$$\begin{cases} \Delta u + u_x = \beta u & \text{in } (0,1)\times(0,1) \\ u(0,y)=0, & y\in[0,1] \\ u(1,y)=0, & y\in[0,1] \\ u(x,0)=0, & x\in[0,1] \\ u(x,1)=w & x\in[0,1] \end{cases}$$

, $w \in C^1([0,1])$, $w(0)=0=w(1)$ e $\beta>0$.

Separando le variabili otteniamo $\frac{X''}{X}+\frac{Y''}{Y}+\frac{X'}{X}=\beta$, da cui $\frac{X''}{X}+\frac{X'}{X}=\beta-\frac{Y''}{Y}=\lambda$. Risolviamo

$$\begin{cases} X''+X'=\lambda X \\ \left.\begin{array}{l} X(0)=0 \\ X(1)=0 \end{array}\right\} \longrightarrow \text{discendono dalle condizioni al bordo } u(0,y)=0=u(1,y). \end{cases}$$

Se $1+4\lambda>0 \Rightarrow \gamma_{1,2} = \begin{cases} -\frac{1}{2}+\frac{\sqrt{1+4\lambda}}{2} \\ -\frac{1}{2}-\frac{\sqrt{1+4\lambda}}{2} \end{cases}$ $V_2=\text{span}\{e^{\gamma_1 x}, e^{\gamma_2 x}\}$, da cui segue

$$\begin{cases} c_1+c_2=0 \\ c_1e^{\gamma_1}+c_2e^{\gamma_2}=0 \end{cases} \Rightarrow \det\begin{bmatrix} 1 & 1 \\ e^{\gamma_1} & e^{\gamma_2}\end{bmatrix}=0 \Leftrightarrow e^{\gamma_2}-e^{\gamma_1}=0 \Leftrightarrow$$

$e^{-\frac{\sqrt{1+4\lambda}}{2}}=e^{\frac{\sqrt{1+4\lambda}}{2}}$ e ciò si verifica solo per $1+4\lambda=0$. Quindi non ci sono autovalori.

Se $1+4\lambda=0 \Rightarrow V_2=\text{span}\{e^{-\frac{1}{2}x}, xe^{-\frac{1}{2}x}\}$. Quindi $\begin{cases} c_1=0 \\ c_2=0 \end{cases}$. Non ci sono autovalori.

Se $1+4\lambda<0 \Rightarrow V_2=\text{span}\left\{e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{|1+4\lambda|}}{2}x\right), e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{|1+4\lambda|}}{2}x\right)\right\}$, allora

$$\begin{cases} c_1=0 \\ c_2e^{-1/2}\sin\left(\frac{\sqrt{|1+4\lambda|}}{2}\right)=0 \end{cases} \Rightarrow \begin{cases} c_1=0 \\ \sin\left(\frac{\sqrt{|1+4\lambda|}}{2}\right)=0 \end{cases} \Rightarrow \frac{\sqrt{|1+4\lambda|}}{2}=k\pi,$$

da cui $\lambda_k=-\frac{1}{4}(1+4k^2\pi^2)$, perché $1+4\lambda<0$ e $|1+4\lambda|=-1-4\lambda$.

Le autosoluzioni sono $e^{-1/2x}\sin(k\pi x)$.

Rimane da risolvere

$$\begin{cases} \beta-\frac{Y''}{Y}=\lambda_k=-\frac{1}{4}(1+4k^2\pi^2) \\ Y(0)=0 \quad (\text{viene da } u(x,0)=0) \end{cases}$$

Ovvero
$$\begin{cases} y'' = y\left(\beta + \frac{1}{4}(1+4k^2\pi^2)\right) \\ y(0)=0 \end{cases}$$

Per ipotesi $\beta>0$, quindi $V_{2,k} = \mathrm{span}\{\sinh(\eta_k y), \cosh(\eta_k y)\}$, dove
$\eta_k = \sqrt{\beta + \frac{1}{4}(1+4k^2\pi^2)}$. Pertanto $A_k \sinh(\eta_k 0) + B_k \cosh(\eta_k 0) = B_k = 0$. Quindi posto $u_k(x,y) = A_k \sinh(\eta_k y) \cdot e^{-\frac{1}{2}x} \sin(k\pi x)$
cerchiamo una soluzione nella forma $\sum_{k=1}^{\infty} A_k e^{-\frac{1}{2}x} \sinh(\eta_k y) \sin(k\pi x)$.
D'altra parte $\sum_{k=1}^{\infty} A_k e^{-\frac{1}{2}x} \sinh(\eta_k) \sin(k\pi x) = w(x)$ ovvero
$\sum_{k=1}^{\infty} A_k \sinh(\eta_k) \sin(k\pi x) = e^{\frac{1}{2}x} w(x)$.

