Esercizio 1

$$\lim_{n\to+\infty} \frac{n^2(e^n+5(n+4)!)}{5^n+n^7+4n^4(n+2)!} = \frac{5}{4}$$

$N \sim 5n^2(n+4)!$, $n\to+\infty$; $D \sim 4n^4(n+2)!$, $n\to+\infty$

$$\frac{N}{D} \sim \frac{5n^2(n+4)!}{4n^4(n+2)!} \sim \frac{5\,\cancel{(n+2)!}\,(n+3)(n+4)}{4\,n^2\,\cancel{(n+2)!}} \underset{n\to+\infty}{\sim} \frac{5}{4}.$$

Esercizio 2 (Vedi testo)

$g'(\pi^2)=7$; $g'\left(\frac{\pi}{2}\right)=\frac{1}{7}$; $K:\mathbb{R}\to\mathbb{R}$

$K(x)=g\left(4x^2+\cos(7x)\right)$; $g:\mathbb{R}\to\mathbb{R}$ derivabile

$K'(x)=g'\left(4x^2+\cos(7x)\right)\left(8x-7\sin(7x)\right)$; calcoliamo $K'\left(\frac{\pi}{2}\right)$.

$$K'\left(\frac{\pi}{2}\right)=g'\left(\pi^2+\cos\frac{7}{2}\pi\right)\left(4\pi-7\sin\frac{7}{2}\pi\right)=g'(\pi^2)\cdot\left(4\pi-7\sin\frac{3}{2}\pi\right)$$
$$=7\cdot(4\pi+7)=28\pi+49$$

Esercizio 3 (Vedi testo)

$f:\left(-\frac{5}{2},0\right)\cup\left(\frac{5}{2},+\infty\right)\to\mathbb{R}$; $f(x)=\log\frac{x}{2|x|-5}$.

$$f'(x)=\frac{2|x|-5}{x}\cdot\frac{2|x|-5-x\cdot 2\,\mathrm{sgn}\,x}{(2|x|-5)^2}=\frac{\cancel{2|x|}-5-\cancel{2|x|}}{x(2|x|-5)}=-\frac{5}{x(2|x|-5)}$$

$$\begin{cases} f'(x)>0 \\ x\in\left(-\frac{5}{2},0\right)\cup\left(\frac{5}{2},+\infty\right)\end{cases} \iff \begin{cases} x(2|x|-5)<0 \\ x\in\left(-\frac{5}{2},0\right)\cup\left(\frac{5}{2},+\infty\right)\end{cases} \iff$$

$$\begin{cases} 2x-5<0 \\ x\in\left(\frac{5}{2},+\infty\right)\end{cases} \vee \begin{cases} x(-2x-5)<0 \\ x\in\left(-\frac{5}{2},0\right)\end{cases} \iff$$

$$\begin{cases} x < \frac{5}{2} \\ x \in \left(\frac{5}{2}, +\infty\right) \end{cases} \vee \begin{cases} 2x+5 < 0 \\ x \in \left(-\frac{5}{2}, 0\right) \end{cases} \iff$$

$$\emptyset \quad \vee \quad \begin{cases} x < -\frac{5}{2} \\ x \in \left(-\frac{5}{2}, 0\right) \end{cases} \iff \emptyset$$

Quindi la derivata prima non è mai positiva nel dominio di f. Pertanto f è decrescente in $\left(-\frac{5}{2}, 0\right)$ e decrescente in $\left(\frac{5}{2}, +\infty\right)$.

Studiamo ora il segno della derivata seconda:

$$f''(x) = 5 \cdot \frac{2|x|-5 + x\, 2\operatorname{sgn} x}{x^2(2|x|-5)^2} = \frac{5(4|x|-5)}{x^2(2|x|-5)^2}$$

$$\begin{cases} f''(x) > 0 \\ x \in \left(-\frac{5}{2}, 0\right) \cup \left(\frac{5}{2}, +\infty\right) \end{cases} \iff \begin{cases} 4|x|-5 > 0 \\ x \in \left(-\frac{5}{2}, 0\right) \cup \left(\frac{5}{2}, +\infty\right) \end{cases}$$

$$\iff \begin{cases} 4x-5 > 0 \\ x \in \left(\frac{5}{2}, +\infty\right) \end{cases} \vee \begin{cases} -4x-5 > 0 \\ x \in \left(-\frac{5}{2}, 0\right) \end{cases} \iff$$

$$\begin{cases} x > \frac{5}{4} \\ x \in \left(\frac{5}{2}, +\infty\right) \end{cases} \vee \begin{cases} x < -\frac{5}{4} \\ x \in \left(-\frac{5}{2}, 0\right) \end{cases} \iff$$

$$x \in \left(\frac{5}{2}, +\infty\right) \cup \left(-\frac{5}{2}, -\frac{5}{4}\right)$$

Pertanto f è convessa in $\left(-\frac{5}{2}, -\frac{5}{4}\right)$ ed è convessa in $\left(\frac{5}{2}, +\infty\right)$. Sarà concava in $\left(-\frac{5}{4}, 0\right)$.

