Es1. $t \mapsto |t-2|e^{-2t^2} \in C(\mathbb{R})$, quindi $t \mapsto |t-2|e^{-2t^2}$ è localmente Riemann integrabile su $\mathbb{R}$ e $f(x) = \int_2^x |t-2|e^{-2t^2}dt$

è derivabile con derivata continua in ogni punto $x \in \mathbb{R}$.

Inoltre per ogni $x \in \mathbb{R}$ $f'(x) = |x-2|e^{-2x^2}$.

Determiniamo gli intervalli di monotonia risolvendo

$$\begin{cases} f'(x) \geq 0 \\ x \in \mathbb{R} \end{cases} \iff \begin{cases} |x-2|e^{-2x^2} \geq 0 \\ x \in \mathbb{R} \end{cases} \iff \begin{cases} |x-2| \geq 0 \\ x \in \mathbb{R} \end{cases} \quad \forall x \in \mathbb{R}$$

Inoltre $f'(x) > 0$ $\forall x \in \mathbb{R}\setminus\{2\}$. Quindi f è monotona strettamente crescente in $\mathbb{R}$. Non esistono punti estremanti e $\lim_{x\to+\infty} f(x) = \int_2^{+\infty} |t-2|e^{-2t^2}dt \in \mathbb{R}$, $\lim_{x\to-\infty} \int_2^{-\infty} |t-2|e^{-2t^2}dt \in \mathbb{R}$

perché sia $\int_2^{+\infty} |t-2|e^{-2t^2}dt$ che $\int_2^{-\infty} |t-2|e^{-2t^2}dt$ sono convergenti. Infatti, dal teorema del confronto, nel primo caso abbiamo

$$|t-2|e^{-2t^2} \leq te^{-2t^2} \quad \text{per ogni } t > 2$$

Quindi da $\int_2^{+\infty} te^{-2t^2}dt = \left[-\frac{e^{-2t^2}}{4}\right]_{t=2}^{t=+\infty} = \frac{e^{-8}}{4}$ segue

la convergenza di $\int_2^{+\infty} |t-2|e^{-2t^2}dt$.

Analogamente $\int_2^{-\infty} |t-2|e^{-2t^2}dt = -\int_{-\infty}^{2} |t-2|e^{-2t^2}dt = \int_{-\infty}^{2} (t-2)e^{-2t^2}dt$

$= -\int_{-\infty}^{2} (2-t)e^{-2t^2}dt$ con $(2-t)e^{-2t^2} \geq 0$ $\forall\, t \leq 2$

Quindi $2-te^{-2t^2} \leq -2te^{-2t^2}$ per ogni $t \leq -2$

e $\int_{-\infty}^{2} (2-t)e^{-2t^2}dt$ converge perché converge

$$-2\int_{-\infty}^{2} te^{-2t^2} = \left[\frac{e^{-2t^2}}{2}\right]_{t=-\infty}^{t=2} = \frac{e^{-8}}{2}.$$

Si noti che $\int_{2}^{-\infty} |t-2|e^{-2t^2}dt$ sarà un numero negativo.

f' è derivabile per ogni $x \in \mathbb{R}\setminus\{2\}$ perché prodotto di funzioni derivabili in $\mathbb{R}\setminus\{2\}$ e $\mathbb{R}$ rispettivamente

