Sapendo che $A=\{(x,y,z)\in\mathbb{R}^3 : x^2+\frac{y^2}{4}+\frac{z^2}{9}\le 1,\ z\ge 0,\ z\le \frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}\}$ (1)

determinare $B\subseteq\mathbb{R}^2$ e $\varphi_1,\varphi_2 : B\to\mathbb{R}$ taliche

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iint_B\left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right)dx\,dy.$$

Sia $A_{x,y}=\{z\in\mathbb{R};\ z^2\le 9\left(1-x^2-\frac{y^2}{4}\right),\ z\ge 0,\ z\le\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}\}$

$$=\left[-3\sqrt{1-x^2-\frac{y^2}{4}},\ 3\sqrt{1-x^2-\frac{y^2}{4}}\right]\cap[0,+\infty[\ \cap\ ]-\infty,\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}]$$

$$=\left[0,3\sqrt{1-x^2-\frac{y^2}{4}}\right]\cap\left[0,\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}\right],\quad \text{per}\quad x^2+\frac{y^2}{4}\le 1$$

(altrimenti $A_{x,y}=\emptyset$). Per stabilire quando

$3\sqrt{1-x^2-\frac{y^2}{4}}\le\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}$ risolviamo la disequazione

$3\sqrt{1-\rho^2}\le\frac{3}{2}+\rho$ dove $\sqrt{x^2+\frac{y^2}{4}}=\rho$.

$$9(1-\rho^2)\le\frac{9}{4}+3\rho+\rho^2 \iff 9-9\rho^2\le\frac{9}{4}+3\rho+\rho^2 \iff$$

$$0\le-\frac{27}{4}+3\rho+10\rho^2 \iff \rho\in\left]-\infty,\frac{-3-\sqrt{9+270}}{20}\right]\cup\left[\frac{-3+\sqrt{279}}{20},+\infty\right[$$

ma $0\le\rho\le 1$, quindi se $\rho\in\left[\frac{-3+\sqrt{279}}{20},1\right]$ allora

$A_{x,y}=\left[0,3\sqrt{1-x^2-\frac{y^2}{4}}\right]$, mentre se $\rho\in\left[0,\frac{-3+\sqrt{279}}{20}\right]$

allora $\left[0,\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}\right]$. Pertanto se

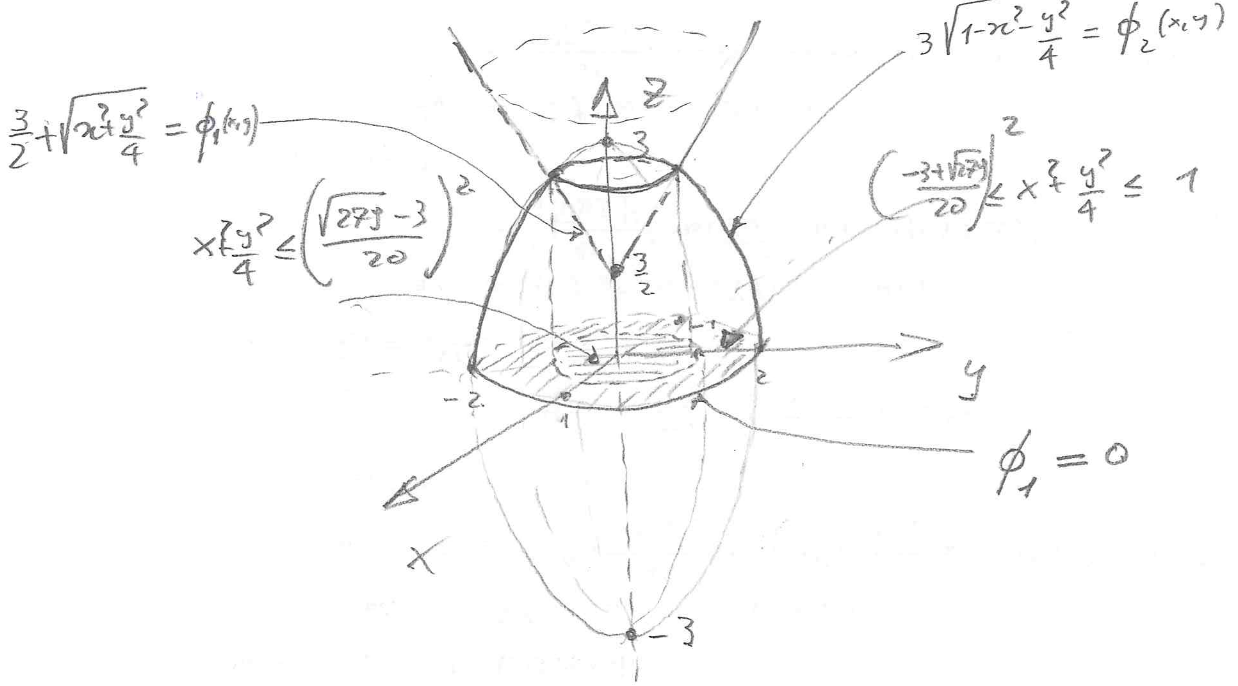
$\sqrt{x^2+\frac{y^2}{4}}\le\frac{-3+\sqrt{279}}{20}$, allora $\phi_1=0$ e $\phi_2=\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}$;

