

$$\text{Siano } \chi_{[-1,1]} = \begin{cases} 1, & x \in [-1,1] \\ 0, & x \in \mathbb{R} \setminus [-1,1] \end{cases}$$

$$\text{e } \chi_{[2,3]} = \begin{cases} 1, & x \in [2,3] \\ 0, & x \in \mathbb{R} \setminus [2,3] \end{cases}$$

Calcolare  $\chi_{[-1,1]} * \chi_{[2,3]}$ .

Le funzioni  $\chi_{[-1,1]}$  e  $\chi_{[2,3]}$  appartengono a  $L^1(\mathbb{R})$ . Quindi la convoluzione è ben definita. In particolare,

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{\mathbb{R}} \chi_{[-1,1]}(x-y) \chi_{[2,3]}(y) dy.$$

$$= \int_2^3 \chi_{[-1,1]}(x-y) dy = \int_{x-2}^{x-3} \chi_{[-1,1]}(t) dt = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt$$

(Cambiamento di variabile)

Caso (1)  $x-3 < x-2 < -1$

Caso (2)  $x-3 < -1 < x-2 < 1$

Caso (3)  $x-3 < -1 < 1 < x-2$

Caso (4)  $-1 < x-3 < 1 < x-2$

$$\text{Caso 5} \quad 1 < x-3 < x-2$$

$$\text{Caso 6} \quad -1 < x-3 < x-2 < 1$$

Ricapitolando

$$\text{Caso 1} \quad (x-3 < x-2 < -1) \Rightarrow x < 1$$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt = 0$$

$$\text{Caso 2} \quad (x-3 < -1 < x-2 < 1) \Rightarrow 1 < x < 2$$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt = \int_{-1}^{x-2} dt = x-2+1 = x-1$$

$$\text{Caso 3} \quad (x-3 < -1 < 1 < x-2) \Rightarrow \emptyset$$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt = \int_{-1}^1 dt = 2$$

$$\text{Caso 4} \quad (1 < x-3 < 1 < x-2) \Rightarrow 3 < x < 4$$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^1 dt = 1-x+3 = 4-x$$

$$\text{Caso 5} \quad (1 < x-3 < x-2) \Rightarrow x > 4$$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt = 0$$

Caso 6  $-1 < x-3 < x-2 < 1 \Rightarrow 2 < x < 3$

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \int_{x-3}^{x-2} \chi_{[-1,1]}(t) dt = \int_{x-3}^{x-2} dt = x-2-x+3 = 1$$

Disegniamo il grafico di  $\chi_{[-1,1]} * \chi_{[2,3]}: \mathbb{R} \rightarrow \mathbb{R}$ .

$$\chi_{[-1,1]} * \chi_{[2,3]}(x) = \begin{cases} 0, & x < 1 \\ x-1, & x \in [1, 2) \\ 1, & x \in [2, 3) \\ 4-x, & x \in [3, 4) \\ 0, & x \geq 4 \end{cases}$$