Inoltre per ogni $x \in \mathbb{R}\setminus\{2\}$

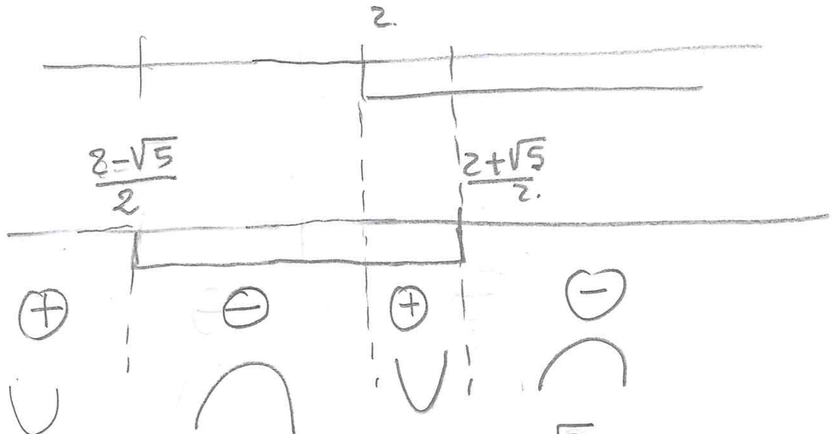
$$f'(x) = \operatorname{sgn}(x-2)\,e^{-2x^2} - 4x|x-2|e^{-2x^2} = \operatorname{sgn}(x-2)\,e^{-2x^2}\left(1-4x(x-2)\right)$$

$$\begin{cases} f'(x) > 0 \\ x \in \mathbb{R}\setminus\{2\} \end{cases} \iff \begin{cases} \operatorname{sgn}(x-2)\left(-4x^2+8x+1\right) > 0 \\ x \in \mathbb{R}\setminus\{2\} \end{cases}$$

ma $-4x^2+8x+1=0 \iff x_{1,2} = \dfrac{-4\pm\sqrt{16+4}}{-4} = \dfrac{-4\pm\sqrt{20}}{-4} = \dfrac{2\mp\sqrt{5}}{2}$

