$\alpha = 2$, $\beta = 5$ $\quad (\alpha^2 < \beta)$

Esercizio 1

$$f(x) = \arctan \frac{x-\alpha}{|x^2-\beta|} - \int_0^x \frac{|t^2-\beta| - t + \alpha}{(t^2-\beta)^2 + |t-\alpha|^2}\, dt$$

$D = \mathbb{R} \setminus \{\pm\sqrt{\beta}\}$ e f è derivabile in D.

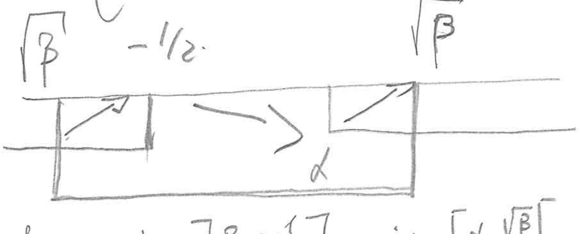
$$f'(x) = \frac{1}{1+\frac{(x-\alpha)^2}{|x^2-\beta|^2}} \, \frac{|x^2-\beta| - 2x\,\mathrm{sgn}(x^2-\beta)\cdot(x-\alpha)}{(x^2-\beta)^2} - \frac{|x^2-\beta| - x + \alpha}{(x^2-\beta)^2 + |x-\alpha|^2}$$

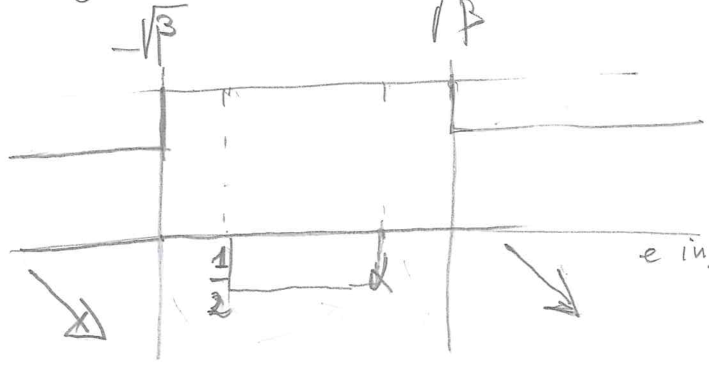
$$= \frac{\cancel{|x^2-\beta|^2}}{|x^2-\beta|^2 + (x-\alpha)^2} \cdot \frac{|x^2-\beta| - 2x\,\mathrm{sgn}(x^2-\beta)\cdot(x-\alpha)}{\cancel{(x^2-\beta)^2}} - \frac{|x^2-\beta| - x + \alpha}{|x^2-\beta|^2 + |x-\alpha|^2}$$

$$= \frac{x-\alpha}{|x^2-\beta|^2 + (x-\alpha)^2} \left(-2x\,\mathrm{sgn}(x^2-\beta) + 1 \right)$$

$$\begin{cases} f' > 0 \\ x \in \mathbb{R} \setminus \{\pm\sqrt{\beta}\} \end{cases} \Longleftrightarrow \begin{cases} (x-\alpha)\left(1 - 2x\,\mathrm{sgn}(x^2-\beta)\right) > 0 \\ x \in \mathbb{R} \setminus \{\pm\sqrt{\beta}\} \end{cases}$$

$$\begin{cases} -\sqrt{\beta} < x < \sqrt{\beta} \\ (x-\alpha)(1+2x) > 0 \end{cases} \lor \begin{cases} x < -\sqrt{\beta} \lor x > \sqrt{\beta} \\ (x-\alpha)(1-2x) > 0 \end{cases}$$