se $\frac{-3+\sqrt{279}}{20}\le\sqrt{x^2+\frac{y^2}{4}}\le 1$, allora $\phi_1=0$ e $\phi_2=3\sqrt{1-x^2-\frac{y^2}{4}}$

con $B = \{(x,y)\in\mathbb{R}^2 :\ x^2+\frac{y^2}{4}\le 1\}$ . Pertanto

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{\left\{x^2+\frac{y^2}{4}\le\left(\frac{-3+\sqrt{279}}{20}\right)^2\right\}} \left(\int_0^{\frac{3}{2}+\sqrt{x^2+\frac{y^2}{4}}} f(x,y,z)\,dz\right)dx\,dy$$

$$+ \iint_{1\le x^2+\frac{y^2}{4}\le\left(\frac{-3+\sqrt{279}}{20}\right)^2} \left(\int_0^{3\sqrt{1-x^2-\frac{y^2}{4}}} f(x,y,z)\,dz\right)dx\,dy$$




Siano $f,g\in C^1(\mathbb{R}^2,\mathbb{R})$ e posto $h:\mathbb{R}^3\to\mathbb{R}$

$h(x,y,z) = f\left(x+y^2+z^3,\ \cos(3x^2+y^3)+3xy+2z,\ g(x^3,y,x)\right)$ calcolare

$Dh(x_0,y_0,z_0)$ .

Applichiamo la regola della catena.

Per semplicità denotiamo $u(x,y,z) = x+y^2+z^3$, $v(x,y,z) = \cos(3x^2+y^3)+3xy+2z$, $w(x,y,z) = g(x^3,y,x)$. Inoltre $u_0 = u(x_0,y_0,z_0)$, $v_0 = v(x_0,y_0,z_0)$, $w_0 = g(x_0^3,y_0,x_0)$ ③

$$\frac{\partial h}{\partial x}(x_0,y_0,z_0) = \frac{\partial f}{\partial u}(u_0,v_0,w_0) + \frac{\partial f}{\partial v}(u_0,v_0,w_0)\left(3y_0 - 6\sin(3x_0^2+y_0^3)x_0\right)$$
$$+ \frac{\partial f}{\partial w}(u_0,v_0,w_0)\left[\frac{\partial g}{\partial x}(x_0^3,y_0,x_0) + \frac{\partial g}{\partial z}(x_0^3,y_0,x_0)\right]$$

$$\frac{\partial h}{\partial y}(x_0,y_0,z_0) = \frac{\partial f}{\partial u}(u_0,v_0,w_0)\,2y_0 + \frac{\partial f}{\partial v}(u_0,v_0,w_0)\left(3x_0 - 9\sin(3x_0^2+y_0^3)y_0^2\right)$$
$$+ \frac{\partial f}{\partial w}(u_0,v_0,w_0)\frac{\partial g}{\partial y}(x_0^3,y_0,x_0)$$

$$\frac{\partial h}{\partial y}(x_0,y_0,z_0) = \frac{\partial f}{\partial u}(u_0,v_0,w_0)\,3z_0 + \frac{\partial f}{\partial v}(u_0,v_0,w_0)\cdot 2z_0.$$

---

$y'' + 4y' + 8y = 3x$

E.O. : $y''+4y'+8y=0$ Eq. Car. $\lambda^2+4\lambda+8=0 \longrightarrow \lambda_1 = -2+\sqrt{4-8} = -2+2i$
$\lambda_2 = -2-i\sqrt{4} = -2-2i$

