tag{4.8}$$

and similarly K_{odd} our operator is $K(j,\beta) = K_{even}K_{odd}$. It is obvious that this K has the same fixed points as equation (4.7). The fact that $\tilde{F}$ is a Ljapounov function for K follows by the convexity of $\tilde{F}$ with respect to every variable M(n).

This allows us to obtain the following (see Appendix B I):

Theorem 4.2.

defined $\beta_c^{MF} = [sup\ \beta\ :\ F_J^{M.F.}(\beta) = 0]$ we have

$$\beta_c^{MF} = \frac{1}{2d} \tag{4.9}$$

5. The fully frustrated case.

It is a well known heuristic fact that the peculiar features of these kind of models are due essentially to the presence of frustrated plaquettes between the configurations on which one perform the average. A plaquette $p \in \mathbb{P}^d$ is said frustrated in a configuration j if $j(p) = \prod_{(n,n') \in p} j(n,n') = -1$. For this reason a good starting point to understand our model seems to be the study of those configurations in which every plaquette is frustrated, the so called fully frustrated (f.f.) configurations j_{ff}.

We recall that fixing the $j(p)$ one fixes an orbit of configurations for the gauge group defined by the action on point's functions, e.g.

$$M \to M' \quad \text{with} \quad M'(n) = M(n)\alpha(n) \quad \text{where} \quad \alpha(n) = \pm 1 \tag{5.1}$$

and on bond's functions, e.g.

$$j \to j' \quad \text{with} \quad j'(n,n') = j(n,n')\alpha(n)\alpha(n') \tag{5.2}$$

It is easy to prove that free energy for a given j, real or mean field, is invariant under the gauge group. The same holds for the density whenever it exists (in fact (2.8) and (4.6) doesn't exclude pathological configurations of measure 0) and, as a consequence, for the critical values of β for a given configuration defined as $\beta_c^{MF}(j) = [sup\ \beta\ :\ F^{M.F.}(j,\beta) = 0]$.

The existence of thermodynamical limit for f.f. configurations is easily obtained as in the ordered case thanks to their homogeneity.

Our main result, for f.f. configurations, is the following:

Theorem 5.1.

$$\beta_c^{MF}(j_{ff}) = \frac{1}{2\sqrt{d}} \tag{5.3}$$

The square root on d is typical for critical parameters in f.f. models (ref [8]).

Proof:

The theorem consist of the upper and lower bound for $\beta_c^{MF}(j_{ff})$. We need the following lemma (see Appendix B II):

Lemma 5.1) Considering a f.f. cube one has:

$$\sum_{(n,n')} j(n,n')m(n)m(n') \leq \frac{\sqrt{d}}{2}\sum_n m(n)^2 \tag{5.4}$$

where the sum are extended to the cube.

From this we obtain

$$\beta|\Lambda|\tilde{F}_\Lambda \geq \sum_{n\in\Lambda}(S_n - 2\beta\sqrt{d}m(n)^2) \tag{5.5}$$

minimizing and performing the thermodynamical limit we find:

$$\beta F^{MF}(j_{tf},\beta) \geq F(2\beta\sqrt{d}) \tag{5.6}$$

where F is the usual mean field free energy density for Ising model i.e. $F(\beta') = \inf_m (S(m) - \beta'\frac{m^2}{2})$ with $\beta'_c = 1$

This means that

$$\beta_c^{MF}(j_{tf}) \geq \frac{1}{2\sqrt{d}} \tag{5.7}$$

The estimate in the other direction have been obtained (ref. [9]) linearising the operator $K(j_{tf},\beta)$. It is interesting to compare this operator to the corresponding one in Ising. Denoting with $\Delta_{(k)}$ the k-step Laplacian in Z^d we have

$$L_{Ising} = \beta(\Delta_{(1)} + 2d\mathbb{1})$$

$$L^2 = \beta^2(\Delta_{(2)} + 4d\mathbb{1}) \tag{5.8}$$

We stress that this form for the linearized operator holds only for f.f. configurations restricted by the condition $j(n,m)j(m,l) = 1$ when n and l are opposite sites with respect to m, and of course it's not gauge invariant; nevertheless the results obtained on critical β and on m.f. free energy are gauge invariant because the latter is so.