Estendiamo $e^{\frac{1}{2}x} w(x)$ come funzione dispari
$$h(x) = \begin{cases} e^{\frac{1}{2}x} w(x), & x \in [0,1] \\ -e^{-\frac{1}{2}x} w(-x), & x \in [-1,0) \end{cases}$$

$h \in C([-1,1])$, perché $w(0)=0$. Inoltre estendendo la funzione periodicamente di periodo 2 risulta $h \in P_2 \cap C(\mathbb{R})$, perché per ipotesi $w(1)=w(-1)=0$, infine $h \in C^1((-1,0)\cup(0,1))$ e in $-1, 0, 1$ esistono le derivate da dx e sx. Risulta
$$\sum_{k=1}^{\infty} h_k \sin(k\pi x) = h(x)$$
e la serie converge uniformemente. Richiediamo allora
$$\sinh(\eta_k) A_k = h_k, \quad \text{cioè} \quad A_k = \frac{h_k}{\sinh(\eta_k)}.$$
Quindi la soluzione formale del problema è
$$u(x,y) = \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x} \sin(k\pi x) \sinh(\eta_k y)$$

Verifichiamo se $u \in C([0,1]\times[0,1])$. Ricordiamo che

$$\frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x}\sin(k\pi x)\sinh(\eta_k y) \in C([0,1]\times[0,1])$$

e $\sum_{k=1}^{\infty}|h_k| < +\infty$, perché sono soddisfatte le ipotesi per la convergenza uniforme della serie di Fourier (e conv. assoluta).

Perché $u \in C([0,1]\times[0,1])$ occorre che $\sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x}\sin(k\pi x)\sinh(\eta_k y)$ converga uniformemente. Se proviamo la convergenza totale avremo provato anche la convergenza uniforme.

$$\sup_{[0,1]\times[0,1]} \left|\frac{h_k e^{-\frac{1}{2}x}\sin(k\pi x)\sinh(\eta_k y)}{|\sinh(\eta_k)|}\right| \leq \max_{[0,1]\times[0,1]} |h_k| \frac{\sinh \eta_k y}{\sinh \eta_k},$$

($\sinh y \geq 0$ per $y \geq 0$). Il sinh è monotona crescente, quindi $\leq |h_k|$. Pertanto la serie in oggetto converge totalmente e uniformemente e $u \in C([0,1]\times[0,1])$.

Verifichiamo ora se $u \in C^1((0,1)\times(0,1))$. Consideriamo

$$\sum_{k=1}^{\infty}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\sin(k\pi x)\frac{h_k}{\sinh(\eta_k)}\sinh(\eta_k y) + k\pi e^{-\frac{1}{2}x} h_k \cos(k\pi x)\frac{\sinh(\eta_k y)}{\sinh(\eta_k)}\right)$$

è sufficiente verificare la convergenza uniforme di

$$\sum_{k=1}^{\infty} k\pi e^{-\frac{1}{2}x} h_k \cos(k\pi x)\frac{\sinh(\eta_k y)}{\sinh(\eta_k)}.$$

(l'altra serie è dello stesso tipo esaminato per la continuità)

Proviamo la convergenza totale in $[0,1]\times[0,1-\varepsilon]$, con $\varepsilon > 0$

$$k\pi e^{-\frac{1}{2}x}|h_k||\cos(k\pi x)|\frac{\sinh(\eta_k y)}{\sinh(\eta_k)} \leq \pi k |h_k| \frac{\sinh(\eta_k y)}{\sinh \eta_k} \leq \pi k |h_k| \frac{\sinh \eta_k (1-\varepsilon)}{\sinh \eta_k}.$$

ma $\frac{\sinh (1-\varepsilon)\eta_k}{\sinh \eta_k} \sim \frac{e^{(1-\varepsilon)\eta_k}}{e^{\eta_k}}$ (per $\eta_k \to +\infty$).

