REPLICA EQUIVALENCE IN THE EDWARDS-ANDERSON MODEL

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Abstract

After introducing and discussing the *link-overlap* between spin configurations we show that the Edwards-Anderson model has a *replica-equivalent* quenched equilibrium state, a property introduced by Parisi in the description of the mean-field spin-glass phase which generalizes ultrametricity. Our method is based on the control of fluctuations through the property of stochastic stability and works for all the finite-dimensional spin-glass models.

In the description of the spin glass phase the standard quantity usually investigated is the *overlap* between Ising spin configurations:

$$q(\sigma,\tau)=\frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i\ . \tag{1}$$

Within the mean-field approach such a quantity gives, in the so called quenched ensemble, a complete description of the system and it is in fact in terms of its distribution properties that the mean-field theory has been developed and understood by Parisi [MPV] and became successively accessible to a rigorous mathematical investigation starting from the seminal paper by Guerra [G]. The mean-field picture is described by two main features: first the overlap quenched distribution has a support that includes the neighbor of zero where the disorder is concentrated, second the distribution fulfills factorization-like properties [MPV] and is, in particular, completely identified by the single overlap probability. The factorization properties have been distinguished in two classes, *replica equivalent* and *ultrametric* (see [P1], [P2] and [PRT]) the first being a subclass of the second in the sense that ultrametricity implies replica-equivalence but, in general, not the viceversa. Replica equivalence is in fact introduced requiring to the overlap algebraic matrix ansatz Q (defined in [MPV]) to have any two rows (or columns) identical up to permutations. Such a condition is clearly satisfied by the replica symmetry breaking ansatz with its ultrametric structure but of course it includes many non-ultrametric instances.

In [AC], [GG] and [C] it has been shown how to derive replica-equivalence for mean-field spin-glass models from elementary thermodynamic properties like boundedness of fluctuations or investigating a new property of invariance under random perturbations called stochastic stability.

In this letter we show that the Edwards-Anderson spin-glass is *replica equivalent* with respect to the *bond-overlap* quenched distribution. Our strategy is based on a reformulation of stochastic stability which holds for finite-dimensional systems. Our method, which applies to every finite-dimensional Gaussian model, shows that the spin glass is fully described by the quenched distribution of a proper overlap and, with respect to it, is replica equivalent. The aim of the paper is then twofold: to establish new features of the realistic spin-glass models and to stress the proper quantity to be investigated.

To illustrate the physical meaning of our result we first derive the lowest order replica equivalence relation from the basic thermodynamic fact that the specific heat per particle is bounded everywhere (except at most on isolated singularities). Second we show that the property of stochastic stability holds, when properly formulated, also in finite-dimensions and implies the entire set of replica equivalent identities at every order.

For definiteness we consider the Edwards-Anderson spin-glass model but our method is largely independent on the details of the interactions and at the end we exhibit a wide class of finite-dimensional spin glass models to which our study apply *sic et simpliciter*. In the d-dimensional square lattice we study the the nearest-neighbors Hamiltonian

$$H(J,\sigma) = - \sum_{(n,n')} J_{n,n'} \sigma_n \sigma_{n'} \ , \tag{2}$$

where the $J_{n,n'}$ represent the quenched disorder and are usually assumed to be independent normal Gaussian variables. While the standard site-overlap between two spin configurations σ and τ is the normalized sum of the local site-overlap (1) the bond-overlap is the normalized sum of the local nearest-neighbor overlap

$$p(\sigma,\tau) = \frac{1}{N_B}\sum_{(n,n')}\sigma_n\tau_n\sigma_{n'}\tau_{n'} \ , \tag{3}$$

