#1

Calcolare

$$\lim_{n\to+\infty}\left(\frac{2e^{n}+10(n+1)n^{19}}{2e^{2+n}+n^{20}}+\frac{(2+n)^2\left(\sqrt{3+n^2}-\sqrt{3+2n^2}\right)(n+1)!}{(n+4)!}\right)=e^{-2}-\frac{1}{1+\sqrt{2}}.$$

$$\frac{2e^{n}+10(n+1)n^{19}}{2e^{2+n}+n^{20}}\underset{n\to+\infty}{\sim}\frac{2e^{n}}{2e^{2+n}}\sim\frac{1}{e^2}$$

$$\frac{(2+n)^2\left(\sqrt{3+n^2}-\sqrt{3+2n^2}\right)(n+1)!}{(n+4)!}\underset{n\to+\infty}{\sim}\frac{(2+n)^2\left(\sqrt{3+n^2}-\sqrt{3+2n^2}\right)}{(n+4)(n+3)(n+2)}$$

$$\sim\frac{\sqrt{3+n^2}-\sqrt{3+2n^2}}{n+2}\sim\frac{\cancel{3}+n^2-\cancel{3}-2n^2}{\left(\sqrt{3+n^2}+\sqrt{3+2n^2}\right)(n+2)}\sim\frac{-1}{1+\sqrt{2}}$$

#2

$$y''+y'+2y=e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)+3$$

$$y''+y'+2y=0\qquad \lambda_{1,2}=\frac{-1\pm\sqrt{1-8}}{2}=\frac{-1\pm i\sqrt{7}}{2}\;\begin{cases}\frac{-1+i\sqrt{7}}{2}\\ \frac{-1-i\sqrt{7}}{2}\end{cases}$$

Integrale generale di $y''+y'+2y=0$: $V_2=\operatorname{span}\left\{e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right),e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)\right\}.$

$$=e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right)+B\cos\left(\frac{\sqrt{7}}{2}x\right)\right).$$

$$'=\frac{1}{2}e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right)+B\cos\left(\frac{\sqrt{7}}{2}x\right)\right)+e^{\frac{1}{2}x}\left(\frac{A\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}x\right)-\frac{B\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}x\right)\right).$$

$$\frac{1}{4}e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right)+B\cos\frac{\sqrt{7}}{2}x\right)+\frac{1}{2}e^{\frac{1}{2}x}\left(\frac{\sqrt{7}}{2}A\cos\left(\frac{\sqrt{7}}{2}x\right)-\frac{\sqrt{7}}{2}B\sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$

$$+\frac{1}{2}e^{\frac{1}{2}x}\left(\frac{A\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}x\right)-\frac{B\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}x\right)\right)+e^{\frac{1}{2}x}\left(-\frac{A7}{4}\sin\left(\frac{\sqrt{7}}{2}x\right)-\frac{B7}{4}\cos\right.$$

Sostituendo

$$\psi'' + \psi' + 2\psi = e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)$$

$$e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right) + B\cos\left(\frac{\sqrt{7}}{2}x\right)\right) + e^{\frac{1}{2}x}\left(\frac{\sqrt{7}}{2}A\cos\left(\frac{\sqrt{7}}{2}x\right) - \frac{\sqrt{7}}{2}B\sin\left(\frac{\sqrt{7}}{2}x\right)\right) + e^{\frac{1}{2}x}\left(-\frac{7}{4}A\sin\left(\frac{\sqrt{7}}{2}x\right) - \frac{7}{4}B\cos\right.$$

$$+ \frac{1}{2}e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right) + B\cos\left(\frac{\sqrt{7}}{2}x\right)\right) + e^{\frac{1}{2}x}\left(\frac{A\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}x\right) - \frac{B\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$

$$+ 2e^{\frac{1}{2}x}\left(A\sin\left(\frac{\sqrt{7}}{2}x\right) + B\cos\left(\frac{\sqrt{7}}{2}x\right)\right) = e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)$$

$$\begin{cases} \frac{1}{4}A - \frac{\sqrt{7}}{2}B - \frac{7}{4}A + \frac{1}{2}A - \frac{\sqrt{7}}{2}B + 2A = 1 \\ \frac{1}{4}B + \frac{\sqrt{7}}{2}A - \frac{7}{4}B + \frac{B}{2} + \frac{A\sqrt{7}}{2} + 2B = 0 \end{cases} \qquad \begin{cases} A - \sqrt{7}B = 1 \\ B + \sqrt{7}A = 0 \end{cases}$$

$$\begin{cases} A - \sqrt{7}B = 1 \\ \sqrt{7}A + B = 0 \end{cases} \qquad A = \frac{\det\begin{bmatrix} 1 & -\sqrt{7} \\ 0 & 1 \end{bmatrix}}{\det\begin{bmatrix} 1 & -\sqrt{7} \\ \sqrt{7} & 1 \end{bmatrix}} = \frac{1}{1+7} = \frac{1}{8}$$

$$B = \frac{\det\begin{bmatrix} 1 & 1 \\ \sqrt{7} & 0 \end{bmatrix}}{8} = -\sqrt{7}\,\frac{1}{8} = -\frac{\sqrt{7}}{8}.$$