Quindi $k \frac{\sinh(1-\varepsilon)\eta_k}{\sinh \eta_k} \sim k e^{-\varepsilon \eta_k}$ e $k e^{-\varepsilon\eta_k} < e^{-\varepsilon \frac{\eta_k}{2}}$ per $k \to +\infty$, notare che $|h_k| \xrightarrow[k\to+\infty]{} 0$ perché $\sum |h_k| < +\infty$.

Pertanto $\displaystyle\sum_{k=1}^{\infty} \frac{\partial}{\partial x}\left(e^{-\frac{x}{2}} h_k \sin(k\pi x) \frac{\sinh(\eta_k y)}{\sinh(\eta_k)}\right) = \frac{\partial}{\partial x} u(x,y)$ in $[0,1]\times[0,1-\varepsilon]$

Analogamente $\displaystyle\sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}\left(e^{-\frac{x}{2}} h_k \sin(k\pi x) \sinh(\eta_k y)\right)$

$$=\sum_{k=1}^{\infty} \frac{\partial}{\partial x}\left(-\frac{1}{2} e^{-\frac{x}{2}}\sin(k\pi x) + k\pi e^{-\frac{x}{2}}\cos(k\pi x)\right)\frac{\sinh(\eta_k y)}{\sinh(\eta_k)} h_k$$

$$=\sum_{k=1}^{\infty} h_k\left(\frac{1}{4} e^{-\frac{x}{2}}\sin(k\pi x) - \frac{k\pi}{2}\cos(k\pi x) e^{-\frac{x}{2}} - \frac{k\pi}{2} e^{-\frac{x}{2}}\cos(k\pi x) - k^2\pi^2 e^{-\frac{x}{2}}\sin(k\pi x)\right)\frac{\sinh(\eta_k y)}{\sinh \eta_k}$$

Sarà sufficiente provare che la serie $\displaystyle\sum_{k=1}^{\infty} h_k k^2\pi^2 e^{-\frac{x}{2}}\sin(k\pi x)\frac{\sinh(\eta_k y)}{\sinh(\eta_k)}$

è totalmente convergente in $[0,1]\times[0,1-\varepsilon]$, $\varepsilon>0$. Infatti

$$\left| h_k k^2\pi^2 e^{-\frac{x}{2}}\sin(k\pi x)\frac{\sinh(\eta_k y)}{\sinh(\eta_k)}\right| \le |h_k| k^2\pi^2 \frac{\sinh(\eta_k y)}{\sinh \eta_k}$$

$\displaystyle\le |h_k| k^2\pi^2 \frac{\sinh(\eta_k(1-\varepsilon))}{\sinh(\eta_k)}$. Notiamo che $\displaystyle\frac{\sinh(\eta_k(1-\varepsilon))}{\sinh \eta_k} \sim e^{-\varepsilon\eta_k}$

e quindi $\displaystyle k^2\pi^2\frac{\sinh(\eta_k(1-\varepsilon))}{\sinh(\eta_k)} \le C\, e^{-\frac{\varepsilon\eta_k}{2}}$

Pertanto abbiamo la convergenza totale e quindi quella uniforme.

Verifichiamo se $\frac{\partial u}{\partial y} \in C^1((0,1)\times(0,1))$ e si può derivare per serie. In particolare

$$\sum_{k=1}^{\infty} \frac{\partial}{\partial y}\left(e^{-\frac{x}{2}} h_k \sin(k\pi x)\frac{\sinh(\eta_k y)}{\sinh \eta_k}\right) = \sum_{k=1}^{+\infty} e^{-\frac{x}{2}} h_k \sin(k\pi x)\frac{\cosh(\eta_k y)}{\sinh(\eta_k)}\eta_k.$$

Studiamo la convergenza totale in $[0,1]\times[0,1-\varepsilon]$

$$\left| e^{-\frac{x}{2}} h_k \sin(k\pi x)\frac{\cosh(\eta_k y)}{\sinh(\eta_k)}\eta_k\right| \le |h_k|\eta_k\frac{\cosh(\eta_k y)}{\sinh \eta_k} \le |h_k|\eta_k\frac{\cosh \eta_k(1-\varepsilon)}{\sinh \eta_k}.$$

osserviamo che $\displaystyle\frac{\cosh \eta_k(1-\varepsilon)}{\sinh \eta_k} \sim \frac{e^{\eta_k(1-\varepsilon)}}{e^{\eta_k}} \sim e^{-\eta_k\varepsilon}$, $k\to+\infty$

perché $\eta_k \to +\infty$. Quindi

$$|h_k|\eta_k\frac{\cosh \eta_k(1-\varepsilon)}{\sinh \eta_k} \le C\,|h_k|\, e^{-\eta_k\frac{\varepsilon}{2}}.$$ e la serie è totalmente convergente e $\displaystyle\frac{\partial u}{\partial y} = \sum_{k=1}^{\infty}\eta_k e^{-\frac{x}{2}} h_k \sin(k\pi x)\frac{\cosh(\eta_k y)}{\sinh(\eta_k)}$.