N_B being the number of nearest neighbor couples (bonds). For a discussion of the relevance of bond-overlap (or link-overlap as it first appeared in [MPRL1]) and for its use in numerical experiments to study the low temperature phase of finite-dimensional Ising spin glasses one may see [MPRLZ] and [MPRL2]. There is an obvious a priori advantage of the quantity (3) with respect to (1): while a spin flip inside a bond-connected region changes $q(\sigma,\tau)N$ of an amount proportional to the region volume it only changes $p(\sigma,\tau)N_B$ of an amount proportional to the region surface. The previous observation fails of course when the connectivity of the space grows with the volume like in the mean field cases. In the Sherrington-Kirkpatrick model for instance the usual overlap and the bond-overlap are related by the algebraic formula $q^2 = 2p + 1/N$ so that it is totally irrelevant which one of the two is studied. On the contrary in finite-dimension it does not exist such a simple relation among the two quantities even if from a bond configuration $\{\sigma_n\sigma_{n'}\}$ one may reconstruct the spin configuration $\{\sigma_n\}$ (up to a global sign) and vice-versa (see for this purpose the treatment of Gauge invariance in [BF] and [N]).

But definitely the deeper reason to introduce the bond overlap is related to the mathematical properties of the Hamiltonian (2). Being a sum of Gaussian variables (the J's) it is a Gaussian variable itself completely identified by its covariance matrix whose elements turn out to be proportional to the bond-overlap $p(\sigma,\tau)$. Indicating by Av the Gaussian average we have in fact:

$$\begin{aligned} Av\left(H(J,\sigma)H(J,\tau)\right) &= \sum_{(n,n'),(m,m')} Av\left(J_{n,n'}J_{m,m'}\right)\sigma_n\sigma_{n'}\tau_m\tau_{m'} \\ &= \sum_{(n,n'),(m,m')} \delta_{(n,n')(m,m')}\sigma_n\sigma_{n'}\tau_m\tau_{m'} \\ &= N_B\, p(\sigma,\tau)\ . \end{aligned} \tag{4}$$

More specifically we will work with the Hamiltionian (2) as with a family of 2^N Gaussian variables $\{H_\sigma\}$ (one for each configuration σ) whose joint distribution is specified by the $2^N \times 2^N$ square matrix of elements $p(\sigma,\tau)$ (for this perspective in the mean-field case see [CDGG]).

The previous observation says that all the typical quantities that are derived from the free energy like the internal energy the specific heat etc. are described by the bond-overlap moments with respect to the quenched measure. The same situation occurs in parallel for the mean-field case with respect to its own covariance matrix which is the square power of the standard site-overlap. Let for completeness show how in the Edwards-Anderson case the internal energy and specific heat are related to the quenched average of the matrix p. As the computation is going to illustrate our result does not depend on the detailed structure of the Hamiltonian as far as it is Gaussian. The quenched internal energy

$$U(\beta) \;=\; Av\left(\frac{\sum_\sigma H_\sigma e^{-\beta H_\sigma}}{\sum_{\sigma'} e^{-\beta H_{\sigma'}}}\right) \tag{5}$$

can be related to the bond-overlap moment using the elementary rule of integration by parts for correlated Gaussian variables $\{\xi_i\}$ with covariances

$c_{i,j}$ which states that for every bounded function f

$$Av\,(\xi_i \cdot f) \;=\; Av\left(\sum_j c_{i,j} \cdot \frac{\partial f}{\partial \xi_j}\right) \ . \tag{6}$$

Applying the (6) to the right hand side of (5) gives

$$Av\left(\frac{H_\sigma e^{-\beta H_\sigma}}{\sum_{\sigma'} e^{-\beta H_{\sigma'}}}\right) \;=\; N_B Av\left(\sum_\tau p(\sigma,\tau)\frac{\partial}{\partial H_\tau}\frac{e^{-\beta H_\sigma}}{\sum_{\sigma'} e^{-\beta H_{\sigma'}}}\right) \ , \tag{7}$$

and after the straightforward computation of the derivative we obtain

$$\frac{U(\beta)}{N_B} \;=\; -\beta\left(1 - Av\left(\frac{\sum_{\sigma,\tau} p(\sigma,\tau) e^{-\beta(H_\sigma + H_\tau)}}{\sum_{\sigma,\tau} e^{-\beta(H_\sigma + H_\tau)}}\right)\right) \ . \tag{8}$$