$V_2 = \operatorname{span}\{e^{-2x}\cos(2x),\ e^{-2x}\sin(2x)\}$

Cerchiamo una sol nella forma $y = ax+b$

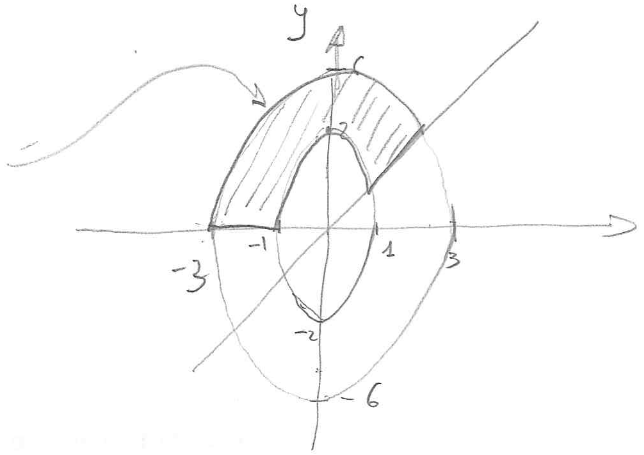
$y' = a$, $y''=0$. Pertanto $4a+8ax+8b = 3x$, da cui

segue $\begin{cases} 8a = 3 \\ 4a+8b=0 \end{cases}$ $\begin{cases} a = \frac{3}{8} \\ \frac{3}{2}+8b=0 \end{cases}$ $\begin{cases} a = \frac{3}{8} \\ b = -\frac{3}{16} \end{cases}$

$$LV_2 = V_2 + \frac{3}{8}x - \frac{3}{16}.$$

Calcolare $\displaystyle\iint_E (2x+3y^2)\,dx\,dy$

$E=\left\{(x,y)\in\mathbb{R}^2;\ 1\le x^2+\frac{y^2}{4}\le 9,\ y\ge 0,\ y\ge x\right\}$




Passiamo in coordinate polari ponendo

$$\frac{x}{3}=\rho\cos\theta,\quad \frac{y}{6}=\rho\sin\theta \quad , \text{ allora}$$

$$1\le 9\rho^2\cos^2\theta+\frac{36\rho^2\sin^2\theta}{4}\le 9 \iff 1\le 9\rho^2\cos^2\theta+9\rho^2\sin^2\theta\le 9 \iff$$

$$\frac{1}{9}\le\rho^2\le 1 \iff \boxed{\frac{1}{3}\le\rho\le 1}.\quad (\rho\ge 0).$$

Inoltre $6\rho\sin\theta\ge 0 \to \rho>0$ e $0\le\theta\le\pi$ e

$$6\rho\sin\theta\ge 3\rho\cos\theta \iff 2\sin\theta\ge\cos\theta \qquad \begin{cases} \operatorname{tg}\theta\ge\frac{1}{2}\\ \theta\in[0,\frac{\pi}{2}[\\ \theta\in]\frac{3}{2}\pi,2\pi[\end{cases} \vee \begin{cases}\operatorname{tg}\theta\le\frac{1}{2}\\ \theta\in]\frac{\pi}{2},\frac{3}{2}\pi[\end{cases}$$

Cioè $\boxed{\arctan\frac{1}{2}\le\theta\le\pi}$ ; $\det JT=\begin{bmatrix}3\cos\theta, & -3\rho\sin\theta\\ 6\sin\theta, & 6\rho\cos\theta\end{bmatrix}=18\rho$

$T(\rho,\theta)=(3\rho\cos\theta,6\rho\sin\theta)$ , $Q=\left[\frac{1}{3},1\right]\times\left[\arctan\frac{1}{2},\pi\right]$

$T(Q)=E$.

$$\iint_E(2x+3y^2)\,dx\,dy=\iint_Q(6\rho\cos\theta+108\rho^2\sin^2\theta)\,18\rho\,d\theta\,d\rho$$

$$=108\iint_Q\rho^2\cos\theta\,d\theta\,d\rho+1940\iint_Q\rho^3\sin^2\theta\,d\theta\,d\rho=108\int_{\arctan\frac{1}{2}}^{\pi}\cos\theta\,d\theta\left[\frac{1}{3}\rho^3\right]_{\rho=\frac{1}{3}}^{\rho=1}$$

$$+1940\int_{\arctan\frac{1}{2}}^{\pi}\sin^2\theta\,d\theta\cdot\left[\frac{1}{4}\rho^4\right]_{\rho=\frac{1}{3}}^{\rho=1}$$