Using the *sup* norm induced by our metric

$$\|L^2\| = \sup_M \frac{\|L^2 M\|}{\|M\|}$$

one finds

$$|(L^2 M)(n)| = \beta^2 |(\sum_{n' \in \tilde{N}(n)} M(n') + 2dM(n))| \leq 4d\beta^2 \|M\|$$

$$\|(L^2 M)(n)\| \leq 4d\beta^2 \|M\| \tag{5.9}$$

$$\|L^2\| \leq 4d\beta^2$$

This inequality is saturated with every periodic $\tilde{M}$ of period 2. In other words we take a d-dimensional cube with arbitrary magnetization in his vertex and replicate it by reflexion with respect to the plaquettes limiting it:

$$(L^2\tilde{M})(n) = 4d\beta^2 \tilde{M}(n) \tag{5.10}$$

which gives

$$\|L^2\| = 4d\beta^2 \tag{5.11}$$

It follows that

$$\beta_c^{MF}(j_{tf}) \leq \frac{1}{2\sqrt{d}} \tag{5.12}$$

which proves the theorem 5.1.

This last result suggests that the natural lattice for f.f. configurations have a doubled step. This implies that the estimate (5.12) can also be obtained with an appropriate Ansatz with period 2; actually one can show that the Ansatz $\tilde{M}$ parameterized by the 2^d value of magnetisations in the cube and replicate by reflexion have the right critical temperature.

6. Comments

Up to now we have obtained a rather clear picture of mean field behaviour for fully frustrated configurations. In the usual Ising case the $\Delta_{(1)}$ binds the magnetization to be constant and only Z_2 symmetry breaking is permitted. In f.f. configurations $\Delta_{(2)}$ leaves "one step" degree of freedom that allows a 2^d multiplicity in the starting directions for mean field solutions when $\beta \geq \frac{1}{2\sqrt{d}}$. How the f.f. behaviour is reflected in the j-averaged mean field is not yet clear. Further developments are needed to establish whether the embryonic degeneracy we have found can be responsible of multivalley picture that one expects from this kind of models.

Appendix A

I)

The (4.4) is a direct consequence of reflection positivity property. We show the proof for

$$\Lambda = \Lambda_1 \cup \Lambda_2$$

with Λ_1 congruent and adjacent to Λ_2. The generalization to multidecomposition is trivial. Defining B(1,2) the set of bonds between Λ_1 and Λ_2 and indicating $M_\Lambda = M_{\Lambda_1} \otimes M_{\Lambda_2}$, we obtain

$$\tilde{F}_\Lambda \left(M_{\Lambda_1} \otimes M_{\Lambda_2}\right) = \frac{1}{2}\left(\tilde{F}_{\Lambda_1}(M_{\Lambda_1}) + \tilde{F}_{\Lambda_2}(M_{\Lambda_2})\right) - \frac{1}{|\Lambda|} \sum_{(n,n')\in B(1,2)} j(n,n')m_1(n)m_2(n') \quad (A.1)$$

it follows, defining $\overline{M}_\Lambda$ with $F_\Lambda^{M.F.}\left(j_{B(\Lambda)},\beta\right) = \tilde{F}_\Lambda\left(\overline{M}_\Lambda, j_{B(\Lambda)},\beta\right)$

$$\tilde{F}_\Lambda \left(\overline{M}_{\Lambda_1} \otimes \overline{M}_{\Lambda_2}\right) = \frac{1}{2}\left(F_{\Lambda_1}^{MF} + F_{\Lambda_2}^{MF}\right) - \frac{1}{|\Lambda|} \sum_{(n,n')\in B(1,2)} j(n,n')m_1(n)m_2(n') \qquad (A.2)$$

