1#

$$\lim_{x\to 0} \frac{\cos(x+\pi)}{x}\left(\frac{\sin^3(6x)-\sinh^3(6x)}{e^{7x}-1-\sin(7x)-\frac{49x^2}{2}-\frac{343x^3}{3}}\right) = \frac{6^5\cdot 24}{7^4}$$

$$\cos(x+\pi) \xrightarrow[x\to 0]{} -1$$

$$\sin^3(6x)-\sinh^3(6x) = (\sin(6x)-\sinh(6x))(\sin^2(6x)+\sin(6x)\sinh(6x)+\sinh^2(6x))$$

$$\sin(6x)-\sinh(6x) \sim \cancel{6x}-\frac{(6x)^3}{3!}-\cancel{6x}-\frac{(6x)^3}{3!}+o(x^4), \quad x\to 0$$

$$\sim -\frac{1}{3}(6x)^3, \quad x\to 0$$

$$\sin^2(6x)+\sin(6x)\sinh(6x)+\sinh^2(6x) \sim 3(6x)^2, \quad x\to 0$$

Pertanto

$$N \sim -\left(-\frac{1}{3}(6x)^3\right)\cdot 3\cdot(6x)^2 \sim (6x)^5$$

$$e^{7x} \sim 1+7x+\frac{(7x)^2}{2}+\frac{(7x)^3}{3!}+\frac{(7x)^4}{4!}+o(x^4), \quad x\to 0$$

$$\sin(7x) \sim 7x-\frac{(7x)^3}{3!}+o(x^4), \quad x\to 0$$

$$D \sim x\left(\cancel{1+7x}+\cancel{\frac{(7x)^2}{2}}+\frac{(7x)^3}{6}+\frac{(7x)^4}{24}+o(x^4)-\cancel{1}-\cancel{7x}+\frac{(7x)^3}{6}-\cancel{\frac{49}{2}x^2}-\frac{343}{3}x^3\right)$$

$$\sim x\left(\cancel{\frac{(7x)^3}{3}}-\cancel{\frac{343}{3}x^3}+\frac{7^4}{24}x^4\right) \sim \frac{7^4}{24}x^5, \quad x\to 0$$

Pertanto

$$\lim_{x\to 0}\frac{N}{D} = \lim_{x\to 0}\frac{6^5x^5}{\frac{7^4}{24}x^5} = \frac{6^5\cdot 24}{7^4}$$

2# $\alpha>0$

$$\int_0^{+\infty}\frac{(x-\sin x)^8}{(e^{\alpha x}-1)^\alpha(x^{2\alpha}+24)}\,dx$$

$$(x-\sin x)^8 \sim \left(\cancel{x}-\cancel{x}+\frac{x^3}{3!}+o(x^4)\right)^8 \sim \frac{x^{24}}{6^8}, \quad x\to 0$$

$$(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24) \underset{x\to 0}{\sim} (1+\alpha x+o(x)-1)^{\alpha}\, 24 \underset{x\to 0}{\sim} 24\alpha^{\alpha} x^{\alpha}$$

Pertanto

$$\frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)} \sim \frac{x^{24}}{6^8}\cdot\frac{1}{24\alpha^{\alpha}x^{\alpha}} \sim \frac{1}{6^8\cdot 24\,\alpha^{\alpha}x^{\alpha-24}}, \quad x\to 0$$

Quindi $\displaystyle\int_0^1 \frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)}\,dx$ converge se e solo se

(per $\alpha>0$) $\quad \alpha-24<1 \iff \alpha<25.$

D'altra parte

$$\frac{(x-1)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)} \leq \frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)} \leq \frac{(x+1)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)}$$

(per $x\geq 1$)

Quindi

$$\frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)} \sim \frac{x^8}{(e^{\alpha x})^{\alpha}x^{2\alpha}}, \quad x\to+\infty$$