Verifichiamo se $\frac{\partial^2 u}{\partial y^2} = \sum_{k=1}^{\infty} \frac{\partial^2}{\partial y^2}\left(e^{-\frac{x}{2}} h_k \sin(k\pi x) \frac{\sinh(\eta_k y)}{\sinh \eta_k} \right)$

Poiché $\sum_{k=1}^{\infty} \frac{\partial^2}{\partial y^2}\left(e^{-\frac{x}{2}} h_k \sin(k\pi x) \frac{\sinh(\eta_k y)}{\sinh \eta_k} \right)$

$= \sum_{k=1}^{\infty} \eta_k^2 e^{-\frac{x}{2}} h_k \sin(\pi k x) \frac{\sinh(\eta_k y)}{\sinh(\eta_k)}$; verifichiamo la convergenza

totale $\sum_{k=1}^{\infty} \left| \eta_k^2 h_k e^{-\frac{x}{2}} \sin(k\pi x) \frac{\sinh(\eta_k y)}{\sinh \eta_k} \right|$

$$\le \sum_{k=1}^{\infty} \eta_k^2 |h_k| \frac{\sinh \eta_k(1-\varepsilon)}{\sinh \eta_k} \le c \sum_{k=1}^{\infty} \eta_k^2 |h_k| e^{-\eta_k \varepsilon} \le c' \sum_{k=1}^{\infty} e^{-\eta_k \frac{\varepsilon}{2}} < +\infty$$

Pertanto $u \in C^2((0,1)\times(0,1)) \cap C([0,1]\times[0,1])$ con

$$u(x,y) = \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x} \sin(k\pi x) \sinh(\eta_k y)$$

Verifichiamo il risultato: in $(0,1)\times(0,1)$

$\Delta u + u_x = \beta u$

$$\Delta u + u_x = \sum_{k=1}^{\infty} \frac{h_k}{\sinh \eta_k} \left[\Delta\left(e^{-\frac{1}{2}x} \sin(k\pi x) \sinh(\eta_k y)\right) + \frac{\partial}{\partial x}\left(e^{-\frac{1}{2}x} \sin(k\pi x) \sinh(\eta_k y)\right)\right]$$

$$= \sum_{k=1}^{\infty} \frac{h_k}{\sinh \eta_k} \left[\frac{1}{4} e^{-\frac{x}{2}} \sin(k\pi x) - \underbrace{\frac{k}{2}\pi \cos(k\pi x) e^{-\frac{x}{2}}}_{A_1} \underbrace{- \frac{k\pi}{2} \cos(k\pi x)}_{A_2} - k^2\pi^2 e^{-\frac{x}{2}} \sin(k\pi x)\right] \sinh(\eta_k y)$$

$$+ \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x} \sin(k\pi x)\, \eta_k^2 \sinh(\eta_k y)$$

$$+ \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} \left(-\frac{1}{2} e^{-\frac{1}{2}x} \sin(k\pi x) \sinh(\eta_k y) + \underbrace{e^{-1/2 x} k\pi \cos(k\pi x) \sinh(\eta_k y)}_{A_3} \right)$$

$$= \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} e^{-\frac{1}{2}x} \left[\frac{1}{4} - k^2\pi^2 + \eta_k^2 - \frac{1}{2} \right] \sinh(\eta_k y) \sin(k\pi x)$$

$$= \sum_{k=1}^{\infty} \frac{h_k}{\sinh(\eta_k)} \left[-\frac{1}{4} - k^2\pi^2 + \beta + \frac{1}{4}(1 + 4k^2\pi^2) \right] \sinh(\eta_k y) \sin(k\pi x)$$

$$= \sum_{k=1}^{\infty} \frac{\beta h_k}{\sinh(\eta_k)} \sinh(\eta_k y) \sin(k\pi y) = \beta u.$$