Correzione

$$\frac{\left(\sqrt[3]{n}-\sqrt[3]{n+2}\right)n}{\sqrt[3]{4n-5}+\sqrt[3]{n+2}} = \frac{(\cancel{n}-\cancel{n}-2)n}{\left(\sqrt[3]{n^2}+\sqrt[3]{n(n+2)}+\sqrt[3]{(n+2)^2}\right)\left(\sqrt[3]{4n-5}+\sqrt[3]{n+2}\right)}$$

$$= \frac{-2n}{\left(\sqrt[3]{n^2}+\sqrt[3]{n(n+2)}+\sqrt[3]{(n+2)^2}\right)\left(\sqrt[3]{(4n-5)}+\sqrt[3]{n+2}\right)},$$ ma

$$\sqrt[3]{n^2}+\sqrt[3]{n(n+2)}+\sqrt[3]{(n+2)^2} \underset{n\to+\infty}{\sim} 3n^{2/3}; \quad \sqrt[3]{4n-5}+\sqrt[3]{n+2} \underset{n\to+\infty}{\sim} (1+4^{1/3})\sqrt[3]{n},$$

quindi

$$\frac{\left(\sqrt[3]{n}-\sqrt[3]{n+2}\right)n}{\sqrt[3]{4n-5}+\sqrt[3]{n+2}} \underset{n\to+\infty}{\sim} \frac{-2n}{3n^{2/3}\cdot(1+4^{1/3})\sqrt[3]{n}} \underset{n\to+\infty}{\sim} \frac{-2\cancel{n}}{3(1+4^{1/3})\cancel{n}},$$

pertanto $$\lim_{n\to+\infty}\frac{\left(\sqrt[3]{n}-\sqrt[3]{n+2}\right)n}{\sqrt[3]{4n-5}+\sqrt[3]{n+2}} = \lim_{n\to+\infty}\frac{-2n}{3(1+4^{1/3})n} = -\frac{2}{3(1+4^{1/3})}$$ $\square$

$$\left(\sqrt{n+1}-\sqrt{n+7}\right)\sqrt{n+4} = \frac{\cancel{n}+1-\cancel{n}-7}{\sqrt{n+1}+\sqrt{n+7}}\cdot\sqrt{n+4},$$

ma $\sqrt{n+1}+\sqrt{n+7} \underset{n\to+\infty}{\sim} 2\sqrt{n}$ e $\sqrt{n+4} \underset{n\to+\infty}{\sim} \sqrt{n}$.

Quindi

$$\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n+7}\right)\sqrt{n+4} = \lim_{n\to+\infty}\frac{-6\sqrt{n+4}}{\sqrt{n+1}+\sqrt{n+7}}$$

$$= \lim_{n\to+\infty}\frac{-6\sqrt{n}}{2\sqrt{n}} = -3.$$ $\square$

$$\frac{\sqrt{n+1}-\sqrt{n-7}}{\sqrt{3n+3}-\sqrt{3n+4}} = \frac{\cancel{n}+1-\cancel{n}+7}{\sqrt{n+1}+\sqrt{n-7}} \cdot \frac{\sqrt{3n+3}+\sqrt{3n+4}}{\cancel{3n}+3-\cancel{3n}-4} = \frac{8\left(\sqrt{3n+3}+\sqrt{3n+4}\right)}{-\left(\sqrt{n+1}+\sqrt{n-7}\right)},$$

ma $\sqrt{3n+3}+\sqrt{3n+4} \underset{n\to+\infty}{\sim} 2\sqrt{3}\sqrt{n}$ e $\sqrt{n+1}+\sqrt{n-7} \underset{n\to+\infty}{\sim} 2\sqrt{n}$.

Allora

$$\lim_{n\to+\infty} \frac{\sqrt{n+1}-\sqrt{n-7}}{\sqrt{3n+3}-\sqrt{3n+4}} = \lim_{n\to+\infty} -8\,\frac{\sqrt{3n+3}+\sqrt{3n+4}}{\sqrt{n+1}+\sqrt{n-7}} = \lim_{n\to+\infty} -8\,\frac{\cancel{2}\sqrt{3}\cancel{\sqrt{n}}}{\cancel{2\sqrt{n}}}$$

$$= -8\sqrt{3}. \ \square$$

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^{n} = \left(\frac{n-1+1}{n-1}\right)^{n} = \left(1+\frac{1}{n-1}\right)^{n}$$

$$= \left(1+\frac{1}{n-1}\right)^{n-1} \cdot \left(1+\frac{1}{n-1}\right),$$

ma $\displaystyle\lim_{n\to+\infty} \left(1+\frac{1}{n-1}\right)^{n-1} = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^{n} = e$,

e $\displaystyle\lim_{n\to+\infty} 1+\frac{1}{n-1} = 1$

Quindi $\displaystyle\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to+\infty} \left(1+\frac{1}{n-1}\right)^{n-1} \cdot \lim_{n\to+\infty} \left(1+\frac{1}{n-1}\right) = e$ $\square$

