Siano $1, x, \sin(x) \in L^2([0,\pi])$.

Calcolare la matrice di Gram associata a questi vettori, determinare la proiezione ortogonale della funzione $x-1$ su $V_3 = \text{span}\{1, x, \sin(x)\}$ infine determinare con il metodo di Gram-Schmidt tre vettori L^2-ortogonali t.c. siano una base per V_3.

Indichiamo con $v_1 = 1$, $v_2 = x$ e $v_3 = \sin(x)$.

La matrice di Gram è $\left(\langle v_i, v_j \rangle_{L^2}\right)_{1 \le i,j \le 3}$.

Pertanto

$\langle v_1, v_1 \rangle_{L^2([0,\pi])} = \int_0^\pi 1\, dx = \pi$; $\langle v_1, v_2 \rangle_{L^2([0,\pi])} = \int_0^\pi 1 \cdot x\, dx$
$= \left[\frac{1}{2}x^2\right]_{x=0}^{x=\pi} = \frac{\pi^2}{2}$; $\langle v_1, v_3 \rangle_{L^2} = \int_0^\pi \sin(x)\, dx = \left[-\cos(x)\right]_{x=0}^{x=\pi}$
$= -\cos(\pi) + 1 = 2$; $\langle v_2, v_1 \rangle_{L^2} = \langle v_1, v_2 \rangle_{L^2}$ perché v_1 e v_2 sono reali ; $\langle v_2, v_2 \rangle_{L^2} = \int_0^\pi x^2 dx = \left[\frac{1}{3}x^3\right]_{x=0}^{x=\pi} = \frac{\pi^3}{3}$

$\langle v_2, v_3 \rangle = \int_0^\pi x \sin(x)\, dx = \left[-x\cos x\right]_{x=0}^{x=\pi} + \int_0^\pi \cos(x) = +\pi + \left[\sin x\right]_{x=0}^{x=\pi}$
$= \pi$. $\langle v_3, v_1 \rangle_{L^2} = \langle v_1, v_3 \rangle_{L^2}$ perché v_1 e v_3 sono reali $\langle v_3, v_2 \rangle_{L^2} = \langle v_2, v_3 \rangle_{L^2}$ perché v_2 e v_3 sono reali, finalmente $\langle v_3, v_3 \rangle_{L^2} = \int_0^\pi \sin^2(x) dx$

$$= \int_0^\pi \frac{1-\cos(2x)}{2} dx = \left[\frac{1}{2}x - \frac{\sin(2x)}{4}\right]_{x=0}^{x=\pi} = \frac{\pi}{2}$$

Quindi

$$\mathcal{G} = \begin{bmatrix} \pi & \frac{\pi^2}{2} & 2 \\ \frac{\pi^2}{2} & \frac{\pi^3}{3} & \pi \\ 2 & \pi & \frac{\pi}{2} \end{bmatrix}$$

La proiezione ortogonale di $x-1$ su V_3 è data da $y = c_1 v_1 + c_2 v_2 + c_3 v_3$ con c_1, c_2, c_3 soluzione di

$$\mathcal{G} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle (x-1), v_1 \rangle_{L^2} \\ \langle (x-1), v_2 \rangle_{L^2} \\ \langle (x-1), v_3 \rangle_{L^2} \end{bmatrix}$$

Calcoliamo

$$\langle x-1, v_1 \rangle_{L^2} = \int_0^\pi x-1 \, dx = \left[\frac{1}{2}(x-1)^2\right]_{x=0}^{x=\pi} = \frac{(\pi-1)^2}{2} - \frac{1}{2}$$

$$\langle x-1, v_2 \rangle_{L^2} = \int_0^\pi (x-1)\, x \, dx = \left[-\frac{1}{2}x^2 + \frac{1}{3}x^3\right]_{x=0}^{x=\pi} = \frac{\pi^3}{3} - \frac{1}{2}\pi^2$$

$$\langle x-1, \sin x \rangle_{L^2} = \int_0^\pi (x-1) \sin(x) dx = \left[-\cos(x) \cdot (x-1)\right]_{x=0}^{x=\pi}$$

$$+ \int_0^\pi \cos(x)\, dx = \left[-\cos(x)(x-1) + \sin(x)\right]_{x=0}^{x=\pi}$$

$$= (\pi-1) - 1 = \pi - 2$$

Risolvendo

$$\begin{bmatrix} \pi & \frac{\pi^2}{2} & 2 \\ \frac{\pi^2}{2} & \frac{\pi^3}{3} & \pi \\ 2 & \pi & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{\pi^2-2\pi}{2} \\ \frac{\pi^3}{3} - \frac{\pi^2}{2} \\ \pi - 2 \end{bmatrix}$$

otteniamo con il metodo di Gauss:

$$\begin{bmatrix} \pi & \frac{\pi^2}{2} & 2 \\ 0 & \frac{\pi^3}{12} & 0 \\ 0 & 0 & \frac{\pi^2-8}{2\pi} \end{bmatrix} \Bigg\| \begin{bmatrix} \frac{\pi^2-2\pi}{2} \\ \frac{\pi^3}{12} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi, & \frac{\pi^2}{2} & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix} \Bigg\| \begin{bmatrix} \frac{\pi^2-2\pi}{2} \\ 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} \pi, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix} \Bigg\| \begin{bmatrix} -\pi \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix} \Bigg\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Pertanto $c_1 = -1$, $c_2 = 1$ e $c_3 = 0$.