observing that

$$\tilde{F}_\Lambda(M_\Lambda) = \tilde{F}_\Lambda(-M_\Lambda) \qquad (A.3)$$

one obtain

$$\tilde{F}_\Lambda \left(\overline{M}_{\Lambda_1} \otimes -\overline{M}_{\Lambda_2}\right) = \frac{1}{2}\left(F_{\Lambda_1}^{MF} + F_{\Lambda_2}^{MF}\right) + \frac{1}{|\Lambda|} \sum_{(n,n')\in B(1,2)} j(n,n')m_1(n)m_2(n') \qquad (A.4)$$

and

$$F_\Lambda^{MF} \leq \frac{1}{2}\left(F_{\Lambda_1}^{MF} + F_{\Lambda_2}^{MF}\right) \qquad (A.5)$$

and finally j-averaging

$$F_{\Lambda,J}^{M.F.}(\beta) \leq F_{\Lambda',J}^{M.F.}(\beta) \qquad (A.6)$$

with Λ' congruent to Λ_1.

II)

To prove (4.5) we first consider those Λ for which it is possible a rational disjoint decomposition i.e.

$$\Lambda = \cup_{i=1}^{N_\Lambda} \tilde{\Lambda}_i \qquad (A.7)$$

where $\tilde{\Lambda}_i$ are congruents to an arbitrary $\tilde{\Lambda}$. From the above lemma generalized to multiple decomposition we obtain:

$$F_{\Lambda}^{MF}(j_{B(\Lambda)}) - \frac{1}{N_\Lambda}\sum_{i=1}^{N_\Lambda} F_{\tilde{\Lambda}_i}^{MF}(j_{B(\tilde{\Lambda}_i)}) \leq 0 \tag{A.8}$$

Estimating boundary terms in (A.1) we find the inequality in the other direction:

$$F_{\Lambda}^{MF}(j_{B(\Lambda)}) - \frac{1}{N_\Lambda}\sum_{i=1}^{N_\Lambda} F_{\tilde{\Lambda}_i}^{MF}(j_{B(\tilde{\Lambda}_i)}) \geq -\frac{|\partial\tilde{\Lambda}|}{|\tilde{\Lambda}|} \tag{A.9}$$

Now one observes that $F_{\tilde{\Lambda}_i}^{MF}(j_{B(\tilde{\Lambda}_i)})$ are mutually independent identically distributed random variables. By strong law of large numbers (ref [10]) we have:

$$\exists \lim_{N_\Lambda\to\infty} \sum_{i=1}^{N_\Lambda} F_{\tilde{\Lambda}_i}^{MF}(j_{B(\tilde{\Lambda}_i)}) = \langle F_{\tilde{\Lambda}}^{MF}(j_{B(\tilde{\Lambda})})\rangle_j = F_{\tilde{\Lambda},J}^{MF} \quad \text{with probability one} \tag{A.10}$$

performing limsup in (A.8) and liminf in (A.9) for $\Lambda\nearrow Z^d$ we find

$$\limsup_{\Lambda\nearrow Z^d} F_{\Lambda}^{MF}(j_{B(\Lambda)}) - F_{\tilde{\Lambda},J}^{MF} \leq 0 \tag{A.11}$$

and

$$\liminf_{\Lambda\nearrow Z^d} F_{\Lambda}^{MF}(j_{B(\Lambda)}) - F_{\tilde{\Lambda},J}^{MF} \geq -\frac{|\partial\tilde{\Lambda}|}{|\tilde{\Lambda}|} \tag{A.12}$$

The last couple of inequalities holds for every $j_{Z^d} \in A_l$ where l indicizes the growth $\tilde{\Lambda}\nearrow Z^d$ Now performing the lim $\tilde{\Lambda}\nearrow Z^d$ and using (4.4) we get

$$\limsup_{\Lambda\nearrow Z^d} F_{\Lambda}^{MF}(j_{B(\Lambda)}) \leq F_J^{MF} \leq \liminf_{\tilde{\Lambda}\nearrow Z^d} F_{\tilde{\Lambda}}^{MF}(j_{B(\tilde{\Lambda})}) \tag{A.13}$$