Quindi $\psi = e^{\frac{1}{2}x}\left(\frac{1}{8}\sin\left(\frac{\sqrt{7}}{2}x\right) - \frac{\sqrt{7}}{8}\cos\left(\frac{\sqrt{7}}{2}x\right)\right)$

$\eta = K \qquad \eta' = 0 \qquad \eta'' = 0 \qquad 2K = 3 \longrightarrow K = \frac{3}{2};\ \eta = \frac{3}{2}$

L'integrale generale è:

$$V_2 = \operatorname{span}\left\{e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right), e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right)\right\} + \psi + \eta.$$

#3
$$\left(z^6+2+3i\right)\left(z^2+(2+i\sqrt{2}+3i)z+(2i-\sqrt{2})3\right)=0$$

$z^6+2+3i=0 \iff z^6=-2-3i$. Allora

$$z_k=(13)^{\frac{1}{12}}e^{i\theta_k},\quad \theta_k=\frac{\arctan\left(\frac{3}{2}\right)+\pi+2k\pi}{6},\quad k=0,1,2,3,4,5.$$

Mentre $z^2+(2+i\sqrt{2}+3i)z+(2i-\sqrt{2})3=0 \iff$

$(z+2+i\sqrt{2})(z+3i)=0 \iff z=-2-i\sqrt{2},\quad z=-3i.$

#4 $\alpha>0$

$$\int_1^{+\infty}\frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t^2-1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\,dt\ ;\quad \frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t^2-1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}=\frac{\left(\sin\frac{t}{2}+2\right)t^{-\left(\frac{1}{2}\right)}}{(t-1)^{\frac{\alpha}{2}}(t+1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}$$

Se $t_0>1$, $t\in(1,t_0]$

$$\frac{c_1(\alpha,t_0)}{(t-1)^{\frac{\alpha}{2}}}\le\frac{t^{-\left(\frac{1}{2}\right)}}{(t-1)^{\frac{\alpha}{2}}(t_0+1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t_0\right)^{\frac{\alpha}{2}}}\le\frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t-1)^{\frac{\alpha}{2}}(t+1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\le\frac{3t^{-\left(\frac{1}{2}\right)}}{2^{\frac{\alpha}{2}}\left(\frac{3}{2}\right)^{\frac{\alpha}{2}}}\cdot\frac{1}{(t-1)^{\frac{\alpha}{2}}}\le c_2(\alpha)\frac{1}{(t-1)^{\frac{\alpha}{2}}}$$

Quindi $\displaystyle\int_1^{t_0}\frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t^2-1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\,dt$ converge se e solo

se $\frac{\alpha}{2}<1 \iff \alpha<2$.

Se $t\ge t_0$

$$\frac{c_4(\alpha)}{t^{\frac{1}{2}+\frac{3}{2}\alpha}}\le\frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t-1)^{\frac{\alpha}{2}}(t+1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\le\frac{c_3(\alpha)}{t^{\frac{1}{2}+\frac{3}{2}\alpha}};$$

quindi $\displaystyle\int_{t_0}^{+\infty}\frac{\left(\sin\left(\frac{t}{2}\right)+2\right)t^{-\left(\frac{1}{2}\right)}}{(t^2-1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\,dt$ converge se e solo se $\frac{1}{2}+\frac{3}{2}\alpha>1 \iff \frac{3}{2}\alpha>\frac{1}{2}$

$\Longleftrightarrow \quad \alpha > \frac{1}{3}.$

Pertanto
$$\int_1^{+\infty} \frac{\left(\sin\frac{t}{2}+2\right) t^{-\frac{1}{2}}}{(t^2-1)^{\frac{\alpha}{2}}\left(\frac{1}{2}+t\right)^{\frac{\alpha}{2}}}\,dt \quad \text{converge}$$

se e solo se $\quad \frac{1}{3} < \alpha < 2.$

\# 5

$$\lim_{x\to 0} \frac{\sinh^2(2x)-\sin^2(2x)}{2x\cos^3(\pi+2x)\left(\cos(3x)+3x+9x^2-e^{3x}\right)} = \frac{2^5}{3^3}$$