Pertanto $\displaystyle\int_1^{+\infty} \frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)}\,dx$ converge se e solo

se converge $\displaystyle\int_1^{+\infty}\frac{x^8}{e^{\alpha^2 x}x^{2\alpha}}\,dx$ e ciò si verifica se

e solo se $\displaystyle\int_1^{+\infty}\frac{1}{e^{\alpha^2 x}x^{2\alpha-8}}\,dx$. D'altra parte

$$\frac{1}{e^{\alpha^2 x}x^{2\alpha-8}} \leq \begin{cases} \dfrac{1}{e^{\alpha^2 x}}, & \text{se } \alpha\geq 4 \\[2ex] \dfrac{C}{e^{\frac{\alpha^2 x}{2}}}, & \text{se } 0<\alpha<4 \end{cases} \longrightarrow \text{perché } \frac{1}{e^{\alpha^2 x}x^{2\alpha-8}} = \frac{1}{e^{\frac{\alpha^2}{2}x}}\cdot\frac{x^{8-2\alpha}}{e^{\frac{\alpha^2}{2}x}} \underset{0<\alpha<4}{\leq} \frac{C}{e^{\frac{\alpha^2}{2}x}} \quad \left(\text{infatti } \frac{x^{8-2\alpha}}{e^{\frac{\alpha^2}{2}x}} \underset{x\to+\infty}{\longrightarrow} 0\right)$$

Pertanto $\displaystyle\int_0^{+\infty}\frac{(x-\sin x)^8}{(e^{\alpha x}-1)^{\alpha}(x^{2\alpha}+24)}\,dx$ converge se e solo se $0<\alpha<25.$

3# $y''+16y = \sin(4x) + 5x$. Consideriamo l'eq. diff omogenea ass.

$$y''+16y=0 \quad \longrightarrow \quad \lambda^2+16=0 \quad \text{eq. caratteristica} \quad \lambda_{1,2}=\pm 4i$$

Quindi $V_2 = \mathrm{span}\{\cos(4x), \sin(4x)\}$.

Cerchiamo una sol. di $y''+16y=\sin(4x)$ con il metodo per simpatia. $f(x)=\sin(4x) = e^{\alpha x}\sin(4x)$ $(\alpha=0, \beta=4)$.

e $\pm i4$ è sol di $\lambda^2+16=0$ con molteplicità uno, quindi

$$\psi_1(x) = x\left(A\sin(4x)+B\cos(4x)\right).$$

Sostituendo

$$\psi_1' = A\sin(4x)+B\cos(4x) + x\left(4A\cos(4x) - 4B\sin(4x)\right)$$

$$\psi_1'' = 4A\cos(4x) - 4B\sin(4x) + 4A\cos(4x) - 4B\sin(4x) + x\left(-16A\sin(4x) - 16B\cos(4x)\right)$$

Pertanto $\psi_1''+16\psi_1 = \sin(4x)$ significa

$$8A\cos(4x) - 8B\sin(4x) = \sin(4x)$$

Quindi $8A=0$ e $-8B=1$, cioè $A=0$, $B=-\frac{1}{8}$

Pertanto $\psi_1 = -\frac{x}{8}\cos(4x)$.

Cerchiamo una soluzione ψ_2 di $y''+16y=5x$ con il metodo per simpatia

$f = e^{0x}5x$ $(\alpha=0$ e $P_1(x)=5x)$. Poiché α non è sol di $\lambda^2+16=0$. Quindi cerchiamo una soluzione nella forma $\psi_2 = Ax+B$. Sostituiamo nell'eq. $y''+16y=5x$ ottenendo

$$16(Ax+B)=5x \iff 16Ax+16B=5x \iff$$

$$\begin{cases}16A=5\\16B=0\end{cases} \iff \begin{cases}A=\frac{5}{16}\\B=0\end{cases} \quad \text{Pertanto} \quad \psi_2=\frac{5}{16}x$$

Quindi $LV_1 = V_2 - \frac{x}{8}\cos(4x) + \frac{5}{16}x$.

$h(x) = f(\pi x + g(x))$ $\quad g(0)=7,\ g'(0)=e^2,\ f'(0)=?,\ f'(7)=e$

$h'(x) = f'(\pi x + g(x))\,(\pi + g'(x))$. Quindi

$h'(0) = f'(g(0))(\pi + g'(0)) = f'(7)\,(\pi + e^2) = e(\pi + e^2)$

Risposta $\boxed{A}$

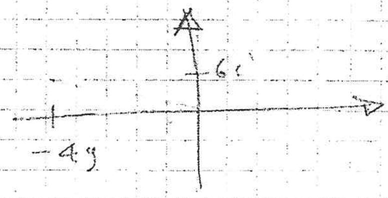
#5 $\quad \left(z^3 - (7-i)z^2 - 7iz\right)\left(z^4 + 49 - 6i\right) = 0$