The last equality holds for every j_{Z^d} belonging to all A_l after some $\bar{l}$ i.e.

$$\forall j_{Z^d} \in A = \cup_{k=1}^{\infty}\cap_{l=k}^{\infty} A_l \tag{A.14}$$

and

$$P(A) = 1 - P(\overline{A}) = 1 - P(\cap_{k=1}^{\infty}\cup_{l=k}^{\infty}\overline{A}_l) = \tag{A.15}$$

$$= 1 - \lim_{k\to\infty}\sum_{l=k}^{\infty} P(\overline{A}_l) = 1.$$

With standard technicality one extends the result to Van Hove limit.

Appendix B

I)

The theorem 4.2 is proved in two steps; first one notes that the operator $K(j, \beta)$ is a contraction for $\beta \leq \frac{1}{2d}$ for every j.In fact

$$|(K(j,\beta)M_1)(n) - (K(j,\beta)M_2)(n)| =$$

$$= |\tanh(\beta \sum_{n' \in N(n)} j(n,n')m_1(n')) - \tanh(\beta \sum_{n' \in N(n)} j(n,n')m_2(n'))| =$$

$$\leq \beta| \sum_{n' \in N(n)} j(n,n')(m_1(n') - m_2(n'))| =$$

$$\leq 2d\beta\rho(M_1, M_2) \tag{B.1}$$

and

$$\rho(KM_1, KM_2) \leq 2d\beta\rho(M_1, M_2) \tag{B.2}$$

This implies

$$\beta_c^{MF} \geq \frac{1}{2d} \tag{B.3}$$

In the other direction we observe that, for the well known properties of mean field Ising free energy

$$F_\Lambda^{M.F.}(j_{Ising}, \beta) < 0 \quad \text{for} \quad \beta > \frac{1}{2d} \tag{B.4}$$

and for monotonicity we get

$$F_J^{M.F.} \leq \langle F_\Lambda^{M.F.}(j_{B(\Lambda)}, \beta)\rangle_J < 0 \quad \text{for} \quad \beta > \frac{1}{2d} \tag{B.5}$$

because in the j-average there is at least the contribution of Ising configurations. So we find

$$\beta_c^{M.F.} \leq \frac{1}{2d} \tag{B.6}$$

and the theorem is proved.

II)

Considered a fully frustrated cube one has:

$$\sum_{(n,n')} j(n,n')m(n)m(n') \leq \frac{\sqrt{d}}{2} \sum_n m(n)^2 \tag{B.7}$$

The inequality is proved reducing to the Jordan form the matrix corresponding to our quadratic form. For simplicity we show the proof in d=2. Keeping in mind the frustrated plaquette we have to prove:

$$m_1 m_2 + m_2 m_3 + m_3 m_4 - m_4 m_1 \le \frac{\mu^2}{2}({m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2) \qquad (B.8)$$

finding the better choice for μ. Performing the transformation:

$$m_1 = a + c$$

$$m_4 = d + b$$

$$m_2 = a - c$$

$$m_3 = d - b$$

the (B.8) becomes

$$a^2 - b^2 - c^2 + d^2 - 2ab - 2cd \le \mu^2(a^2 + b^2 + c^2 + d^2) \qquad (B.9)$$

and the associated matrix

$$B = \begin{pmatrix} \mu^2 - 1 & 1 & 0 & 0 \\ 1 & \mu^2 + 1 & 0 & 0 \\ 0 & 0 & \mu^2 + 1 & 1 \\ 0 & 0 & 1 & \mu^2 - 1 \end{pmatrix}$$

We immediately find from the $B \ge 0$ condition

$$(\mu^4 - 2)^2 \ge 0 \qquad (B.10)$$

which gives as best choice

$$\mu^2 = \sqrt{2} \qquad (B.11)$$

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