$$= 108\left(\operatorname{sen}\pi - \operatorname{sen}\left(\arctan\tfrac{1}{2}\right)\right)\left(\tfrac{1}{3}-\tfrac{1}{27}\right) + 1940\int_{\arctan\frac{1}{2}}^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= -108\operatorname{sen}\left(\operatorname{arctg}\tfrac{1}{2}\right)\frac{9-1}{27} + 1940\left[\tfrac{1}{2}\theta - \frac{\operatorname{sen}(2\theta)}{4}\right]_{\theta=\operatorname{arctg}\frac{1}{2}}^{\theta=\pi}$$

$$= -32\operatorname{sen}\left(\operatorname{arctg}\tfrac{1}{2}\right) + 1940\left(\frac{\pi}{2} - \frac{\operatorname{arctg}\frac{1}{2}}{2} + \frac{\operatorname{sen} 2\left(\operatorname{arctg}\frac{1}{2}\right)}{4}\right).$$

Sia $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (x^2+y^2-4)(y-3x)$. Classificare i punti critici

$$\frac{\partial f}{\partial x} = 2x(y-3x) - 3(x^2+y^2-4) \;; \qquad \frac{\partial f}{\partial y} = 2y(y-3x) + x^2+y^2-4$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x(y-3x) - 3(x^2+y^2-4) = 0 \\ 2y(y-3x) + x^2+y^2-4 = 0 \end{cases} \qquad \begin{cases} 2x(y-3x) + 6y(y-3x) = 0 \\ x^2+y^2-4 = -2y(y-3x) \end{cases}$$

$$\begin{cases} (y-3x)(x+3y) = 0 \\ x^2+y^2-2 = -2y(y-3x) \end{cases} \Longleftrightarrow \mathrm{I}\begin{cases} y = 3x \\ x^2+y^2-4 = 0 \end{cases} \vee \;\mathrm{II}\begin{cases} x = -3y \\ 9y^2+y^2-4 = -20y^2 \end{cases}$$

$$\mathrm{I}\begin{cases} y = 3x \\ 10x^2 = 4 \end{cases} \quad \begin{cases} y = \pm 3\sqrt{\frac{2}{5}} \\ x = \pm\sqrt{\frac{2}{5}} \end{cases} \qquad \mathrm{II}\begin{cases} x = -3y \\ 30y^2 = 4 \end{cases} \quad \begin{cases} x = \mp 3\sqrt{\frac{2}{15}} \\ y = \pm\sqrt{\frac{2}{15}} \end{cases}$$

$$Hf = \begin{bmatrix} -2(y-3x)-6x-6x\,, & 2x-6y \\ -6y+2x & ,\; 2(y-3x)+2y+2y \end{bmatrix}$$

$$Hf\left(\pm\sqrt{\tfrac{2}{5}}, \pm 3\sqrt{\tfrac{2}{5}}\right) = \begin{bmatrix} \mp 12\sqrt{\frac{2}{5}}, & \mp 16\sqrt{\frac{2}{5}} \\ \mp 16\sqrt{\frac{2}{5}}, & \pm 6\sqrt{\frac{2}{5}} \end{bmatrix} \Rightarrow \det Hf\left(\sqrt{\tfrac{2}{5}}, 3\sqrt{\tfrac{2}{5}}\right) < 0$$

$\boxed{\text{Selle}}$

$$Hf\left(-3\sqrt{\tfrac{2}{15}}, \sqrt{\tfrac{2}{15}}\right) = \begin{bmatrix} 2\left(\sqrt{\tfrac{2}{15}}+9\sqrt{\tfrac{2}{15}}\right)+48\sqrt{\tfrac{2}{15}}, & -6\sqrt{\tfrac{2}{15}}-6\sqrt{\tfrac{2}{15}} \\ -12\sqrt{\tfrac{2}{15}}, & 2\left(\sqrt{\tfrac{2}{15}}+9\sqrt{\tfrac{2}{15}}\right)+4\sqrt{\tfrac{2}{15}} \end{bmatrix}$$

$$= \begin{bmatrix} 68\sqrt{\tfrac{2}{15}}, & -12\sqrt{\tfrac{2}{15}} \\ -12\sqrt{\tfrac{2}{15}}, & 24\sqrt{\tfrac{2}{15}} \end{bmatrix}$$

$$\det Hf\left(-3\sqrt{\tfrac{2}{15}}, \sqrt{\tfrac{2}{15}}\right) = \frac{2}{15}\det\begin{bmatrix} 68 & -12 \\ -12 & 24 \end{bmatrix} = \frac{8}{15}\det\begin{bmatrix} 17, & -12 \\ -3, & 24 \end{bmatrix}$$

$$= \frac{32}{15}\det\begin{bmatrix} 17 & -3 \\ -3 & 6 \end{bmatrix} = \frac{96}{15}\det\begin{bmatrix} 17 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{96}{15}\cdot 31 > 0$$