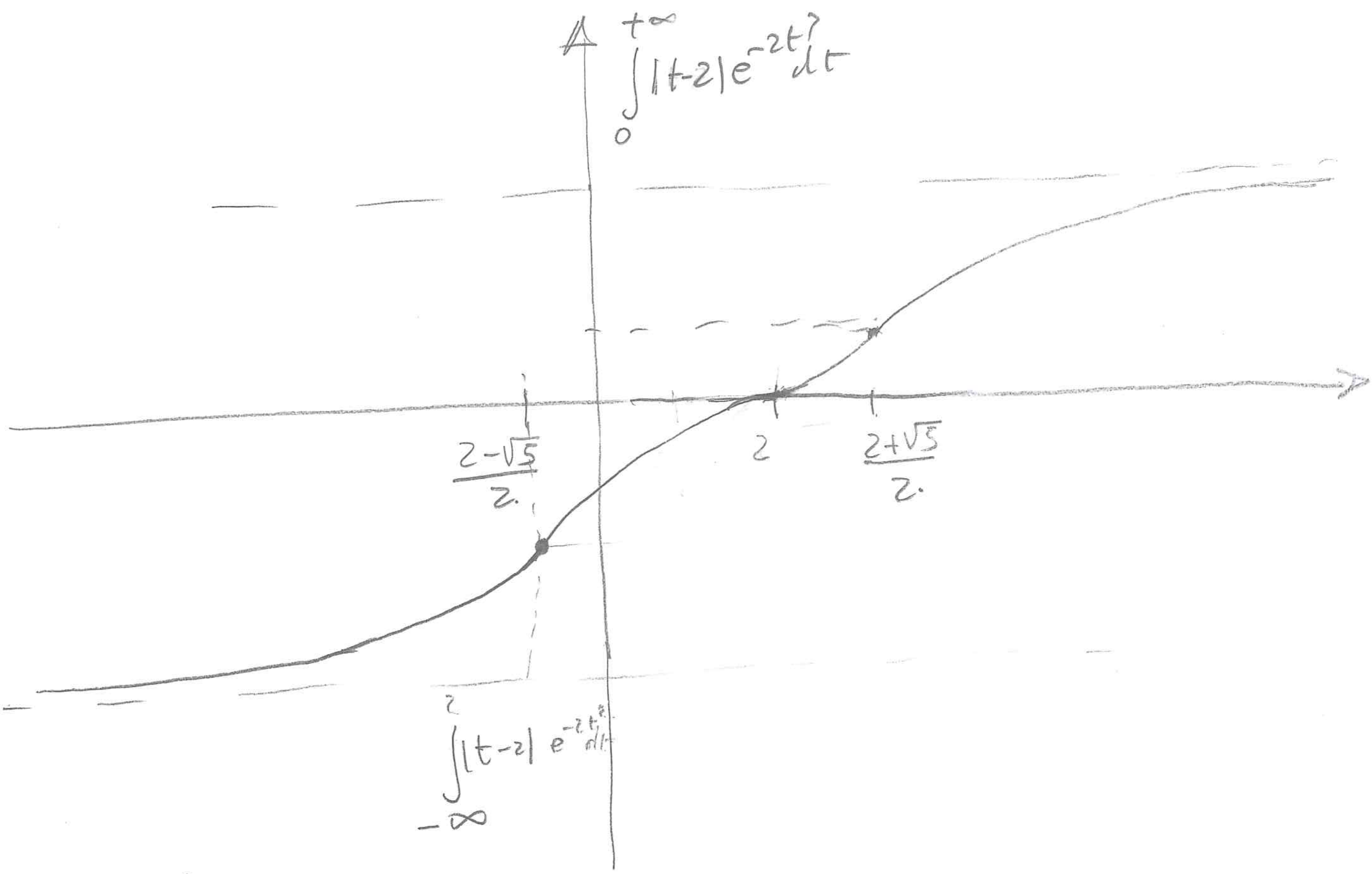
F_1 $\operatorname{sgn}(x-2) > 0$

F_2 $-4x^2+8x+1 > 0$



La funzione f è convessa in $\left]-\infty, \frac{2-\sqrt{5}}{2}\right]$, $\left[2, \frac{2+\sqrt{5}}{2}\right]$ mentre f è concava in $\left[\frac{2-\sqrt{5}}{2}, 2\right]$, $\left[\frac{2+\sqrt{5}}{2}, +\infty\right[$

$\frac{2-\sqrt{5}}{2}$, 2 , $\frac{2+\sqrt{5}}{2}$ sono punti di flesso



Se studiamo $g: x \mapsto |x-2| e^{-2x^2} - f(x)$ otteniamo che $\forall x \in \mathbb{R} \setminus \{2\}$

$$g'(x) = \mathrm{sgn}(x-2) e^{-2x^2} - 4x|x-2| e^{-2x^2} - |x-2| e^{-2x^2}$$

$$= e^{-2x^2} \mathrm{sgn}(x-2) \left(1 - 4x(x-2) - (x-2) \right)$$

$$= e^{-2x^2} \mathrm{sgn}(x-2) \left(1 - 4x^2 + 8x - x + 2 \right)$$

$$= e^{-2x^2} \mathrm{sgn}(x-2) \left(-4x^2 + 7x + 3 \right)$$

$$\begin{cases} g'(x) > 0 \\ x \in \mathbb{R} \setminus \{2\} \end{cases} \iff \begin{cases} \mathrm{sgn}(x-2)(-4x^2+7x+3) > 0 \\ x \in \mathbb{R} \setminus \{2\} \end{cases}$$

