La correzione degli esercizi 1,2,3 è la stessa del preappello del 18/12/2014.

Esercizio 4

$$\mathcal{F}\chi_{[-1,1]}(\xi) = \int_{-1}^{1} e^{-ix\xi}\, d\xi = \left[-\frac{e^{-ix\xi}}{i\xi}\right]_{x=-1}^{x=1} = \frac{-e^{-i\xi}+e^{i\xi}}{i\xi} = \frac{2}{\xi}\left(\frac{e^{i\xi}-e^{-i\xi}}{2i}\right)$$

$$= \frac{2}{\xi}\sin(\xi).$$

Inoltre

$$(\chi_{[-1,1]} * \chi_{[-1,3]})(x) = \int_{\mathbb{R}} \chi_{[-1,1]}(y)\,\chi_{[-1,3]}(x-y)\,dy$$

$$= \int_{-1}^{1} \chi_{[-1,3]}(x-y)\,dy = -\int_{x+1}^{x-1} \chi_{[-1,3]}(s)\,ds = \int_{x-1}^{x+1} \chi_{[-1,3]}(s)\,ds$$

$$x-y=s$$
$$ds=-dy$$

per ogni $x\in\mathbb{R}$ $\quad x-1<x+1$. Quindi

$$[x-1,x+1]\cap[-1,3] = \begin{cases} \emptyset, & \text{se } x+1<-1 \iff \boxed{x<-2} \quad (1^\circ) \\ [-1,x+1], & \text{se } -1\le x+1\le 3 \text{ e } x-1\le -1. \quad (2^\circ) \\ [-1,3], & \text{se } x-1\le -1 \wedge x+1\ge 3 \quad (3^\circ) \\ [x-1,x+1], & -1\le x-1 \wedge x+1\le 3. \quad (4^\circ) \\ [x-1,3], & -1\le x-1 \wedge x+1\ge 3 \wedge x-1\le 3 \quad (5^\circ) \\ \emptyset, & x-1>3 \iff \boxed{x>4} \quad (6^\circ) \end{cases}$$

(2°) $\begin{cases} -2\le x\le 2 \\ x\le 0 \end{cases} \iff \boxed{-2\le x\le 0}$; (3°) $\begin{cases} x\le 0 \\ x\ge 2 \end{cases} \iff \emptyset$

(4°) $\begin{cases} 0\le x \\ x\le 2 \end{cases} \iff 0\le x\le 2$; (5°) $\begin{cases} 0\le x \\ x\ge 2 \\ x\le 4 \end{cases} \iff 2\le x\le 4$.

Riassumendo.

$$(\chi_{[-1,1]} * \chi_{[-1,3]})(x) = \begin{cases} 0, & \text{se} \quad x < -2 \\ \displaystyle\int_{-1}^{x+1} dt, & \text{se} \quad -2 \leq x \leq 0 \\ \displaystyle\int_{x-1}^{x+1} dt, & \text{se} \quad 0 \leq x \leq 2 \\ \displaystyle\int_{x-1}^{3} dt & \quad 2 \leq x \leq 4 \\ 0 & \quad x \geq 4 \end{cases}$$

$$= \begin{cases} 0, & \text{se} \quad x < -2 \\ x+2, & \text{se} \quad -2 \leq x \leq 0 \\ 2, & \text{se} \quad 0 \leq x \leq 2 \\ 4-x, & \text{se} \quad 2 \leq x \leq 4 \\ 0, & \text{se} \quad x \geq 4 \end{cases}$$