Calcolare l'integrale doppio $\iint_K \frac{7}{y+4}\,dxdy$ dove $K=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 4-x^2\}$

$K_x=\{y\in\mathbb{R} : 0\le y\le 4-x^2\}$; $\Pi_1(K)=\{x\in\mathbb{R} : K_x\ne\phi\}$

$\Pi_1(K)=\{x\in\mathbb{R} : 4-x^2\ge 0\}=[-2,2]$. Quindi

$$\iint_K \frac{7}{y+4}\,dxdy = 7\int_{-2}^{2}\left(\int_0^{4-x^2}\frac{1}{y+4}\,dy\right)dx = 7\int_{-2}^{2}\Big[\log|y+4|\Big]_{y=0}^{y=4-x^2}dx$$

$$=7\int_{-2}^{2}\left(\log|8-x^2|-\log 4\right)dx = 7\int_{-2}^{2}\log\frac{|8-x^2|}{4}\,dx$$

$$=7\left[x\log\frac{8-x^2}{4}\right]_{x=-2}^{x=2}-7\int_{-2}^{2}\frac{4x}{8-x^2}\left(\frac{-2x}{4}\right)dx = 7\left(2\log 1+2\log 1\right)+14\int_{-2}^{2}\frac{x^2}{8-x^2}\,dx$$

$$=14\int_{-2}^{2}\frac{x^2-8+8}{8-x^2}\,dx = 14\int_{-2}^{2}\frac{8}{8-x^2}\,dx-14\int_{-2}^{2}dx = 14\int_{-2}^{2}\frac{8}{(2\sqrt2-x)(2\sqrt2+x)}\,dx$$

$$-14\cdot 4 = 112\int_{-2}^{2}\frac{A}{(2\sqrt2-x)}+\frac{B}{(2\sqrt2+x)}\,dx-56.$$

$A(2\sqrt2+x)+B(2\sqrt2-x)=1$; $2\sqrt2(A+B)+x(A-B)=1$

$$\begin{cases}2\sqrt2(A+B)=1\\ A-B=0\end{cases}\quad\begin{cases}2A=\frac{1}{2\sqrt2}\\ A=B\end{cases}\quad\begin{cases}A=\frac{1}{4\sqrt2}\\ B=\frac{1}{4\sqrt2}\end{cases}$$

$$=\frac{112}{4\sqrt2}\left(\int_{-2}^{2}\frac{1}{2\sqrt2-x}\,dx+\int_{-2}^{2}\frac{1}{2\sqrt2+x}\,dx\right)-56=\frac{28}{\sqrt2}\Big[-\log|2\sqrt2-x|+\log|2\sqrt2+x|\Big]_{x=-2}^{x=2}$$

$$-56=\frac{28}{\sqrt2}\left[\log\frac{|2\sqrt2+x|}{|2\sqrt2-x|}\right]_{x=-2}^{x=2}-56=\frac{28}{\sqrt2}\left(\log\frac{2(\sqrt2+1)}{2(\sqrt2-1)}-\log\frac{2(\sqrt2-1)}{2(\sqrt2+1)}\right)-56$$

$$=\frac{28}{\sqrt2}\log\left(\frac{\sqrt2+1}{\sqrt2-1}\cdot\frac{\sqrt2+1}{\sqrt2-1}\right)-56=\frac{28}{\sqrt2}\log\left(\frac{\sqrt2+1}{\sqrt2-1}\right)^2-56=56\left(\frac{1}{\sqrt2}\log\frac{\sqrt2+1}{\sqrt2-1}-1\right).$$

Sviluppiamo il precedente calcolo applicando il Teorema di riduzione con K considerato come dominio x-semplice

$K_y = \{x \in \mathbb{R} : x^2 \leq 4 ; x^2 \leq 4-y\}$ e $\pi_2(K) = \{y \in \mathbb{R} : K_y \neq \phi\}$

$= \{y \in \mathbb{R} : y \geq 0, 0 \leq 4-y\} = [0,4]$. Osserviamo che $K_y = [-\sqrt{4-y}, \sqrt{4-y}]$.

Quindi

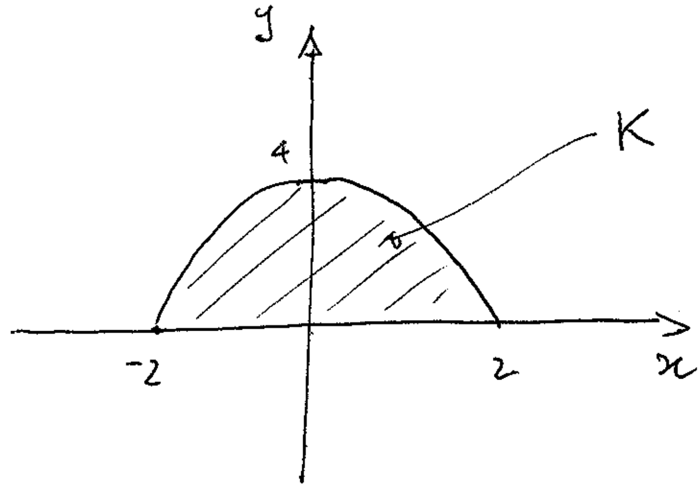
$$I = \iint_K \frac{7}{y+4}\,dxdy = \int_0^4 \left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{7}{y+4}\,dx \right) dy = 7\int_0^4 \frac{2\sqrt{4-y}}{y+4}\,dx = 14 \int_0^4 \frac{\sqrt{4-y}}{y+4}\,dy,$$

