Correzione

$$\frac{\left(\sqrt[3]{n+2}-\sqrt[3]{n-4}\right)n^3}{\sqrt{n+5}} = \frac{\left(\sqrt[3]{n+2}-\sqrt[3]{n-4}\right)}{\sqrt{n+5}}\frac{\left(\sqrt[3]{(n+2)^2}+\sqrt[3]{(n+2)(n-4)}+\sqrt[3]{(n-4)^2}\right)}{\left(\sqrt[3]{(n+2)^2}+\sqrt[3]{(n+2)(n-4)}+\sqrt[3]{(n-4)^2}\right)}\cdot n^3$$

$$= \frac{\left(\cancel{n}+2-\cancel{n}+4\right)n^3}{\sqrt{n+5}\left(\sqrt[3]{(n+2)^2}+\sqrt[3]{(n+2)(n-4)}+\sqrt[3]{(n-4)^2}\right)} = \frac{6n^3}{\sqrt{n+5}\left(\sqrt[3]{(n+2)^2}+\sqrt[3]{(n+2)(n-4)}+\sqrt[3]{(n-4)^2}\right)}$$

Quindi

$$\frac{\left(\sqrt[3]{n+2}-\sqrt[3]{n-4}\right)n^3}{\sqrt{n+5}} \underset{n\to+\infty}{\sim} \frac{6n^3}{n^{1/2}\cdot 3n^{\frac{2}{3}}} \sim \frac{2n^3}{n^{7/6}},$$ da cui

$$\lim_{n\to+\infty}\frac{\sqrt[3]{n+2}-\sqrt[3]{n-4}}{\sqrt{n+5}}\cdot n^3 = \lim_{n\to+\infty} 2n^{\frac{11}{6}} = +\infty. \quad \square$$

$$\frac{n^5+\sqrt[7]{n}-\log n}{3n^3\cdot\sqrt{(n+5)^2}} \sim \frac{n^5}{3\cdot n^4} \sim \frac{n}{3}.$$ Quindi

$$\lim_{n\to+\infty}\frac{n^5+\sqrt[7]{n}-\log n}{3n^3\cdot\sqrt{(n+5)^2}} = \lim_{n\to+\infty}\frac{n}{3} = +\infty.$$

Infatti

$\sqrt[7]{n} = o(n^5)$, $n\to+\infty$ (cioè $\lim_{n\to+\infty}\frac{\sqrt[7]{n}}{n^5}=0$)

$\log n = o(n^5)$, $n\to+\infty$ (cioè $\lim_{n\to+\infty}\frac{\log n}{n^5}=0$) $\square$

Utilizziamo il criterio del rapporto, ponendo

$a_n = \frac{n^5}{e^n}$. Allora

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^5}{e^{n+1}} \cdot \frac{e^n}{n^5} = \frac{1}{e} \cdot \left(\frac{n+1}{n}\right)^5 \xrightarrow[n\to+\infty]{} \frac{1}{e} < 1$$

perché dall'algebra dei limiti $\lim_{n\to+\infty} \left(\frac{n+1}{n}\right)^5 = 1$

Pertanto $\exists \lim_{n\to+\infty} \frac{n^5}{e^n} = 0$. In particolare $n^5 = o(e^n)$, $n\to+\infty$ $\square$

Utilizziamo il criterio del rapporto. Poniamo $a_n = \frac{n!}{e^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{(n+1)\cancel{n!}}{e \cdot \cancel{n!}} = \frac{n+1}{e} \xrightarrow[n\to+\infty]{} +\infty$$

Allora $\exists \lim_{n\to+\infty} \frac{n!}{e^n} = +\infty$. Pertanto $e^n = o(n!)$ $n\to+\infty$

perché $\lim_{n\to+\infty} \frac{e^n}{n!} = 0$. $\square$

Utilizzando il criterio del rapporto $a_n = \frac{5^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \xrightarrow[n\to+\infty]{} 0$$

Quindi $\lim_{n\to+\infty} \frac{5^n}{n!} = 0$, cioè $5^n = o(n!)$. $\square$

Utilizzando il criterio del rapporto $a_n = \frac{n^7}{n!}$

$$\frac{(n+1)^7}{(n+1)!} \cdot \frac{n!}{n^7} = \left(\frac{n+1}{n}\right)^7 \cdot \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0$$ (algebra dei limiti)