$\left(z^3 - (7-i)z^2 - 7iz\right) = 0 \iff z\left(z^2 - (7-i)z - 7i\right) = 0$

$\boxed{z_4 = 0} \vee \left(z^2 - (7-i)z - 7i\right) = 0$

$(z-7)(z+i) = 0 \iff \boxed{z_5 = 7} \vee \boxed{z_6 = -i}$

$z^4 = -49 + 6i \qquad |-49+6i| = \sqrt{49^2 + 6^2} = \sqrt{2437}$



$\arg(-49+6i) = -\arctan\frac{6}{49} + \pi$

Posto $\varphi = -\arctan\frac{6}{49} + \pi$, quindi

$z_k = (2437)^{1/8}\, e^{i\theta_k}$, con $\theta_k = \dfrac{\varphi + 2k\pi}{4}$, $k = 0, 1, 2, 3.$

#6

$f(x) = \log\left(|x-11| - \sqrt{x^2+4}\right)$

Dominio d'esistenza $\quad |x-11| - \sqrt{x^2+4} > 0 \iff |x-11| > \sqrt{x^2+4} \iff$

$\cancel{x^2} - 22x + 121 > \cancel{x^2} + 4 \iff 117 > 22x \iff x < \frac{117}{22}$

$D = \left]-\infty, \frac{117}{22}\right[$. Quindi

$f: \left]-\infty, \frac{117}{22}\right[\longrightarrow \mathbb{R}$

Inoltre $f: \left]-\infty, \frac{117}{22}\right[\longrightarrow \mathbb{R}$ è derivabile in $\left]-\infty, \frac{117}{22}\right[\setminus \{11\}$ ma $\frac{117}{22} < 11$, quindi f è derivabile in $\left]-\infty, \frac{117}{22}\right[$, perché f è composizione di funzioni derivabili e $\forall x \in \left]-\infty, \frac{117}{22}\right[$

$f'(x) = \dfrac{1}{|x-11| - \sqrt{x^2+4}} \cdot \left(\operatorname{sgn}(x-11) - \dfrac{x}{\sqrt{x^2+4}}\right)$ e

Studiamo il segno di f':

$$\begin{cases} f'(x) > 0 \\ x \in \left]-\infty, \frac{117}{22}\right[\end{cases} \iff \begin{cases} \operatorname{sgn}(x-11) - \frac{x}{\sqrt{x^2+4}} > 0 \\ x \in \left]-\infty, \frac{117}{22}\right[\end{cases}$$

(si noti che $|x-11| - \sqrt{x^2+4} > 0$ nel dominio perché l'argomento del logaritmo deve essere positivo)

$\operatorname{sgn}(x-11) > 0$ se $x > 11$

$\operatorname{sgn}(x-11)$ $\quad - \; - \; - \; - \; \Big[\; - \; - \; - \; + \; + \; + \; + \; + \; + \; +$

$\qquad\qquad\qquad\qquad\quad \frac{117}{22} \qquad\qquad 11$

Quindi se $x < \frac{117}{22}$

$$\begin{cases} -1 - \frac{x}{\sqrt{x^2+4}} > 0 \\ x \in \left]-\infty, \frac{117}{22}\right[\end{cases} \iff \begin{cases} -\sqrt{x^2+4} > x \\ x \in \left]-\infty, \frac{117}{22}\right[\end{cases} \iff \begin{cases} -x > \sqrt{x^2+4} \\ x \in \left]-\infty, \frac{117}{22}\right[\end{cases}$$