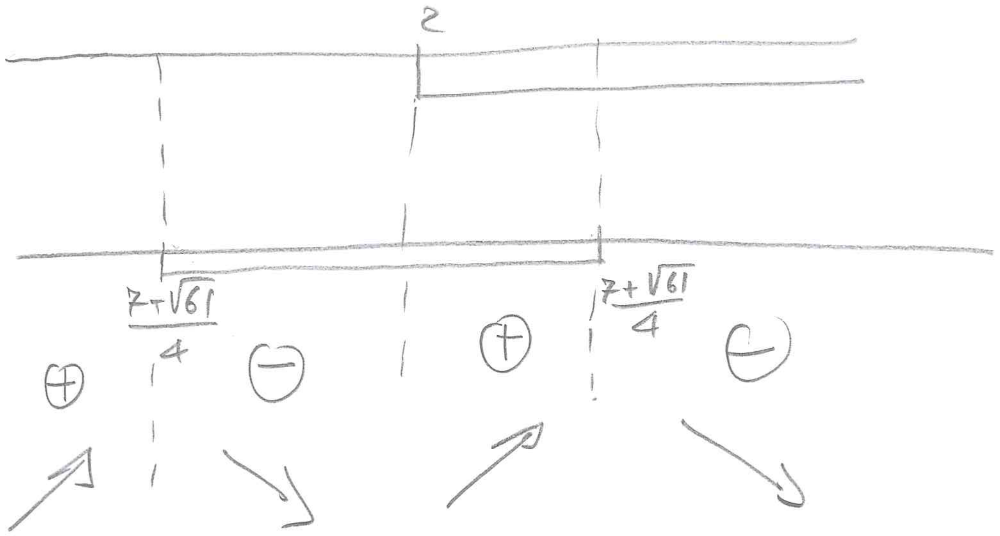
ma $-4x^2+7x+3 = 0$

$$\iff x_{1,2} = \frac{-7 \pm \sqrt{49+12}}{-4}$$

$$\iff x_{1,2} = \frac{-7 \pm \sqrt{61}}{-4} = \frac{7 \mp \sqrt{61}}{4}$$

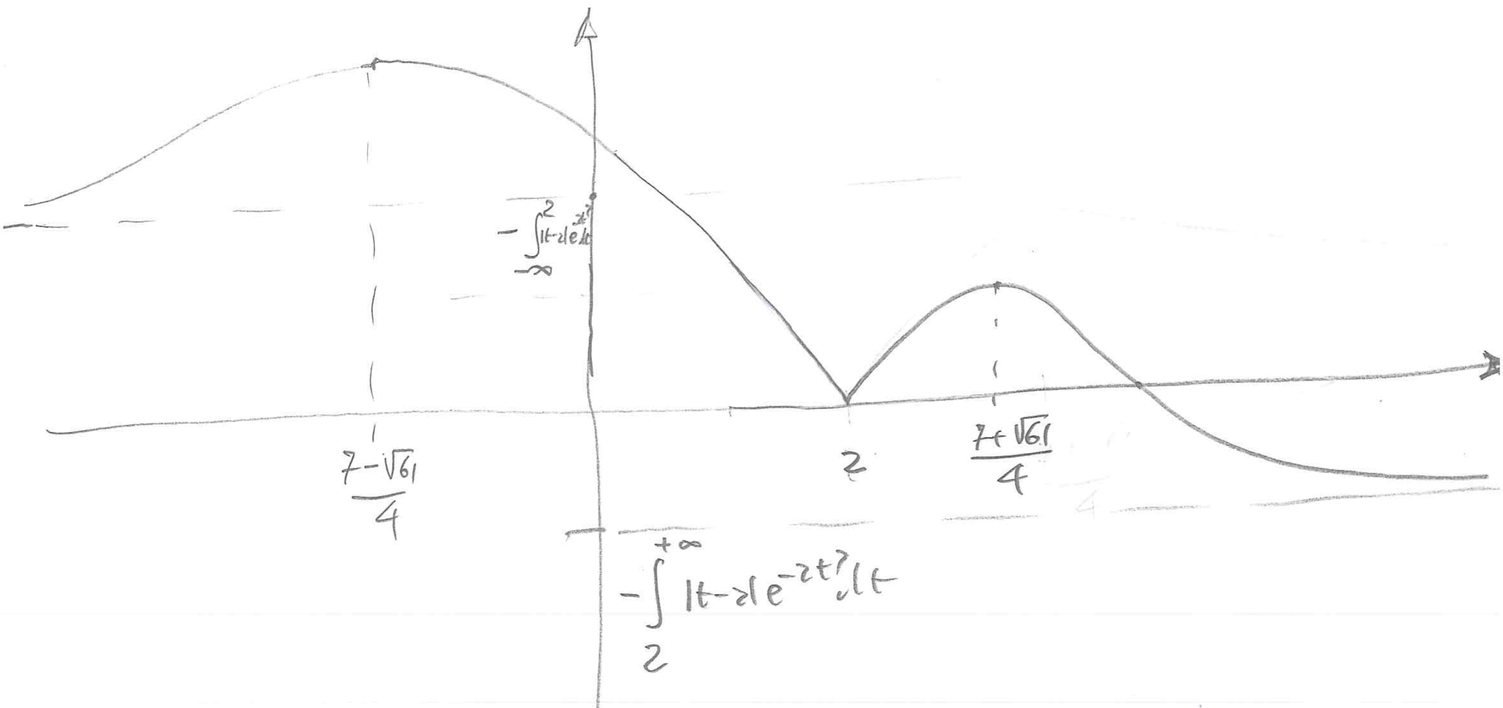
$F_1 = \operatorname{sgn}(x-2)$

$F_2 = -4x^2+7x+3$



$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} -f(x) = -\int_{-\infty}^{2} |t-2| e^{-2t^2} dt$$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} -f(x) = -\int_{2}^{+\infty} |t-2| e^{-2t^2} dt$$



L'equazione ha due soluzioni una è in 2 e l'altra è data da un numero strettamente maggiore di $\frac{7+\sqrt{61}}{4}$. Infatti $f\left(\left[\frac{7+\sqrt{61}}{4}, +\infty\right[\right) = \left]-\int_{2}^{+\infty} |t-2| e^{-2t^2} dt, f\left(\frac{7+\sqrt{61}}{4}\right)\right]$ con $-\int_{2}^{+\infty} |t-2| e^{-2t^2} dt < 0$ e $f\left(\frac{7+\sqrt{61}}{4}\right) > 0$.

Es 2

$$\int_3^9 \frac{t}{t^2-3t+9}\,dt = \frac{1}{2}\int_3^9 \frac{2t}{t^2-3t+9}\,dt = \frac{1}{2}\int_3^9 \frac{2t-3}{t^2-3t+9}\,dt$$

$$+ \frac{3}{2}\int_3^9 \frac{1}{t^2-3t+9}\,dt,$$

$$\Delta = 9-36=-27<0$$

Quindi

$$= \frac{1}{2}\Big[\log|t^2-3t+9|\Big]_{t=3}^{t=9} + \frac{3}{2}\int_3^9 \frac{1}{(t-\frac{3}{2})^2+9-\frac{9}{4}}\,dt$$

