Se osserviamo che $e^{(\mathrm{Re}\,a + i\,\mathrm{Im}\,a)t} = e^{\mathrm{Re}\,a\,t}\, e^{i\,\mathrm{Im}\,a\,t}$.

$= e^{\mathrm{Re}\,a\,t}(\cos \mathrm{Im}\,a\,t + i \sin \mathrm{Im}\,a\,t)$ allora

$$\mathrm{Im} \int_0^t e^{a\xi}(t-\xi)\, d\xi = \int_0^t e^{\mathrm{Re}\,a\,\xi} \sin[(\mathrm{Im}\,a)\,t]\,(t-\xi)\, d\xi.$$

Quindi $$\int_0^t e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\xi}(t-\xi)\,d\xi = \left[\frac{1}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}\, e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\xi}(t-\xi)\right]_{\xi=0}^{\xi=t} + \int_0^t \frac{e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\xi}}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}\,d\xi$$

$$= -\frac{t}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} + \left[\frac{e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\xi}}{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2}\right]_{\xi=0}^{\xi=t} = -\frac{t}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} + \frac{e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)t}-1}{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2}$$

da cui

$$\mathrm{Im}\int_0^t e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\xi}(t-\xi)\,d\xi = \mathrm{Im}\left(\frac{t}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} + \frac{e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)t}-1}{\frac{1}{4}-\frac{i\sqrt{3}}{2}-\frac{3}{4}}\right)$$

$$= \mathrm{Im}\left(-\frac{t\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{\frac{1}{4}+\frac{3}{4}} + \frac{e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right)-1}{-\frac{1}{2}-i\frac{\sqrt{3}}{2}}\right)$$

$$= +\frac{\sqrt{3}}{2}t + \mathrm{Im}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left[\left(e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right)-1\right]$$

$$= +\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{e^{-\frac{1}{2}t}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}$$

Per cui $$\boxed{y(t) = 2t + 2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - 2}$$

(sapendo che $y = 2\left(x * \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)\right)(t)$, come calcolato precedentemente).

Utilizzando le formule del ritardo e del riscalamento della trasformata di Laplace per determinare f t.c. $\mathcal{L}f = \frac{1}{s^2+s+1}$:

$$\frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{4}{3}\,\frac{1}{\left[\frac{2}{\sqrt{3}}(s+\frac{1}{2})\right]^2+1}$$

. Posto $g(t) = e^{-\frac{1}{2}t}h\left(\frac{\sqrt{3}}{2}t\right)$

si ottiene

$$\mathcal{L}g(s) = \mathcal{L}\left(h\left(\frac{\sqrt{3}}{2}t\right)\right)\left(s+\frac{1}{2}\right) = \frac{2}{\sqrt{3}}\,\mathcal{L}h\left(\frac{2}{\sqrt{3}}\left(s+\frac{1}{2}\right)\right)$$

$$\mathcal{L}(e^{at}q(t)) = \mathcal{L}q(s-a)\,; \qquad \mathcal{L}(q(ct)) = \frac{1}{c}\mathcal{L}g\left(\frac{s}{c}\right)$$

Quindi è sufficiente conoscere h t.c. $\mathcal{L}h(s) = \frac{1}{1+s^2}$, cioè $h(t) = \sin t$. Pertanto $f = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

Calcolare la soluzione di

$$\begin{cases} y''+y'+y = 2t \\ y(0)=0 \\ y'(0)=0 \end{cases} \qquad (1+s+s^2)\mathcal{L}y = 2\mathcal{L}t$$

$\mathcal{L}y = 2\mathcal{L}t \cdot \frac{1}{1+s+s^2}$, ma $\mathcal{L}y = 2\mathcal{L}t \cdot \mathcal{L}\left(\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin_+\left(\frac{\sqrt{3}}{2}t\right)\right)$

$= \mathcal{L}\left[2t_+ * \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin_+\left(\frac{\sqrt{3}}{2}t\right)\right]$. Quindi

$$y(t) = 2\left(x_+ * \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin_+\left(\frac{\sqrt{3}}{2}x\right)\right)(t) = 2\cdot\int_0^t \frac{2}{\sqrt{3}}e^{-\frac{1}{2}\xi}\sin\left(\frac{\sqrt{3}}{2}\xi\right)(t-\xi)\,d\xi.$$

(e 0 se $t<0$)

Supponiamo di dover calcolare $\int_0^t e^{a\xi}\xi\,d\xi = \left[\frac{e^{a\xi}}{a}\xi\right]_{\xi=0}^{\xi=t} - \int_0^t \frac{e^{a\xi}}{a}\,d\xi$

$$= \left[\frac{e^{a\xi}}{a}\xi\right]_{\xi=0}^{\xi=t} - \left[\frac{e^{a\xi}}{a^2}\right]_{\xi=0}^{\xi=t} = \frac{e^{at}t}{a} - \frac{e^{at}}{a^2} + \frac{1}{a^2}$$