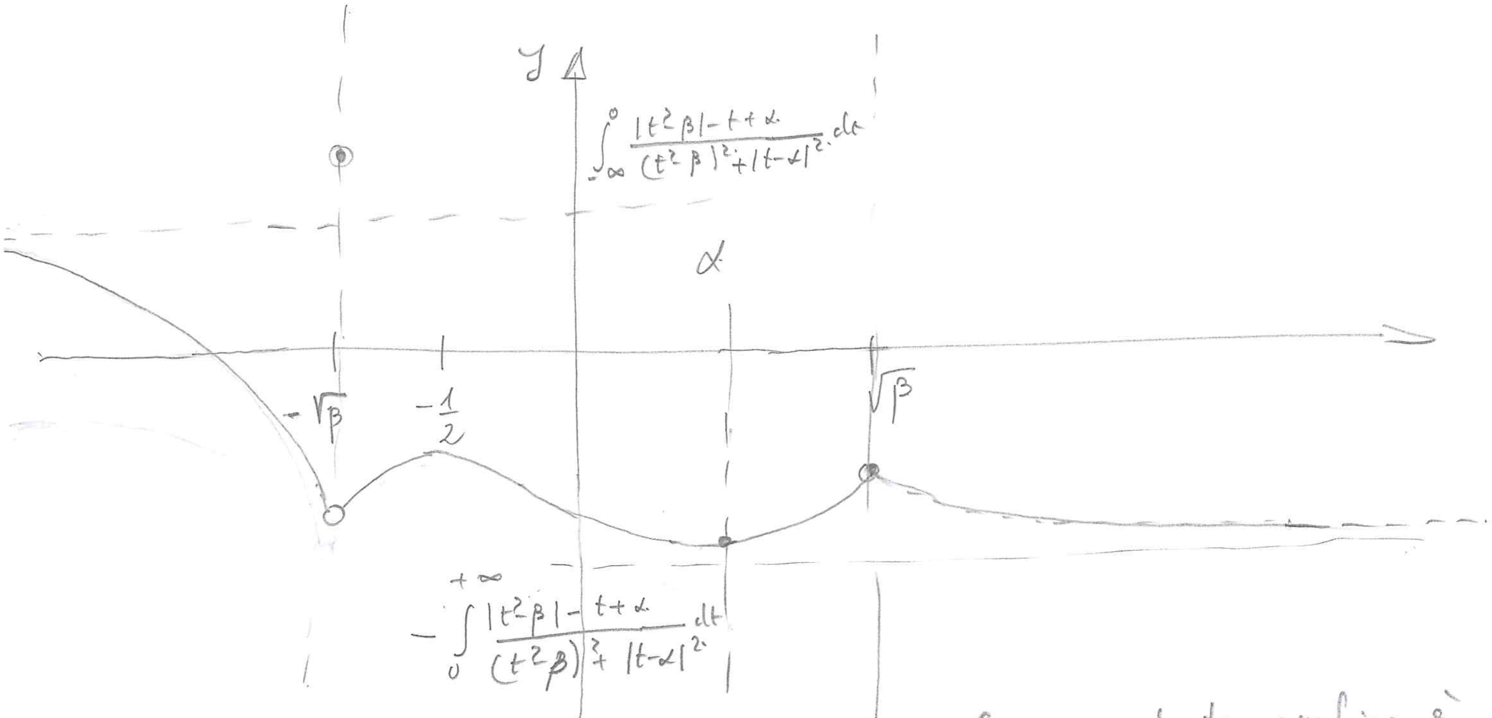
f cresce in $]\beta, -\frac{1}{2}]$ e in $[\alpha, \sqrt{\beta}[$ e in $]\sqrt{\beta}, +\infty[$