$$= \frac{1}{2}\log\frac{63}{9} + \frac{3}{2}\int_3^9 \frac{1}{\frac{27}{4}\left\{\left[\frac{2}{\sqrt{27}}(t-\frac{3}{2})\right]^2+1\right\}}\,dt$$

$$= \frac{1}{2}\log 7 + \frac{2}{9}\int_3^9 \frac{1}{1+\left[\frac{2}{\sqrt{27}}(t-\frac{3}{2})\right]^2}\,dt$$

$$= \frac{1}{2}\log 7 + \frac{2}{9}\left[\frac{\sqrt{27}}{2}\arctan\left(\frac{2}{\sqrt{27}}(t-\frac{3}{2})\right)\right]_{t=3}^{t=9}$$

$$= \frac{1}{2}\log 7 + \frac{\sqrt{3}}{3}\left(\arctan\frac{2}{3\sqrt{3}}\cdot\frac{15}{2} - \arctan\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{2}\log 7 + \frac{\sqrt{3}}{3}\left(\arctan\frac{5}{\sqrt{3}} - \frac{\pi}{6}\right)$$

ES.3

$$\int_0^{+\infty} \frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}}\,dx = \int_0^1 \frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}}\,dx + \int_1^{+\infty} \frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}}\,dx$$

Se $x \to 0$

$$\frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}} \sim_{x\to 0} \begin{cases} \dfrac{1}{x^{\alpha+\frac{1}{12\alpha}}}, & \text{se } \alpha+\frac{1}{12\alpha}<8 \quad (i) \\ \dfrac{1}{2x^8}, & \text{se } \alpha+\frac{1}{12\alpha}=8 \quad (ii) \\ \dfrac{1}{x^8}, & \text{se } \alpha+\frac{1}{12\alpha}>8 \quad (iii) \end{cases}$$