$\left(-3\sqrt{\tfrac{2}{15}}, \sqrt{\tfrac{2}{15}}\right)$ è p.to di minimo (perché $a_{11} = 68\sqrt{\tfrac{2}{15}}$)

$$Hf\left(3\sqrt{\tfrac{2}{15}}, -\sqrt{\tfrac{2}{15}}\right) = \begin{bmatrix} 2\left(-\sqrt{\tfrac{2}{15}}-9\sqrt{\tfrac{2}{15}}\right)-48\sqrt{\tfrac{2}{15}}, & 6\sqrt{\tfrac{2}{15}}+6\sqrt{\tfrac{2}{15}} \\ 12\sqrt{\tfrac{2}{15}}, & 2\left(-\sqrt{\tfrac{2}{15}}-9\sqrt{\tfrac{2}{15}}\right)-4\sqrt{\tfrac{2}{15}} \end{bmatrix}$$

$$= \begin{bmatrix} -68\sqrt{\tfrac{2}{15}}, & 12\sqrt{\tfrac{2}{15}} \\ 12\sqrt{\tfrac{2}{15}}, & -24\sqrt{\tfrac{2}{15}} \end{bmatrix} \longrightarrow \det Hf\left(3\sqrt{\tfrac{2}{15}}, -\sqrt{\tfrac{2}{15}}\right) = \frac{2}{15}\cdot 4\cdot 12 \det\begin{bmatrix} -17 & 1 \\ 3 & -2 \end{bmatrix}$$

$$= \frac{96}{15}\cdot 31 > 0$$, ma $a_{11} = -68\sqrt{\tfrac{2}{15}} < 0$

quindi $\left(3\sqrt{\tfrac{2}{15}}, -\sqrt{\tfrac{2}{15}}\right)$ è un p.to di massimo

Sia $\alpha > 0$. Per quali valori di $\alpha$ converge il seguente integrale?
$$\int_0^{+\infty} \frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin(x^3)}{x^{\frac{4}{3}} + 3}\, dx$$

$$\int_0^1 \frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3}\, dx$$ converge se e solo se

$\frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3}$ è int. in $(0,1)$. Poiché

$\frac{e^{\frac{1}{1+x}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3} > 0$ in $(0,1)$, applichiamo il criterio del confronto

$$\frac{e^{\frac{1}{1+x}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3} \sim \frac{e\, x^{3-\alpha}}{3}, \quad x \to 0.$$

Quindi $\int_0^1 \frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3}\, dx < +\infty$ se $\alpha - 3 < 1$

cioè se $\alpha < 4$.

Studiamo ora $\int_1^{+\infty} \frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3}\, dx$. La funzione integranda non è positiva. Ricerchiamo allora la conv. assoluta dell'integrale

$$\left| \frac{e^{\frac{1}{x+1}} x^{-\alpha} \sin x^3}{x^{\frac{4}{3}}+3} \right| \leq \frac{e^{\frac{1}{2}} x^{-\alpha}}{x^{\frac{4}{3}}} = \frac{e^{\frac{1}{2}}}{x^{\frac{4}{3}+\alpha}}$$

Pertanto, essendo $\int_1^{+\infty} \frac{1}{x^{\frac{4}{3}+\alpha}}\, dx < +\infty$ per ogni $\alpha > -\frac{1}{3}$ concludiam che l'integrale generalizzato convergerà per $\alpha > 0$ se e solo se $\alpha \in (0,4)$.

$(z^5 + 2 + 3i)(z^2 + 6iz - 13) = 0$

$z^5 = 4e^{i\varphi}$ con $\varphi = \operatorname{arctg}\frac{3}{2} + \pi$. Quindi

$z_k = 4^{\frac{1}{5}} e^{i\theta_k}$, $\theta_k = \dfrac{\varphi + 2\pi k}{5}$, $k = 0, 1, 2, 3, 4$.

$z^2 + 6iz - 13 = 0 \iff z = -3i \pm \sqrt{-9 + 13} = -3i \pm 2$, da cui segue

$z_6 = 2 - 3i$, $z_7 = -2 - 3i$.