f decresce in $[-\frac{1}{2}, \alpha]$ e in $]-\infty, -\sqrt{\beta}[$

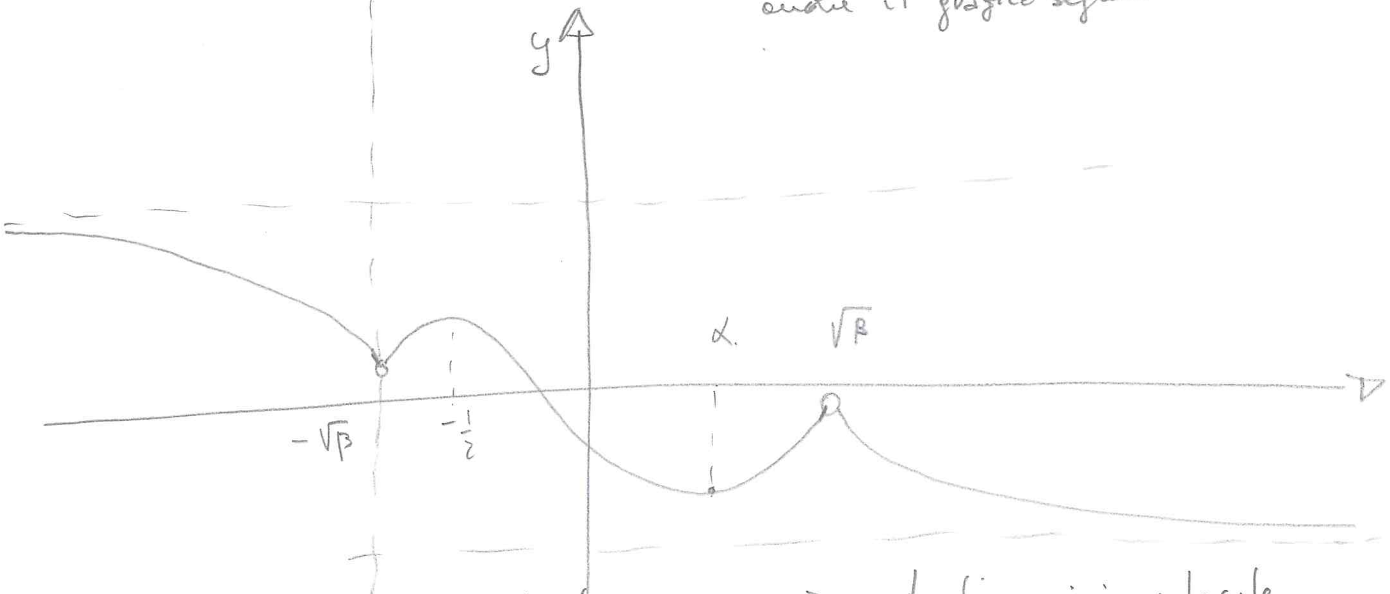
$$\lim_{x\to+\infty} f(x) = -\int_0^{+\infty} \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2}\,dt < 0 \quad \text{perché} \quad \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2} \sim \frac{1}{t^2} \quad \text{per } t\to\pm\infty$$

$$\lim_{x\to-\infty} f(x) = -\int_0^{-\infty} \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2}\,dt = \int_{-\infty}^{0} \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2}\,dt > 0.$$

Quindi $y = -\int_0^{+\infty} \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2}\,dt$ e $y = \int_{-\infty}^{0} \frac{|t^2-\beta|-t+\alpha}{(t^2-\beta)^2+|t-\alpha|^2}\,dt$ sono rispettivamente gli asintoti per $x\to+\infty$ e $x\to-\infty$



Poiché non sono richiesti i valori estremanti il precedente grafico è solo indicativo. Per esempio è comunque accettabile anche il grafico seguente.



$-\frac{1}{2}$ è punto di max locale, α è punto di minimo locale.

Es 2.

$$I = \int_{-\pi+4}^{\pi+4} \sin(4t-16)\cos(3t-12)\,dt = \int_{-\pi+4}^{\pi+4} \sin(4(t-4))\cos(3(t-4))\,dt$$

$t-4=s \qquad dt=ds$

$$= \int_{-\pi}^{\pi} \sin(4s)\cos(3s)\,ds.$$ Quindi integrando per parti

$$I = \left[-\frac{\cos(4s)}{4}\cos(3s) \right]_{s=-\pi}^{s=\pi} - \int_{-\pi}^{\pi} \frac{3}{4}\cos(4s)\sin(3s)\,ds$$

$$= \left[-\frac{\cos(4s)}{4}\cos(3s) \right]_{s=-\pi}^{s=\pi} - \frac{3}{4}\left\{ \left[+\frac{1}{4}\sin(4s)\sin(3s) \right]_{s=-\pi}^{s=\pi} - \int_{-\pi}^{\pi} \frac{3}{4}\sin(4s)\cos \right.$$

$$= \left[-\frac{\cos(4s)}{4}\cos(3s) - \frac{3}{16}\sin(4s)\sin(3s) \right]_{s=-\pi}^{s=\pi} + \frac{9}{16} I$$

Quindi
$$7I = I - \frac{9}{16} I = -\frac{\cos(4\pi)}{4}\cos(3\pi) + \frac{\cos(4\pi)}{4}\cos(3\pi) = 0.$$

cioè $I = 0$.