Se (i), allora

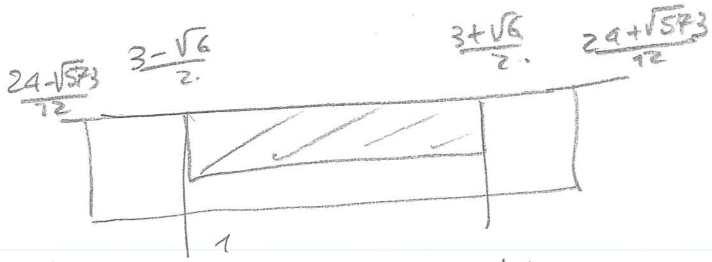
$$\begin{cases} 12\alpha^2+1<96\alpha \\ \alpha>0 \end{cases} \iff \begin{cases} 12\alpha^2-96\alpha+1<0 \\ \alpha>0 \end{cases}$$

$$\alpha_{1,2} = \frac{48\pm\sqrt{48^2-12}}{12} = \frac{48\pm\sqrt{2292}}{12} = \frac{48\pm 2\sqrt{573}}{12} = \frac{24\pm\sqrt{573}}{12}$$

$\frac{24-\sqrt{573}}{12} < \alpha < \frac{24+\sqrt{573}}{12}$. Quindi avremo convergenza dell'integrale se $\alpha + \frac{1}{12\alpha} < 1 \iff 12\alpha^2+1-12\alpha<0$

$$\alpha_{1,2} = \frac{6\pm\sqrt{36-12}}{12} \begin{cases} \dfrac{6+2\sqrt6}{12} = \dfrac{3+\sqrt6}{6} \\ \dfrac{6-2\sqrt6}{12} = \dfrac{3-\sqrt6}{6} \end{cases} \iff \frac{3-\sqrt6}{6} < \alpha < \frac{3+\sqrt6}{6}$$

Da cui



$$\boxed{\frac{3-\sqrt6}{2} < \alpha < \frac{3+\sqrt6}{2}}$$

(ii) e (iii) $\int_0^1 \frac{1}{x^8}\,dx$ diverge.

Invece $\int_1^{+\infty} \frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}}\,dx$ dal criterio del confronto

$$\frac{e^{-\alpha x}}{x^8+x^{\alpha+\frac{1}{12\alpha}}} \le \frac{e^{-\alpha x}}{x^8} \le \frac{e^{-\frac{\alpha}{2}x}}{x^8}\, e^{-\frac{\alpha}{2}x} \le M e^{-\frac{\alpha}{2}x}$$

deduciamo che l'int. gen. converge per ogni $\alpha>0$

perché $e^{-\frac{\alpha}{2}x}$ è int. in s.g. su $[1,+\infty[$ per ogni $\alpha>0$.

Es.4

$$\lim_{x\to 0} \frac{e^{2x}-\cosh(2x)-2x-\frac{(2x)^2}{6}}{x^2\left(\sin^2(3x^2+2x)-(2x)^2\right)\sin\left(\frac{\pi}{6}+x\right)}$$

$$N \underset{x\to 0}{\sim} 1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\frac{(2x)^5}{5!}-\left(1+\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}+o(x^5)\right)-2x-\frac{(2x)^2}{6}$$

$$\sim \frac{(2x)^5}{5!}$$

$$D \sim x^2\left(\sin(3x^2+2x)-2x\right)\left(\sin(3x^2+2x)+2x\right)\sin\frac{\pi}{6}$$

$$\sim x^2\left(3x^2+\cancel{2x}+o(x^2)-\cancel{2x}\right)\left(2x+2x+o(x)\right)\frac{1}{2}$$

$$\sim \frac{12x^5}{2} \sim 6x^5$$

Quindi $\lim_{x\to 0}\frac{N}{D} = \lim_{x\to 0} \frac{2^5}{5!}\cancel{x^5}\frac{1}{6\cancel{x^5}} = \frac{2^5}{6\cdot 5!}$

Es. 5 Sia $f: I \longrightarrow \mathbb{R}$ $I \subseteq \mathbb{R}$, I intervallo

Sia $\varphi \in C^1([\alpha, \beta], \mathbb{R})$ t.c. $\varphi([\alpha, \beta]) \subset I$.

Se $f \in C(I)$, allora

$$\int_{\alpha}^{\beta} f(\varphi(t))\, \varphi'(t)\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(\tau)\, d\tau .$$