The (8) shows that the internal energy can be computed by first averaging the $p(\sigma,\tau)$ with respect to the random Gibbs-Boltzmann state over *two copies* of the system and then quenching the disorder by the Gaussian average. The final resulting operation is a probability measure (the so called quenched state E) over the matrix element $p(\sigma,\tau)$

$$E(p_{1,2}) \;=\; Av\left(\frac{\sum_{\sigma,\tau} p(\sigma,\tau) e^{-\beta(H_\sigma + H_\tau)}}{\sum_{\sigma,\tau} e^{-\beta(H_\sigma + H_\tau)}}\right) \ , \tag{9}$$

$$\frac{U(\beta)}{N_B} \;=\; -\beta\,(1 - E(p_{1,2})) \ . \tag{10}$$

More generally for higher order quantities like the specific heat etc. one introduces the quenched measure over an arbitrary number r of copies. For instance the computation of the specific heat is related, among others, to the moment

$$E(p_{1,2}p_{2,3}) \;=\; Av\left(\frac{\sum_{\sigma_1,\sigma_2,\sigma_3} p(\sigma_1,\sigma_2)p(\sigma_2,\sigma_3) e^{-\beta(H_{\sigma_1}+H_{\sigma_2}+H_{\sigma_3})}}{\sum_{\sigma_1,\sigma_2,\sigma_3} e^{-\beta(H_{\sigma_1}+H_{\sigma_2}+H_{\sigma_3})}}\right) \ . \tag{11}$$

From the definition of the specific heat per particle and using again just the rule of integration by parts (6) we get from (10)

$$
\begin{aligned}
c(\beta) &= \frac{d}{d\beta}\frac{U(\beta)}{N_B} = -\left(1 - E(p_{1,2})\right) + \beta \frac{d}{d\beta} E(p_{1,2}) = \\
&= -\left(1 - E(p_{1,2})\right) + 2\beta N_B E(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4}) \ .
\end{aligned}
\tag{12}
$$

Due to the convexity of the free energy the specific heat per particle is a bounded quantity (a part at most on isolated singularities [Ru, G]) and the (11) shows that in the thermodynamic limit ($N_B \to \infty$) the quenched state has to fulfill the identity

$$
E(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4}) = 0 \ , \tag{13}
$$

with a rate of decrease of at least N^{-1}. The previous relation is the lowest order replica equivalence identity (see [P1] and [P2]). Before introducing a general criterion which reproduce the whole set of those identities we want to stress that the previous discussion shows that the $r \times r$ matrix P of elements

$$
p_{l,m} \ = \ p(\sigma^{(l)}, \sigma^{(m)}) \tag{14}
$$

together with its probability measure E fully describes the Edwards-Anderson model in the sense that the moments of P like $E(p_{1,2})$, $E(p_{1,2}^2 p_{1,3})$ etc. represent the entire set of physical observables of the theory.

Let now develop an approach to stochastic stability for finite dimensional systems which runs parallel to the one introduced in [AC] for the mean field case. The starting point is the observation that the addition to the Hamiltonian of an independent Gaussian term of finite size:

$$
h(\tilde{J}, \sigma) = \frac{1}{\sqrt{N_B}} \sum_{(n,n')} \tilde{J}_{n,n'} \sigma_n \sigma_{n'} \tag{15}
$$

amounts to a slight change in the temperature. In fact for the sum law of independent Gaussian variables one has that in distribution it holds the relation

$$\beta H(J,\sigma)+\lambda h(\tilde{J},\sigma) \ = \beta'(\beta,\lambda)H(J',\sigma)\ , \tag{16}$$

with

$$\beta'(\beta,\lambda)\ =\ \sqrt{\beta^2+\frac{\lambda^2}{N_B}}\ , \tag{17}$$

so that, indicating by $E_\lambda^{(\beta)}$ the quenched state of Hamiltonian $\beta H(J,\sigma)+\lambda h(\tilde{J},\sigma)$ the (16) implies