Esercizio 4. $K: \mathbb{R} \to \mathbb{R}$, $K(x) = \dfrac{x^4 e^{5x}}{x^4 + e^{5x}}$

$$K'(x) = \frac{(4x^3 e^{5x} + 5x^4 e^{5x})(x^4 + e^{5x}) - x^4 e^{5x}(4x^3 + 5e^{5x})}{(x^4 + e^{5x})^2}$$

Pertanto $K'(1) = \dfrac{9e^5(1+e^5) - e^5(4+5e^5)}{(1+e^5)^2} = \dfrac{e^5(9+9e^5-4-5e^5)}{(1+e^5)^2}$

$$= \frac{e^5(5+4e^5)}{(1+e^5)^2}.$$

Esercizio 5.

Calcolare il limite

$$\lim_{x \to 0} \frac{\sin(x^2+6x) - \sinh(6x) - x^2 + 72x^3}{\sin(2+x)\sinh^2(7x)(\cosh(7x) - 1)} = -\frac{36}{7^4 \sin 2}$$

$$\sin(x^2+6x) \sim x^2 + 6x - \frac{(6x)^3 + 3(6x)^2 \cdot x^2}{3!} + o(x^4), \quad x \to 0$$

$$\sinh(6x) \sim 6x + \frac{(6x)^3}{3!} + o(x^4), \quad x \to 0$$

$$\sinh 7x \sim 7x + o(x^2), \quad x \to 0$$

$$\cosh 7x \sim 1 - \frac{(7x)^2}{2} + o(x^3), \quad x \to 0.$$ Pertanto

$$N \sim \cancel{x^2} + \cancel{6x} - \cancel{\frac{(6x)^3}{3!}} - 18x^4 - \cancel{6x} - \cancel{\frac{(6x)^3}{3!}} - \cancel{x^2} + \cancel{72x^3} \sim -18x^4, \quad x \to 0$$

$$D \sim \sin(2) \cdot 49x^2 \cdot \frac{(7x)^2}{2}, \quad x \to 0$$

Esercizio 6. $\displaystyle\int_{-\frac{1}{2}}^{0} (3x+36)\cosh(6x+3)\,dx = I$

Integrando per parti:

$$I = \left[(3x+36)\frac{\sinh(6x+3)}{6} \right]_{x=-\frac{1}{2}}^{x=0} - \int_{-\frac{1}{2}}^{0} \cancel{3}\,\frac{\sinh(6x+3)}{\cancel{6}\,2}\,dx$$

$$= 6\sin 3 - \frac{1}{2}\left[\frac{\cosh(6x+3)}{6}\right]_{x=-\frac{1}{2}}^{x=0}$$

$$= 6\sin 3 - \frac{1}{12}\cosh 3 + \frac{1}{12}.$$

Esercizio 7.

$$\int_0^3 \frac{2x^3}{x^4+2x^2+17}\,dx \underset{\substack{x^2=t \\ dt=2x\,dx}}{=} \int_0^9 \frac{t}{t^2+2t+17}\,dt$$

$$= \frac{1}{2}\int_0^9 \frac{2t+2-2}{t^2+2t+17}\,dt = \frac{1}{2}\int_0^9 \frac{2t+2}{t^2+2t+17}\,dt - \int_0^9 \frac{1}{(t+1)^2+16}\,dt$$

$$= \frac{1}{2}\Big[\log|t^2+2t+17|\Big]_{t=0}^{t=9} - \frac{1}{16}\int_0^9 \frac{1}{\left(\frac{t+1}{4}\right)^2+1}\,dt$$

$$= \frac{1}{2}\log\frac{116}{17} - \frac{1}{16}\left[4\arctan\left(\frac{t+1}{4}\right)\right]_{t=0}^{t=9}$$

$$= \frac{1}{2}\log\frac{116}{17} - \frac{1}{4}\left(\arctan\frac{5}{2} - \arctan\frac{1}{4}\right).$$