$\displaystyle\left(1+\frac{9}{n}\right)^{n} = \left(\left(1+\frac{9}{n}\right)^{\frac{n}{9}}\right)^{9}$, ma $\displaystyle\lim_{n\to+\infty} \left(1+\frac{9}{n}\right)^{\frac{n}{9}} = e$

quindi $\displaystyle\lim_{n\to+\infty} \left(1+\frac{9}{n}\right)^{n} = \lim_{n\to+\infty} \left(\left(1+\frac{9}{n}\right)^{\frac{n}{9}}\right)^{9} = e^{9}$.

$\square$

Ricordando la disuguaglianza di Bernoulli

$$2^x \ge 2^{[x]} = (1+1)^{[x]} > 1 + 1\cdot[x] > x$$

perché $[x]$ è la parte intera di x e $\quad [x] \le x < [x]+1$

Quindi $$\log x < \log 2^x = x \log 2$$

In particolare se $x = n^{\alpha/2}$ allora

$$\frac{\alpha}{2}\log n = \log n^{\alpha/2} < n^{\frac{\alpha}{2}} \log 2 .$$ Inoltre

$$\frac{\log n}{n^{\frac{\alpha}{2}}} < \frac{2}{\alpha}\log 2 .$$

Quindi, utilizzando l'ultima disuguaglianza otteniamo

$$0 < \frac{\log n}{n^{\alpha}} = \frac{\log n}{n^{\frac{\alpha}{2}} n^{\frac{\alpha}{2}}} < \frac{2}{\alpha}\,\frac{\log 2}{n^{\alpha/2}}, \qquad \forall n \in \mathbb{N}, n>0$$

Allora dal Teorema del confronto segue che

$$\lim_{n\to+\infty} \frac{\log n}{n^{\alpha}} = 0 ,\ \text{perché}$$

$$\frac{\log 2}{n^{\frac{\alpha}{2}}} \xrightarrow[n\to+\infty]{} 0 .$$

$\square$

$\frac{n!}{n^n} = a_n .$ Applico il criterio del rapporto

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{\cancel{(n+1)}}{\cancel{n+1}}\cdot\left(\frac{n}{n+1}\right)^n = \left(\frac{n+1-1}{n+1}\right)^n = \left(1-\frac{1}{n+1}\right)^n$$

$$= \left(1-\frac{1}{n+1}\right)^{n+1}\cdot\left(1-\frac{1}{n+1}\right)^{-1} \xrightarrow[n\to+\infty]{} e^{-1} < 1 .$$

Quindi $\lim_{n\to+\infty} \frac{n!}{n^n} = 0$

$$\frac{\sqrt{n^3}+5n^{3/2}+\sqrt[5]{n-1}}{\sqrt[5]{n+3}+\sqrt[6]{n^9+1}} \underset{n\to+\infty}{\sim} \frac{5n^{3/2}}{n^{3/2}} \underset{n\to+\infty}{\sim} 5$$, cioè

$$\lim_{n\to+\infty} \frac{\sqrt{n^3}+5n^{3/2}+\sqrt[5]{n-1}}{\sqrt[5]{n+3}+\sqrt[6]{n^9+1}} = 5.$$ Infatti

$\sqrt{n^3} = o(5n^{3/2})$, $\sqrt[5]{n-1} = o(5n^{3/2})$, $n\to+\infty$ e

$\sqrt[5]{n+3} = o(\sqrt[6]{n^9+1})$, $n\to+\infty$ dove $\sqrt[6]{n^9+1} \underset{n\to+\infty}{\sim} n^{\frac{9}{6}}$. $\square$

$$\frac{(n+3)!+n^5}{n!\sqrt{n^6+3}+n^6} \sim \frac{(n+3)!}{n!\sqrt{n^6+3}}$$, $n\to+\infty$, perché

