1# (1)

Determinare $\alpha \in \mathbb{R}^+$ per cui $\displaystyle\int_0^{+\infty} \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}}\,dx$ converge.

$$\frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}} \underset{x\to 0}{\sim} \frac{(3x)^{3\alpha}}{6^\alpha \cdot x^{3+\alpha}} \underset{x\to 0}{\sim} \frac{3^{3\alpha}}{6^\alpha\, x^{3-2\alpha}}$$

quindi $\displaystyle\int_0^1 \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}}\,dx < +\infty \iff 3-2\alpha<1 \iff \alpha>1$

Inoltre.

$$\frac{(\sin h3x-3x)^\alpha}{e^{6x}x^{3+\alpha}} \underset{x\to+\infty}{\sim} \frac{e^{3\alpha x}}{e^{6x}x^{3+\alpha}} \underset{x\to+\infty}{\sim} \frac{1}{e^{(6-3\alpha)x}x^{3+\alpha}}$$

Se $6-3\alpha>0$ allora $\displaystyle\int_1^{+\infty} \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}} < +\infty$

Se $6-3\alpha<0$ allora $\displaystyle\int_1^{+\infty} \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}}$ diverge

perché se $\alpha>2$ allora $\displaystyle\frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}} \xrightarrow[x\to+\infty]{} +\infty$

Se $6-3\alpha=0$ $\displaystyle\frac{(\sin h3x-3x)^\alpha}{e^{6x}x^{3+\alpha}} \underset{x\to+\infty}{\sim} \frac{1}{x^5}$. Quindi

per $\alpha\leq 2$ $\displaystyle\int_1^{+\infty} \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}}\,dx$ converge.

Pertanto $\alpha>0$

$\displaystyle\int_0^{+\infty} \frac{(\sin h(3x)-3x)^\alpha}{e^{6x}x^{3+\alpha}}\,dx$ converge se e solo se

$\alpha \in (1,2]$.

2 #

$$\left(z^2+(3-4i)z+21+9i\right)\left(z^4+5-21i\right)=0.$$

$$z^2+(3-4i)z+21+9i=0 \qquad z_{1,2}=\frac{-3+4i\pm\sqrt{9-24i-16-84-36i}}{2}$$

$$=\frac{-3+4i\pm\sqrt{-91-60i}}{2}$$

ma $60i=2\cdot 30i$ e i divisori di 30 sono 5 e 6, $\underline{10\ e\ 3}$, 15 e 2

Pertanto $(10i)^2+3^2=-100+9=-91$ e

$$-91-60i=(10i-3)^2.$$ Quindi

$$z_{1,2}=\frac{-3+4i\pm(10i-3)}{2}=\frac{-3+4i\pm 10i\mp 3}{2}\left\langle\begin{array}{l}\dfrac{-3+4i+10i-3}{2}\\[2ex]\dfrac{\cancel{-3}+4i-10i\cancel{+3}}{2}\end{array}\right.$$

$$z_1=\frac{-6+14i}{2}=-3+7i\,;\quad z_2=-3i$$

Inoltre da $z^4+5-21i=0$ segue $z^4=-5+21i$

$$|-5+21i|=\sqrt{25+441}=\sqrt{466} \quad \text{e} \quad \arg(-5+21i)=-\arctan\frac{21}{5}+\pi$$

Quindi

$$z_k=(466)^{1/8}e^{i\theta_k}, \quad \text{dove} \quad \theta_k=\frac{-\arctan\frac{21}{5}+\pi+2k\pi}{4}$$

$k=0,1,2,3.$

(3)

\#3 Calcolare

$$\int_0^1 (4x^2-3)\cos\left(\frac{\pi}{4}x\right)dx = \left[+\frac{4}{\pi}\,\mathrm{sen}\left(\frac{\pi}{4}x\right)(4x^2-3)\right]_{x=0}^{x=1} - \frac{4}{\pi}\int_0^1 8x\,\mathrm{sen}\frac{\pi}{4}x\,dx$$

$$= \frac{4}{\pi}\frac{\sqrt{2}}{2} - \frac{32}{\pi}\left\{\left[-\frac{4}{\pi}\cos\left(\frac{\pi}{4}x\right)x\right]_{x=0}^{x=1} + \frac{4}{\pi}\int_0^1 \cos\frac{\pi}{4}x\,dx\right\}$$

$$= \frac{4}{\pi}\frac{\sqrt{2}}{2} - \frac{32}{\pi}\left\{-\frac{4}{\pi}\frac{\sqrt{2}}{2} + \frac{4}{\pi}\left[\mathrm{sen}\left(\frac{\pi}{4}x\right)\frac{4}{\pi}\right]_{x=0}^{x=1}\right\}$$