Quindi $\lim_{n\to+\infty} \frac{n^7}{n!} = 0$, cioè $n^7 = o(n!)$. $\square$

(ii)

$$\frac{|x-5|+2}{|x|-3} < \operatorname{sgn} x \iff \underset{\text{I}}{\begin{cases} \dfrac{|x-5|+2}{|x|-3} < 1 \\ x>0 \end{cases}} \vee \underset{\text{II}}{\begin{cases} \dfrac{|x-5|+2}{|x|-3} < -1 \\ x<0 \end{cases}}$$

I

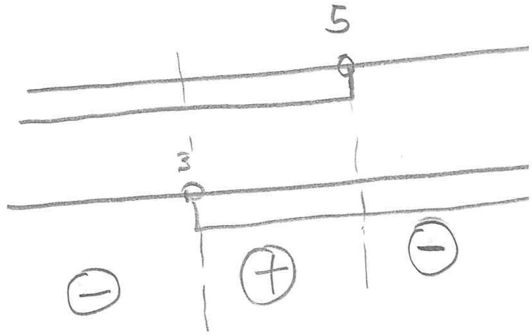
$$\begin{cases} \dfrac{|x-5|+2}{x-3} < 1 \\ x>0 \end{cases} \iff \underset{\text{I}_1}{\begin{cases} \dfrac{5-x+2}{x-3} < 1 \\ 0<x\leq 5 \end{cases}} \vee \underset{\text{I}_2}{\begin{cases} \dfrac{x-5+2}{x-3} < 1 \\ x>5 \end{cases}}$$

I_1

$$\begin{cases} \dfrac{7-x-x+3}{x-3} < 0 \\ 0<x\leq 5 \end{cases} \iff \begin{cases} \dfrac{10-2x}{x-3} < 0 \\ 0<x\leq 5 \end{cases}$$

$N_1>0 \iff 10-2x>0 \iff 5>x$

$D_1>0 \iff x-3>0 \iff x>3$



$$\iff \begin{cases} x<3 \vee 5<x \\ 0<x\leq 5 \end{cases} \iff 0<x<3, \text{ cioè}$$

$S_{\text{I}_1} =]0,3[.$

I_2

$$\begin{cases} \dfrac{x-3}{x-3} < 1 \\ x>5 \end{cases} \quad \begin{cases} 1<1 \\ x>5 \end{cases} \quad S_{\text{I}_2} = \emptyset$$

Pertanto $S_{\text{I}} = S_{\text{I}_1} \cup S_{\text{I}_2} =]0,3[.$

Inoltre

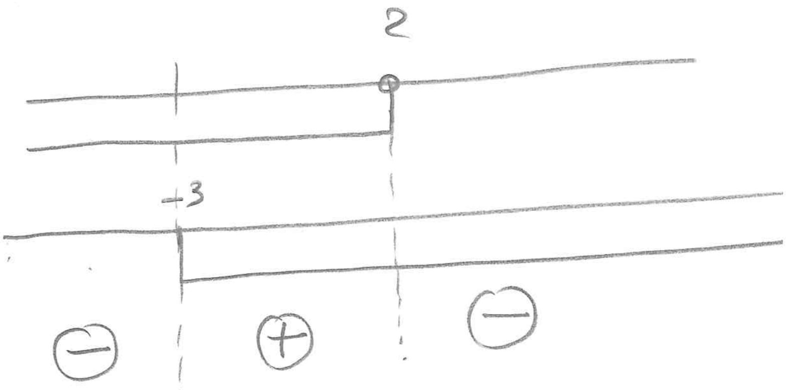
II $\begin{cases} \dfrac{|x-5|+2}{|x|-3} < -1 \\ x<0 \end{cases} \iff \begin{cases} \dfrac{|x-5|+2}{-x-3} < -1 \\ x<0 \end{cases} \iff$

$\begin{cases} \dfrac{|x-5|+2}{x+3} > 1 \\ x<0 \end{cases} \iff \begin{cases} \dfrac{5-x+2}{x+3} > 1 \\ x<0 \end{cases} \iff \begin{cases} \dfrac{7-x-x-3}{x+3} > 0 \\ x<0 \end{cases}$