Es. 3

$$\int_0^{+\infty} \frac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x}\,dx$$ se $x \to 0$ allora $\dfrac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x} \sim$
(i) $\dfrac{2x^{2+\alpha}}{x^{2\alpha}}$ se $2\alpha <$
(ii) $\dfrac{2x^{2+\alpha}}{(1+2\alpha)x}$, se $2\alpha = 1$
(iii) $\dfrac{2x^{2+\alpha}}{2\alpha x}$, se $2\alpha >$

(i) e (ii) $\begin{cases} \alpha - 2 < 1 \\ \alpha \le \frac{1}{2} \\ \alpha > 0 \end{cases} \iff \begin{cases} \alpha < 3 \\ \alpha \le \frac{1}{2} \\ \alpha > 0 \end{cases} \iff 0 < \alpha \le \frac{1}{2}.$

(iii) $\begin{cases} \alpha > \frac{1}{2} \\ -1-\alpha < -1 \end{cases} \iff \alpha > \frac{1}{2}$

Quindi converge per ogni $\alpha > 0$ l'int. gen. $\displaystyle\int_0^1 \frac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x}\,dx.$

Mentre se

$x \to +\infty$

$$\frac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x} \sim \begin{cases} (i) & \dfrac{2x^{2+\alpha}}{x^{2\alpha}} \quad \text{se } 2\alpha>1 \\ (ii) & \dfrac{2x^{2+\alpha}}{(1+2\alpha)x} \quad \text{se } 2\alpha=1 \\ (iii) & \dfrac{2x^{2+\alpha}}{2\alpha x} \quad \text{se } 2\alpha<1 \end{cases}$$

Conv. se

(i) e (ii) $\begin{cases} 2\alpha \geq 1 \\ 2\alpha-\alpha-2>1 \\ \alpha>0 \end{cases} \Longleftrightarrow \begin{cases} \alpha \geq \frac{1}{2} \\ \alpha>3 \end{cases} \Longleftrightarrow \alpha>3$

(iii) $\begin{cases} 2\alpha<1 \\ -1-\alpha>1 \end{cases} \Longleftrightarrow \begin{cases} \alpha<\frac{1}{2} \\ \alpha<-2 \end{cases} \Longleftrightarrow S=\phi$

Quindi $\displaystyle\int_1^{+\infty} \frac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x}\,dx < +\infty \Longleftrightarrow \alpha>3.$

Pertanto $\displaystyle\int_0^{+\infty} \frac{2x^{2+\alpha}}{x^{2\alpha}+2\alpha x}\,dx$ conv se e solo $\alpha>3$.

Es 4

$$\lim_{x\to 0} \frac{\sin^2(4x)-(4x+7x^3)^2}{(1-\cos(7x^2))\sin(\frac{\pi}{3}+x)} = \frac{\left(\frac{4^3}{6}-7\right)8}{-\frac{7^2}{4}\cdot\sqrt{3}}$$

perché

$N \sim (\sin 4x-4x-7x^3)(\sin 4x+4x+7x^3) \sim \left(\frac{(4x)^3}{6}-7x^3\right)8x \sim \left(\frac{4^3}{6}-7\right)8x$

$D \sim -\frac{(7x^2)^2}{2}\cdot\frac{\sqrt{3}}{2}$

Es. 5

Sia $f: I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$, $x_0 \in D(I) \cap I$, f derivabile in x_0 e esiste $\varepsilon > 0$ t.c. $]x_0-\varepsilon, x_0+\varepsilon[\subset I$. Se x_0 è punto estremante locale per f (massimo o minimo), allora $f'(x_0) = 0$.