posto $\sqrt{4-y} = t$, allora $4-y = t^2$, $y = 4-t^2$, $dy = -2t\,dt$

$$I = -28 \int_2^0 \frac{t^2}{8-t^2}\,dt = 28 \int_0^2 \frac{t^2-8+8}{8-t^2}\,dt = 224 \int_0^2 \frac{1}{8-t^2}\,dt - 56$$

$$= 224 \left(\int_0^2 \frac{\frac{1}{4\sqrt{2}}}{2\sqrt{2}-t}\,dt + \int_0^2 \frac{\frac{1}{4\sqrt{2}}}{2\sqrt{2}+t}\,dt \right) - 56 = \frac{224}{4\sqrt{2}} \left[\log \frac{2\sqrt{2}+t}{2\sqrt{2}-t} \right]_{t=0}^{t=2} - 56$$

$$= \frac{56}{\sqrt{2}} \log \frac{2(\sqrt{2}+1)}{2(\sqrt{2}-1)} - 56 = 56 \left(\frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} - 1 \right).$$



Calcolare $\displaystyle\iint_A y^2 x\,dx$ dove $A=\{(x,y)\in\mathbb{R}^2:\ x^2\le y\le x\}$

$A_x=\{y\in\mathbb{R}:\ x^2\le y\le x\}$ e $\pi_1(A)=\{x\in\mathbb{R}:\ A_x\neq\emptyset\}=\{x\in\mathbb{R}:\ x^2\le x\ ,\ x\ge 0\}$

$\begin{cases} x^2\le x \\ x\ge 0\end{cases} \iff \begin{cases} x(x-1)\le 0 \\ x\ge 0\end{cases} \iff 0\le x\le 1$. Quindi $\pi_1(A)=[0,1]$ e

$A_x=[x^2,x]$.

$$\iint_A y^2x\,dxdy=\int_0^1\left(\int_{x^2}^{x} y^2x\,dy\right)dx = \int_0^1 x\left[\tfrac{1}{3}y^3\right]_{y=x^2}^{y=x}dx$$

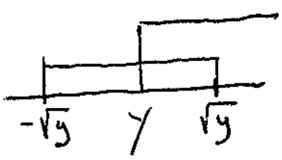
$$=\int_0^1 x\left(\tfrac{1}{3}x^3-\tfrac{1}{3}x^6\right)dx=\frac{1}{3}\int_0^1(x^4-x^7)dx=\frac{1}{3}\left[\tfrac{1}{5}x^5-\tfrac{1}{8}x^8\right]_{x=0}^{x=1}$$

$$=\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)=\frac{1}{40}.$$

Procediamo in modo alternativo considerando A come dominio x-semplice

$A_y=\{x\in\mathbb{R}:\ x^2\le y$ e $y\le x\}$, $\pi_2(A)=\{y\in\mathbb{R}:\ A_y\neq\emptyset\}$; osserviamo che $x^2\le y$ e $y\le x$ implicano che $y\ge 0$, inoltre risulta che

$\begin{cases} -\sqrt{y}\le x\le\sqrt{y} \\ (y\ge 0) \\ y\le x\end{cases}$



quindi affinché vi siano soluzioni in x del precedente sistema occorre che $y\le\sqrt{y}$, $y\ge 0$ da cui segue

$\begin{cases} y^2\le y \\ y\ge 0\end{cases}$ $\begin{cases} 0\le y\le 1 \\ y\ge 0\end{cases}$ cioè $\pi_2(A)=[0,1]$ e $A_y=[y,\sqrt{y}]$.

Pertanto $\displaystyle\iint y^2x\,dx=\int_0^1\left(\int_y^{\sqrt{y}} y^2x\,dx\right)dy=\int_0^1 y^2\left[\tfrac{1}{2}x^2\right]_{x=y}^{x=\sqrt{y}}dy=\frac{1}{2}\int_0^1 y^2(y-y^2)dy$

$$=\frac{1}{2}\left[\tfrac{1}{4}y^4-\tfrac{1}{5}y^5\right]_{y=0}^{y=1}=\frac{1}{2}\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{1}{2}\cdot\frac{1}{20}=\frac{1}{40}.$$