$n^5 = o\,(n+3)!$ e $n^6 = o(n!\sqrt{n^6+3})$ per $n\to+\infty$

Quindi

$$\lim_{n\to+\infty} \frac{(n+3)!}{n!\sqrt{n^6+3}} = 1$$ perché

$$\frac{(n+3)!}{n!\sqrt{n^6+3}} = \frac{(n+3)(n+2)(n+1)\cancel{n!}}{\cancel{n!}\sqrt{n^6+3}}$$, ma

$$\frac{(n+3)(n+2)(n+1)}{\sqrt{n^6+3}} \sim 1$$, $n\to+\infty$ perché

$$\sqrt{n^6+3} \underset{n\to+\infty}{\sim} n^{\frac{6}{2}}.$$ $\square$

$$\frac{\log(n^{10}+5)+\sqrt{3n+1}}{\sqrt[3]{n+1}-\sqrt{n}+\log(5n^{10}+6)} \underset{n\to+\infty}{\sim} \frac{\sqrt{3n+1}}{-\sqrt{n}} \underset{n\to+\infty}{\sim} -\sqrt{3} .$$

Infatti $\log(n^{10}+5)=o\left(\sqrt{3n+1}\right)$, $n\to+\infty$,

$\sqrt[3]{n+1}=o(\sqrt{n})$, $\log(5n^{10}+6)=o(\sqrt{n})$ per $n\to+\infty$.

Quindi $\displaystyle\lim_{n\to+\infty}\frac{\log(n^{10}+5)+\sqrt{3n+1}}{\sqrt[3]{n+1}-\sqrt{n}+\log(5n^{10}+6)}=-\sqrt{3}$. $\square$

$$\frac{2^n+e^{\frac{n}{2}}}{4^{\frac{n}{2}}+e^{\frac{n+1}{2}}} \sim \frac{2^n}{2^n}$$, infatti

$e^{\frac{n}{2}}=o(2^n)$, $n\to+\infty$ (si noti che $e^{\frac{n}{2}}=(\sqrt{e})^n$ ma $\sqrt{e}<2$)

$e^{\frac{n+1}{2}}=o(4^{\frac{n}{2}})$, $n\to+\infty$, perché $e^{\frac{n+1}{2}}=e^{\frac{1}{2}}e^{\frac{n}{2}}=e^{\frac{1}{2}}(\sqrt{e})^n$ mentre $4^{\frac{n}{2}}=2^n$, da cui $e^{\frac{1}{2}}(\sqrt{e})^n=o(2^n)$ perché $\sqrt{e}<2$. $\square$

$$\frac{5^{\frac{n}{2}}+3e^n}{4e^n+\sqrt{5^n}} \sim \frac{3e^n}{4e^n}$$, quindi $\displaystyle\lim_{n\to+\infty}\frac{5^{\frac{n}{2}}+3e^n}{4e^n+\sqrt{5^n}}=\frac{3}{4}$.

E' sufficiente osservare che $\sqrt{5}<e$.

$\square$

$$\frac{5^n+n!}{25^{\frac{n}{2}}+3n!} \sim \frac{n!}{3n!}, \text{ da cui } \lim_{n\to+\infty} \frac{5^n+n!}{25^{\frac{n}{2}}+3n!} = \frac{1}{3}$$

perché $5^n = o(n!)$, e $(25)^{\frac{n}{2}} = 5^n = o(3n!)$ per $n \to +\infty$. $\square$

$\longmapsto$

$\lim_{n\to+\infty} \frac{\sin n}{n} = 0$, infatti $|\sin n| \leq 1$, cioè $\{\sin n\}_{n\in\mathbb{N}}$ è limitata, quindi il risultato segue dal teorema che generalizza l'algebra dei limiti. $\square$

$\longmapsto$

L'esistenza di $\bar{n} \in \mathbb{N}$: $n^3+n^5+1 > n^2$ è banale conseguenza della divergenza della seguente successione $\left\{\frac{n^3+n^5+1}{n^2}\right\}_{n\in\mathbb{N}}$; infatti

$$\frac{n^3+n^5+1}{n^2} \sim \frac{n^5}{n^2} \sim n^3, \quad \text{cioè } \lim_{n\to+\infty} \frac{n^3+n^5+1}{n^2} = +\infty$$
$n \to +\infty$

Ciò significa, in particolare, che per $M = 1$ $\exists$ $\bar{n}(1)$: $\frac{n^5+1+n^3}{n^2} > 1$, $\forall n \in \mathbb{N} : n > \bar{n}(1)$.

Il numero $\bar{n}$ cercato è quello determinato dall'applicazione della definizione di succ. divergente positivamente. $\square$

$\longmapsto$