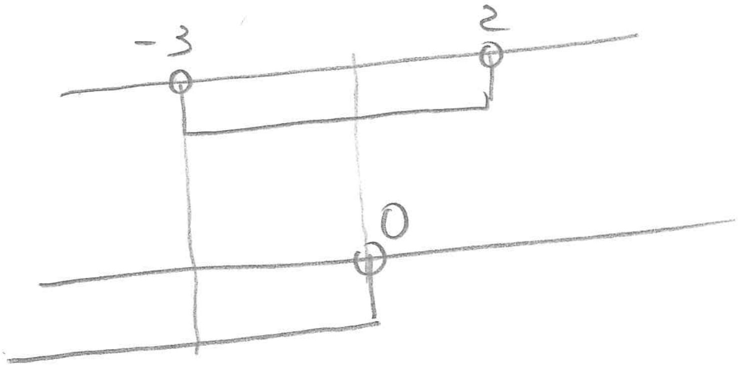
$\iff \begin{cases} \dfrac{4-2x}{x+3} > 0 \\ x<0 \end{cases}$

$N_1 = 4-2x > 0 \iff 2 > x$

$D_1 = x+3 > 0 \iff x > -3$



$\iff \begin{cases} -3 < x < 2 \\ x<0 \end{cases} \iff$



$S_{II} = \,]-3,0[$

Concludiamo allora che la soluzione

$S = S_I \cup S_{II} = \,]0,3[\, \cup \,]-3,0[$

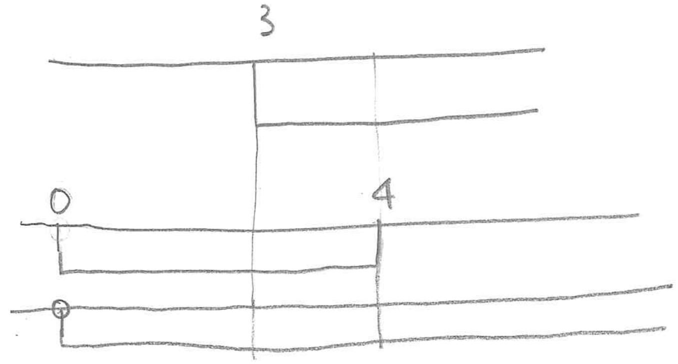
Risolvere $\sqrt{|x-2|-1} \le \operatorname{sgn} x \iff$

$$\begin{cases} |x-2|-1 \ge 0 \\ |x-2|-1 \le 1 \\ x > 0 \end{cases} \iff \begin{cases} 0 \le |x-2|-1 \le 1 \\ x > 0 \end{cases}$$

$$\iff \begin{cases} 1 \le |x-2| \le 2 \\ x > 0 \end{cases} \iff \begin{cases} 1 \le x-2 \lor x-2 \le -1 \\ -2 \le x-2 \le 2 \\ x > 0 \end{cases}$$

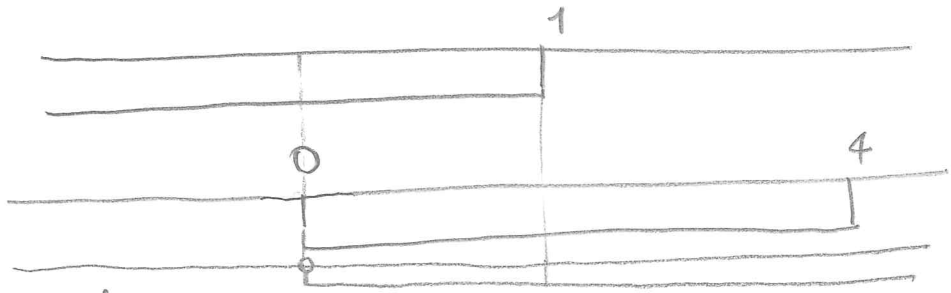
$$\iff \underbrace{\begin{cases} x \ge 3 \\ 0 \le x \le 4 \\ x > 0 \end{cases}}_{\text{I}} \lor \underbrace{\begin{cases} x \le 1 \\ 0 \le x \le 4 \\ x > 0 \end{cases}}_{\text{II}}$$

I



$S_{I} = [3, 4]$

II



$S_{II} =]0, 1]$

Pertanto $S = S_{I} \cup S_{II} =]0, 1] \cup [3, 4]$