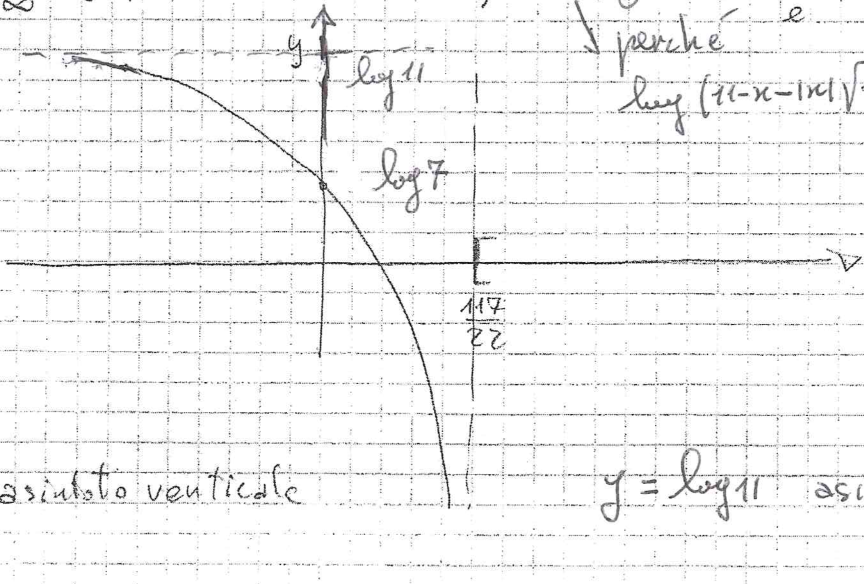
Se $x > 0$ non ha soluzione, se $\begin{cases} x \leq 0 \\ \cancel{x^2} > \cancel{x^2} + 4 \end{cases}$,

non ha soluzione, quindi $f'(x) < 0$ in $\left]-\infty, \frac{117}{22}\right[$, cioè f è monotona decrescente su $\left]-\infty, \frac{117}{22}\right[$. Inoltre

$$\lim_{x \to \left(\frac{117}{22}\right)^-} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \log\left(11 - x - \sqrt{x^2+4}\right)$$

$$= \lim_{x \to -\infty} \log\left(11 - x - |x|\sqrt{1 + \frac{4}{x^2}}\right) = \log 11.$$ Il grafico qualitativo è

perché $\log\left(11 - x - |x|\sqrt{1+\frac{4}{x^2}}\right) \underset{x \to -\infty}{\sim} \log\left(11 - \frac{2}{|x|}\right)$



$x = \frac{117}{22}$ è asintoto verticale $\qquad$ $y = \log 11$ asintoto orizzontale.

#7 $\displaystyle\lim_{n\to+\infty} \frac{n^{\dots}-3\dots+2\dots}{7n^4+3^{\frac{n-4n^4}{7+n}}-2^{\frac{n+n^4}{7+n}}+1}$

$$N \sim n^4-3^{\frac{n-n^4}{7+n}}+2^{\frac{n+n^4}{7+n}} \sim 2^{n^3} \quad \text{per } n\to+\infty$$

$$D \sim 7n^4+3^{\frac{n-4n^4}{7+n}}-2^{\frac{n+n^4}{7+n}+1} \sim -2^{n^3+1}$$

Pertanto $\displaystyle\lim_{n\to+\infty}\frac{N}{D}=-\frac{1}{2}$

#8 $$\int_{-7+\sqrt{5\pi}}^{-7+\sqrt{4\pi}} (t+7)^3 e^{(t+7)^2}\,dt = \int_{\sqrt{5\pi}}^{\sqrt{4\pi}} s^3 e^{s^2}\,ds = \frac{1}{2}\int_{\sqrt{5\pi}}^{\sqrt{4\pi}} s^2 e^{s^2}\,2s\,ds$$

$t+7=s$

$$=\frac{1}{2}\int_{5\pi}^{4\pi} \tau e^{\tau}\,d\tau = \frac{1}{2}\left[\tau e^{\tau}\right]_{\tau=5\pi}^{\tau=4\pi}-\frac{1}{2}\int_{5\pi}^{4\pi} e^{\tau}\,d\tau$$

$s^2=\tau \quad d\tau=2s\,ds$

$$=\frac{1}{2}\left(4\pi e^{4\pi}-5\pi e^{5\pi}\right)-\frac{1}{2}\left[e^{\tau}\right]_{\tau=5\pi}^{\tau=4\pi}$$

$$=\frac{1}{2}\left(4\pi e^{4\pi}-5\pi e^{5\pi}\right)-\frac{1}{2}\left(e^{4\pi}-e^{5\pi}\right)$$

$$=\frac{1}{2}e^{4\pi}(4\pi-1)-\frac{1}{2}e^{5\pi}(5\pi-1).$$