Quindi $y = -1 + X$. Infatti $x - 1 \in V_3$ come si può banalmente verificare direttamente.

costruiamo ora un sistema ortonormale di vettori $\{e_1, e_2, e_3\} \subseteq L^2([0,\pi])$ tale che $\mathrm{Span}\{e_1, e_2, e_3\} = V_3$ con il metodo di Gram-Schmidt

Posto $z_1 = 1$ allora $e_1 = \frac{z_1}{\|z_1\|} = \frac{1}{\sqrt{\pi}}$

Definiamo

$$z_2 = v_2 - \langle v_2, e_1\rangle e_1 = x - \int_0^\pi x \cdot \frac{1}{\sqrt{\pi}}\,dx \cdot \frac{1}{\sqrt{\pi}} = x - \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_{x=0}^{x=\pi}$$

$$= x - \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) = x - \frac{\pi}{2}$$

$$e_2 = \frac{z_2}{\|z_2\|} = \frac{x-\frac{\pi}{2}}{\left(\int_0^\pi \left(x-\frac{\pi}{2}\right)^2 dx\right)^{1/2}} = \frac{x-\frac{\pi}{2}}{\left(\left[\frac{1}{3}\left(x-\frac{\pi}{2}\right)^3\right]_{x=0}^{x=\pi}\right)^{1/2}}$$

$$= \frac{x-\frac{\pi}{2}}{\left\{\frac{1}{3}\left(\left(\frac{\pi}{2}\right)^3 + \left(\frac{\pi}{2}\right)^3\right)\right\}^{1/2}} = \frac{\left(x-\frac{\pi}{2}\right)}{\left(\frac{1}{3}\frac{\pi^3}{2^2}\right)^{1/2}} = \frac{x-\frac{\pi}{2}}{\left(\frac{\pi^3}{12}\right)^{1/2}} = \frac{2\sqrt{3}\left(x-\frac{\pi}{2}\right)}{\pi^{3/2}}$$

$$z_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2$$

$$= \sin x - \int_0^\pi \sin x \cdot \frac{1}{\sqrt{\pi}}\,dx\,\frac{1}{\sqrt{\pi}} - \int_0^\pi \sin x \cdot \frac{2\sqrt{3}}{\pi^{3/2}}\left(x-\frac{\pi}{2}\right)dx \cdot \frac{2\sqrt{3}}{\pi^{3/2}}\left(x-\frac{\pi}{2}\right)$$

$$= \sin x + \frac{1}{\pi}\left[\cos x\right]_{x=0}^{x=\pi} - \frac{4\cdot 3}{\pi^3}\int_0^\pi \sin x\left(x-\frac{\pi}{2}\right)dx\left(x-\frac{\pi}{2}\right)$$

$$= \sin x + \frac{1}{\pi}(-1-1) - \frac{12}{\pi^3}\left\{\left[-\cos x\left(x-\frac{\pi}{2}\right)\right]_{x=0}^{x=\pi} + \int_0^\pi \cos x\,dx\right\}\left(x-\frac{\pi}{2}\right)$$

$$= \sin x - \frac{2}{\pi} - \frac{12}{\pi^3}\left\{-\cos\pi\left(\pi-\frac{\pi}{2}\right) + \cos 0\left(-\frac{\pi}{2}\right) + \left[\sin x\right]_{x=0}^{x=\pi}\right\}\left(x-\frac{\pi}{2}\right)$$

$$= \sin x - \frac{2}{\pi} - \frac{12}{\pi^3}\left(+\frac{\pi}{2} - \frac{\pi}{2} + \sin\pi - \sin 0\right)\left(x-\frac{\pi}{2}\right)$$

$$= \sin x - \frac{2}{\pi} - \frac{12}{\pi^3}$$

Calcoliamo ora

$$\|\varepsilon_3\|_{L^2}^2 = \int_0^\pi \left(\sin x - \frac{2}{\pi} - \frac{12}{\pi^3}\right)^2 dx = \int_0^\pi \sin^2 x - 2\sin x\left(\frac{2}{\pi} + \frac{12}{\pi^3}\right) + \left(\frac{2}{\pi} + \frac{12}{\pi^3}\right)^2 dx$$

$$= \int_0^\pi \sin^2 x \, dx + \left(\frac{2}{\pi} + \frac{12}{\pi^3}\right)^2 \pi = \frac{\pi}{2} + \left(\frac{2}{\pi} + \frac{12}{\pi^3}\right)^2 \pi$$

Pertanto $\|\varepsilon_3\|_{L^2} = \sqrt{\frac{\pi}{2} + \frac{1}{\pi}\left(2 + \frac{12}{\pi^2}\right)^2}$

e finalmente

$$e_3 = \frac{\sin x - \frac{2}{\pi} - \frac{12}{\pi^3}}{\sqrt{\frac{\pi}{2} + \frac{1}{\pi}\left(2 + \frac{12}{\pi^2}\right)^2}} .$$

Per rispondere alla domanda (d) conviene utilizzare $\{e_1, e_2, e_3\}$. Infatti la matrice di Gram è diagonale

e i coefficienti si riducono a

$$c_1 = \int_0^\pi \frac{1}{\sqrt{\pi}} |x-1| \, dx, \quad c_2 = \int_0^\pi \frac{2\sqrt{3}}{\pi^{3/2}} \left(x - \frac{\pi}{2}\right) |x-1| \, dx$$

$$c_3 = \int_0^\pi \frac{\sin x - \frac{2}{\pi} - \frac{12}{\pi^3}}{\sqrt{\frac{\pi}{2} + \frac{1}{\pi}\left(2 + \frac{12}{\pi^2}\right)^2}} |x-1| \, dx$$

e la proiezione sarà

$$c_1 e_1 + c_2 e_2 + c_3 e_3 .$$