$$= \frac{2}{\pi}\sqrt{2} + \frac{64}{\pi^2}\sqrt{2} - \frac{128}{\pi^2}\cdot\frac{4}{\pi}\,\frac{\sqrt{2}}{2} = \frac{2}{\pi}\sqrt{2} + \frac{64}{\pi^2}\sqrt{2} - \frac{256}{\pi^3}\sqrt{2}.$$

$$= \frac{2}{\pi}\sqrt{2}\left(1 + \frac{32}{\pi} - \frac{128}{\pi^2}\right).$$

\#4 Calcolare

$$\lim_{n\to+\infty} \frac{n^{10} - 4^n + 7\cdot 3^n - 6n^7}{100\cdot 3^n + 10n^7 + 5\cdot 4^n} = \lim_{n\to+\infty} \frac{-4^n}{5\cdot 4^n} = -\frac{1}{5}$$

$$n^{10} - 4^n + 7\cdot 3^n - 6n^7 \underset{n\to+\infty}{\sim} -4^n$$

$$100\cdot 3^n + 10n^7 + 5\cdot 4^n \sim 5\cdot 4^n$$

\#5 Calcolare

$$\lim_{x\to 0} \frac{\sqrt{1+16x^2} + \cos(4x) - 2}{\mathrm{senh}(x^2-5)\left(\mathrm{sen}^2(5x) - \mathrm{senh}^2(5x)\right)}$$

$\sqrt{1+t} \underset{t\to 0}{\sim} 1 + \frac{1}{2}t - \frac{1}{8}t^2 + o(t^2)$; $\sqrt{1+16x^2} \sim 1 + 8x^2 - 32x^4 + o(x^4)$

$\mathrm{sen}\, t \sim t - \frac{t^3}{3!} + o(t^4)$, $t\to 0$; $\mathrm{senh}\, t \sim t + \frac{t^3}{3!} + o(t^4)$ $t\to 0$

$\cos t \sim 1 - \frac{t^2}{2} + \frac{t^4}{4!} + o(t^5)$, $t\to 0$; $\cos(4x) \underset{x\to 0}{\sim} 1 - \frac{16}{2}x^2 + \frac{(4x)^4}{4!}$

$$\left(\operatorname{sen}^2(5x) - \operatorname{senh}^2(5x)\right) = \left(\operatorname{sen}(5x) - \operatorname{senh}(5x)\right)\left(\operatorname{sen}(5x)+\operatorname{senh}(5x)\right)$$ 4

Pertanto

$$\operatorname{sen}^2(5x) - \operatorname{senh}^2(5x) \underset{x\to 0}{\sim} \frac{-2(5x)^3}{3!} \cdot 10x \underset{x\to 0}{\sim} -\frac{5^4}{3}\cdot 2x^4$$

Quindi

$$\lim_{x\to 0}\frac{N}{D} = \lim_{x\to 0}\frac{\cancel{1+8x^2} - 32x^4 \cancel{+1-8x^2} + \frac{\cancel{4}^{\,3} x^4}{6} \cancel{-2}}{+\operatorname{senh}(5)\cdot\frac{5^4}{3}\cdot 2x^4} = \frac{\left(-32+\frac{32}{3}\right)}{\operatorname{senh}(5)\cdot\frac{5^4}{3}\cdot 2}$$

$$= \frac{-\frac{2}{3}\cdot 32}{\operatorname{senh}5\cdot 5^4\cdot\frac{2}{3}} = -\frac{32}{5^4\cdot\operatorname{senh}5}$$

$\boxed{\#6}$

$$y'' + 2y' + 3y = e^{-x}$$

$$y'' + 2y' + 3y = 0 \longrightarrow \lambda^2 + 2\lambda + 3 = 0 \longrightarrow \lambda_{1,2} = -1 \pm \sqrt{1-3} = -1 \pm i\sqrt{2}$$

$$V_2 = \operatorname{span}\{e^{-x}\operatorname{sen}(\sqrt{2}x),\ e^{-x}\cos(\sqrt{2}x)\}.$$

Cerchiamo una soluzione di $y''+2y'+3y=e^{-x}$ nella forma $\bar{y}=ke^{-x} \to \bar{y}' = -ke^{-x}$; $\bar{y}'' = ke^{-x}$ da cui segue sostituendo

$$(k-2k+3k)e^{-x} = e^{-x} \iff 2k = 1 \iff k = \frac{1}{2}.$$

Pertanto

$$LV_2 = \operatorname{span}\{e^{-x}\operatorname{sen}(\sqrt{2}x),\ e^{-x}\cos(\sqrt{2}x)\} + \frac{1}{2}e^{-x}$$