$$\sinh^2(2x)-\sin^2(2x) = (\sinh(2x)-\sin(2x))(\sinh(2x)+\sin(2x)).$$

$$\sinh(2x)-\sin(2x) \sim \frac{(2x)^3}{3!}+\frac{(2x)^3}{3!} \sim \frac{(2x)^3}{3}, \quad x\to 0$$

$$\sinh(2x)+\sin(2x) \sim 2x+2x \sim 4x, \quad x \to 0;$$

quindi $\sinh^2(2x)-\sin^2(2x) \sim \frac{2^5}{3}x^4, \quad x\to 0.$

$$\cos(3x)+3x+9x^2-e^{3x} \underset{x\to 0}{\sim} \cancel{1}-\cancel{\frac{(3x)^2}{2}}+o(x^3)+\cancel{3x}+\cancel{9x^2}-\cancel{1}-\cancel{3x}-\cancel{\frac{9x^2}{2}}-\frac{27x^3}{3!}$$

$$\sim -\frac{27}{6}x^3 \sim -\frac{9}{2}x^3, \quad x\to 0; \quad \text{quindi}$$

$$2x\cos^3(\pi+2x)\left(\cos(3x)+3x+9x^2-e^{3x}\right) \underset{x\to 0}{\sim} (-2x)\left(-\frac{9}{2}x^3\right) \sim 9x^4.$$

Pertanto
$$\frac{N}{D} \sim \frac{2^5}{3}\cdot x^4 \cdot \frac{1}{9x^4} \sim \frac{2^5}{3^3}$$

#6 $f(x) = \log\left(\frac{1-2x}{1+3|x|}\right)$

- Il dominio d'esistenza è $D = \{x \in \mathbb{R} : \frac{1-2x}{1+3|x|} > 0\} = (-\infty, \frac{1}{2})$

Infatti $\frac{1-2x}{1+3|x|} > 0 \iff 1-2x > 0$ ($1+3|x|$ è sempre >0).

$\iff x < \frac{1}{2}$.

- $f : (-\infty, \frac{1}{2}) \longrightarrow \mathbb{R}$, $f(x) = \log\left(\frac{1-2x}{1+3|x|}\right)$ è derivabile in $(-\infty, \frac{1}{2}) \setminus \{0\}$ perché composizione di funzioni derivabili. Verifichiamo se nel punto 0 è derivabile: poiché per ogni $x \in (-\infty, \frac{1}{2}) \setminus \{0\}$

$$f'(x) = \frac{1+3|x|}{1-2x} \cdot \frac{-2(1+3|x|) - (1-2x)3\,\mathrm{sgn}(x)}{(1+3|x|)^2}$$ e

$\lim_{x \to 0^-} f'(x) = \frac{-2+3}{1} = \frac{1}{1}$; $\lim_{x \to 0^+} \frac{-2-3}{1} = -5$.

D'altra parte $f \in C((-\infty, \frac{1}{2}))$, quindi f non può essere derivabile in 0 perché $f'^-(0) = 1 \neq -5 = f'^+(0)$.

- Studiamo il segno di f' in $(-\infty, \frac{1}{2}) \setminus \{0\}$.

$$\begin{cases} f'(x) > 0 \\ x \in (-\infty, \frac{1}{2}) \setminus \{0\} \end{cases} \iff \begin{cases} \frac{1+3|x|}{1-2x} \cdot \left(-2(1+3|x|) - 3(1-2x)\,\mathrm{sgn}\,x\right) > 0 \\ x \in (-\infty, \frac{1}{2}) \setminus \{0\} \end{cases} \iff$$

$$\begin{cases} \frac{1-3x}{1-2x} \left(-2 + \cancel{6x} + 3 - \cancel{6x}\right) > 0 \\ x < 0 \end{cases} \vee \begin{cases} \frac{1+3x}{1-2x} \left(-2 - \cancel{6x} - 3 + \cancel{6x}\right) > 0 \\ x \in (0, \frac{1}{2}) \end{cases}$$

$$\begin{cases} \dfrac{1-3x}{1-2x} > 0 \\ x < 0 \end{cases} \quad \vee \quad \begin{cases} -\dfrac{1+3x}{1-2x} > 0 \\ x \in (0, \frac{1}{2}) \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{cases} x < \frac{1}{3} \vee x > \frac{1}{2} \\ x < 0 \end{cases} \quad \vee \quad \begin{cases} x < -\frac{1}{3} \vee x > \frac{1}{2} \\ x \in (0, \frac{1}{2}) \end{cases}$$

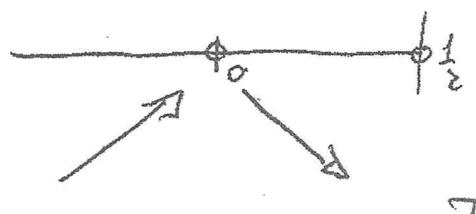
$$\Downarrow \qquad\qquad \Downarrow$$

$$x \in (-\infty, 0) \qquad\qquad \emptyset$$

Pertanto $\begin{cases} f'(x) > 0 \\ x \in (-\infty, \frac{1}{2}) \end{cases} \iff x \in (-\infty, 0)$.