$$E_\lambda^{(\beta)}\ =\ E^{(\beta')}\ \ . \tag{18}$$

Taking the thermodynamic limit (see [CG]) of the (18) and observing that

$$\lim_{N_B\to\infty}\beta'=\beta \tag{19}$$

we obtain

$$E_\lambda^{(\beta)}\ =\ E^{(\beta)}\ \ , \tag{20}$$

for all values of β a part, at most, isolated singularities. Such a property, introduced in the mean field case in [AC], is called *stochastic stability* [P3] and was later investigated in [FMPP1] and [FMPP2] to determine a relation between the off-equilibrium dynamics and the static properties. Stochastic stability has important consequences for the quenched state. In particular it says that the second derivative of the λ-deformed moments have to be zero (the first derivative being zero for antisymmetry). Let apply it for instance to the moment $E(p_{1,2})$. The elementary computation which uses only the Wick rule gives

$$\frac{d^2}{d\lambda^2}E_\lambda(p_{1,2})|_{\lambda=0}=2E(p_{1,2}^2-4p_{1,2}p_{2,3}+3p_{1,2}p_{3,4})\ , \tag{21}$$

so that from (20) we have (in the infinite volume limit)

$$E(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4}) \;=\; 0 \;. \tag{22}$$

Applying analogously stochastic stability to the moment $E(p_{1,2}p_{2,3})$ we obtain

$$E(2p_{1,2}^2p_{2,3}+p_{1,2}p_{2,3}p_{3,1}-6p_{1,2}p_{2,3}p_{3,4}+6p_{1,2}p_{2,3}p_{4,5}-3p_{1,2}p_{2,3}p_{2,4}) \;=\; 0 \;. \tag{23}$$

The (22) and (23) are two of the possible identities found in the framework of replica-equivalence [P1]. The application of stochastic stability to the whole set of moments produces the whole set of replica equivalence identities (this is a purely combinatorial argument and it may be seen in [C]).

We want to observe that our result is different from the one mentioned in [G]. The author there (at the end of section 4) suggests a method to prove some factorization identities for the standard overlap (1) for a general spin model. The result is achieved with the addition to the Hamiltonian of a term proportional to a mean-field spin-glass interaction and sending the interaction strength to zero after the thermodynamic limit is considered. The resulting state turns out to be replica-equivalent for the standard site-overlap but the addition of a mean field perturbation could, in principle, select a sub-phase of the whole equilibrium state. Our result instead obtains replica equivalence with respect to the bond-overlap for the whole quenched state.

Our scheme allows the treatment of the spin-glass Hamiltonians which include many-body interactions as well as unbounded spins variables:

$$H(J,\sigma) = -\sum_X J_X\sigma_X \;, \tag{24}$$

where

$$\sigma_X = \prod_{i\in X}\sigma_i \;, \tag{25}$$

and the J_X's are independent Gaussian variables with zero mean and variance Δ_X^2. A simple calculation like the one in (4) gives

$$\begin{aligned} Av(H(J,\sigma)H(J,\tau)) &= \sum_{X,Y} Av(J_X J_Y)\sigma_X\tau_Y \\ &= \sum_X \Delta_X^2 \sigma_X \tau_X \ . \end{aligned} \tag{26}$$

The replica equivalence identities that have been derived in the mean field case [AC] in terms of the standard site-overlap and that we obtained here in terms of the bond-overlap for the Edwards-Anderson model can be proved for the model of Hamiltonian (24) in terms of the generalized multi-overlap:

$$\widetilde{P}(\sigma,\tau) = \frac{1}{\mathcal{N}}\sum_X \Delta_X^2 \sigma_X \tau_X \ , \tag{27}$$

where we have indicated by $\mathcal{N}$ the number of interacting subsets.

Summarizing we have shown that replica-equivalence holds in the Edwards-Anderson model as well as in all Gaussian finite-dimensional spin-glass models when properly formulated in terms of the relative overlap. As the physical intuition suggests the result is robust enough to remain true also when the disorder J is chosen at random from a ± 1 Bernoulli disorder or for more general centered distributions (see [CG]). Our result holds at every temperature a part at most at isolated singularities and, in particular, it cannot make predictions at exactly $T = 0$. We want to remark moreover that replica equivalence does not identify uniquely the low temperature phase. It is in fact compatible with both the so called droplet picture (see [FH], [NS]) and with the replica symmetry breaking one [MPV]. Nevertheless it provides a rigorous proof of an infinite family of identities among overlap moments and, especially, it clearly points out the role and the importance of the suitable overlap to describe the model with.

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