#7 $\quad f : \mathbb{R} \to \mathbb{R} \quad f(x) = \sinh(3x|x| - 5x^4)$ ⑤

(1) f è derivabile in $\mathbb{R}\setminus\{0\}$ perché in $\mathbb{R}\setminus\{0\}$ f è composizione di funzioni derivabili. Inoltre $\forall x \in \mathbb{R}\setminus\{0\}$

$$f'(x) = \cosh(3x|x| - 5x^4) \cdot (3|x| + 3x\,\mathrm{sgn}\,x - 20x^3)$$
$$= \cosh(3x|x| - 5x^4)\,(6|x| - 20x^3)$$

e $\lim_{x\to 0^+} f'(x) = 0$ come $\lim_{x\to 0^-} f'(x) = 0$

Pertanto f è derivabile su tutto $\mathbb{R}$.

(2) $f'(x) \geq 0 \iff \begin{cases} 6|x| - 20x^3 \geq 0 \\ x \in \mathbb{R} \end{cases}$

$$\begin{cases} 6x - 20x^3 \geq 0 \\ x > 0 \end{cases} \quad \vee \quad \begin{cases} -6x - 20x^3 \geq 0 \\ x \leq 0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\begin{cases} 3 - 10x^2 \geq 0 \\ x > 0 \end{cases} \qquad \underbrace{\begin{cases} 3 + 10x^2 \geq 0 \\ x < 0 \end{cases} \quad \vee \quad x = 0}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

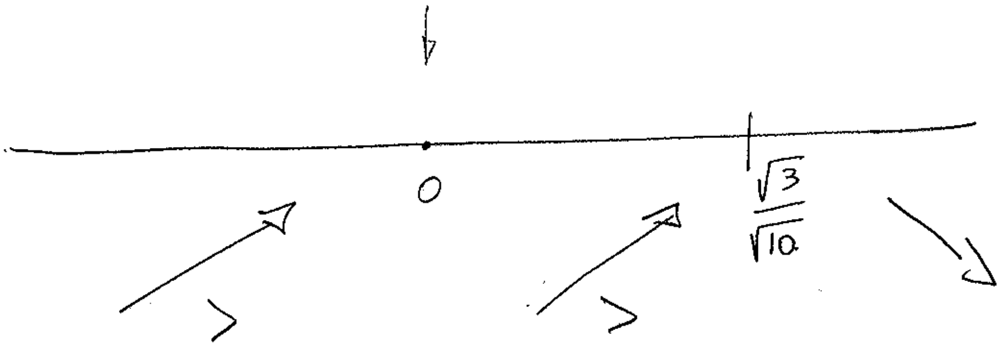
$$\begin{cases} -\sqrt{\frac{3}{10}} \leq x \leq +\sqrt{\frac{3}{10}} \\ x > 0 \end{cases} \qquad\qquad x \leq 0$$

$$\Downarrow$$

$$x \in \left(0, \frac{\sqrt{3}}{\sqrt{10}}\right]$$

Pertanto

la derivata è zero in 0



La funzione è strettamente crescente in $\left(-\infty, \frac{\sqrt{3}}{\sqrt{10}}\right]$ perché è strettamente crescente in $(-\infty, 0]$ e è strettamente crescente in $\left[0, \frac{\sqrt{3}}{\sqrt{10}}\right]$. Inoltre è continua in 0 e quindi è strettamente crescente in $\left(-\infty, \frac{\sqrt{3}}{\sqrt{10}}\right]$.

$\frac{\sqrt{3}}{\sqrt{10}}$ è punto di massimo assoluto per f.

#8

$\sigma : (2, +\infty) \to \mathbb{R}$, $\sigma(x) = \cosh\left(3e^{2x} + \sqrt{\frac{3x+2}{x^2-4}}\right)$

$$\sigma'(x) = \sinh\left(3e^{2x} + \sqrt{\frac{3x+2}{x^2-4}}\right) \cdot \left(6e^{2x} + \frac{1}{2}\left(\frac{3x+2}{x^2-4}\right)^{-1/2} \frac{3(x^2-4) - 2x(3x+2)}{(x^2-4)^2}\right)$$

$$\sigma'(4) = \sinh\left(3e^{8} + \sqrt{\frac{14}{12}}\right) \cdot \left(6e^{8} + \frac{1}{2}\left(\frac{6}{7}\right)^{1/2} \frac{36 - 112}{144}\right)$$

$$= \sinh\left(3e^{8} + \sqrt{\frac{7}{6}}\right) \cdot \left(6e^{8} - \frac{19}{72} \cdot \left(\frac{6}{7}\right)^{1/2}\right).$$

$\square$