Analogamente $\begin{cases} f'(x) < 0 \\ x \in (-\infty, \frac{1}{2}) \end{cases} \iff x \in (0, \frac{1}{2})$.

Quindi f è monotona strettamente crescente in $(-\infty, 0)$, mentre f è monotona strettamente decrescente in $(0, \frac{1}{2})$



0 è punto di massimo assoluto. Non esistono punti di minimo per f.

Infatti $\lim_{x \to -\infty} f(x) = \log \frac{2}{3}$ e $\lim_{x \to (\frac{1}{2})^-} = -$

7 $f:(0,\frac{\pi}{2}) \to \mathbb{R}$

$$f(x) = \frac{\sin^{\cos(x)}(2x)}{\sin^2(3x)+x^3} = \frac{e^{\cos(x)\log(\sin(2x))}}{\sin^2(3x)+x^3}$$

$$f'(x) = \frac{\left[-\sin x \log(\sin(2x)) + \frac{\cancel{2}\cos(2x)}{\cancel{2}\sin(x)}\right] e^{\cos(x)\log(\sin(2x))} \cdot (\sin^2(3x)+x^3) - (6\sin(3x)\cos(3x)+3x^2)\cdot e^{\cos(x)\log(\sin(2x))}}{(\sin^2(3x)+x^3)^2}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\left(-\frac{\sqrt{2}}{2}\log\left(\sin\frac{\pi}{2}\right) + \frac{\cancel{2}\cos\frac{\pi}{2}}{\cancel{2}\sin\left(\frac{\pi}{4}\right)}\right)\cdot e^{\cos\frac{\pi}{4}\log\left(\sin\frac{\pi}{2}\right)} \cdot \left(\sin^2\left(\frac{3}{4}\pi\right)+\left(\frac{\pi}{4}\right)^3\right) - \left(6\sin\left(\frac{3}{4}\pi\right)\cos\left(\frac{3}{4}\pi\right)+\frac{3\pi^2}{16}\right)\cdot e^{\cos\frac{\pi}{4}\cdot\log\left(\sin\frac{\pi}{2}\right)}}{\left(\sin^2\left(\frac{3}{4}\pi\right)+\left(\frac{\pi}{4}\right)^3\right)^2}$$

$$= \frac{-\left(6\sin\frac{3}{4}\pi\cos\frac{3}{4}\pi+\frac{3}{16}\pi^2\right)}{\left(\sin^2\left(\frac{3}{4}\pi\right)+\left(\frac{\pi}{4}\right)^3\right)^2} = \frac{-\left(-6\,\frac{2}{4}+\frac{3}{16}\pi^2\right)}{\left(\frac{2}{4}+\left(\frac{\pi}{4}\right)^3\right)^2}$$

$$= \frac{3-\frac{3}{16}\pi^2}{\left(\frac{1}{2}+\left(\frac{\pi}{4}\right)^3\right)^2}.$$

#8

$$I = \int_{\sqrt{3}}^{\sqrt{6}} \frac{(t^2-3)t}{(3+(t^2-3)^2)^{\frac{5}{2}}} dt$$

posto $(t^2-3)^2 = x$ $\quad dx = 2(t^2-3)\cdot 2t\, dt$

$dx = 4t(t^2-3)\,dt$

$$I = \frac{1}{4}\int_0^9 \frac{dx}{(3+x)^{\frac{5}{2}}} = \frac{1}{4}\int_0^9 (3+x)^{-\frac{5}{2}} dx = \frac{1}{4}\left[-\frac{2}{3}(x+3)^{-\frac{3}{2}}\right]_{x=0}^{x=9}$$

$$= \frac{1}{4}\left(-\frac{2}{3}\, 12^{-\frac{3}{2}} + \frac{2}{3}\cdot 3^{-\frac{3}{2}}\right) = -\frac{1}{6}\left(12^{-\frac{3}{2}} - 3^{-\frac{3}{2}}\right)$$

$$= \frac{1}{6}\left(3^{-\frac{3}{2}} - 12^{-\frac{3}{2}}\right).$$