APPENDIX A

Quasilinear Parabolic Equations on Manifolds

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This is the second order case in the paper **[72]** by Huisken and Polden (or in the Ph.D. Thesis of Polden **[103]**).

Let (M, g) be a smooth, compact, n–dimensional Riemannian manifold. Let us consider the following PDE problem with a smooth initial datum $u_0 : M \to \mathbb{R}$,

$$\begin{cases} u_t = L(u) & \text{in } M \times [0, T] \\ u(\,\cdot\,, 0) = u_0 & \text{on } M\,, \end{cases} \tag{PDE}$$

where the operator L is a second order quasilinear differential operator defined in $M \times [0, \mathcal{T})$ (for some $\mathcal{T} > 0$), acting on a function $u : M \times [0, T] \to \mathbb{R}$ with $T < \mathcal{T}$ as follows (in a coordinate chart),

$$L(u) = Q^{ij}(p, t, u, \nabla u)\nabla^2_{ij}u + b(p, t, u, \nabla u)\,,$$

where Q^{ij} and b are smooth functions. Moreover, the operator L is locally elliptic, that is, around every point $p \in M$ there is a coordinate chart such that $Q^{ij} = Q^{ij}(q, s, z, w)$ is a positive definite matrix with lowest eigenvalue uniformly bounded from below away from zero for (q, s, z, w) in some neighborhood of any (p, t, x, v), with $t \in [0, \mathcal{T})$, $x \in \mathbb{R}$ and $v \in T^*_pM$.
It is easy to check that these assumptions on the quasilinear operator are independent of the choice of the coordinate charts.

In order to show the existence of a solution of problem (PDE) in some positive (small) time interval, first we show the existence of a weak solution if the system is linear, then we show its regularity, finally we will deal with the quasilinear case by means of a linearization procedure.

A.1. The Linear Case

In all this section we assume that system (PDE) is linear, that is, $L(u) = Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b$ and

$$\begin{cases} u_t - Q^{ij}\nabla^2_{ij}u - R^k\nabla_k u - Su = b \\ u(\,\cdot\,, 0) = u_0\,, \end{cases} \tag{A.1.1}$$

where Q^{ij}, R^k, S and b are smooth functions dependent only on $p \in M$ and on $t \in [0, +\infty)$ (not on the function u or its gradient). Moreover, we suppose that

- the smooth functions R^k, S, b are bounded in C^∞,
- the functions Q^{ij} are bounded in C^∞ and there exists a uniform ellipticity constant $\lambda > 0$ of Q^{ij}.

We set $\widetilde{L}(u) = L(u) - b$. Integrating by parts and using Peter–Paul inequality, we have the following standard Gårding's inequality, for every smooth u,

$$-\int_M u\widetilde{L}(u)\,d\mu \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(M)} - C\|u\|^2_{L^2(M)}\,, \tag{A.1.2}$$

for every $t \in [0, +\infty)$, where the constant $C > 0$ depends only on the C^1–norms of the functions Q^{ij}, R^k and S (see **[72]** for details).

If u is solution with smooth initial data u_0 of problem (A.1.1), then $v = ue^{-Ct}$ is a solution of problem

$$\begin{cases} v_t - Q^{ij}\nabla^2_{ij}v - R^k\nabla_k v - Sv - Cv = be^{-Ct} \\ v(\,\cdot\,,0) = u_0\,. \end{cases}$$

and viceversa. Moreover, choosing C larger than the constant appearing in the Gårding's inequality (A.1.2), the associated linear elliptic operator $\widetilde{L}'(v) = Q^{ij}\nabla^2_{ij}v + R^k\nabla_k v + Sv + Cv$ satisfies the analogous inequality

$$-\int_M v\widetilde{L}'(v)\,d\mu \geq \frac{\lambda}{2}\|v\|^2_{W^{1,2}(M)}$$

without the negative term.

By the definition that follows, it will be clear that all the spaces and regularity issues (that we are going to discuss in the next sections) are not affected by multiplication for exponential functions depending only on time, hence, we will assume from now on that Gårding's inequality for our parabolic problem operator holds as follows,

$$-\int_M u\widetilde{L}(u)\,d\mu \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(M)}\,. \tag{A.1.3}$$

We define now the function spaces where we will prove the existence of a weak solution of problem (A.1.1).

DEFINITION A.1.1. Given $a > 0$, for every pair of functions $f, g \in C^\infty_c(M \times [0, +\infty))$ we set

$$\begin{aligned} \langle f, g\rangle_{LL_a(M)} &= \int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t), g(\,\cdot\,,t)\rangle_{L^2(M)}dt\,, \\ \langle f, g\rangle_{LW_a(M)} &= \int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t), g(\,\cdot\,,t)\rangle_{W^{1,2}(M)}dt\,, \\ \langle f, g\rangle_{WW_a(M)} &= \langle f, g\rangle_{LW_a(M)} + \langle f_t, g_t\rangle_{LL_a(M)} \end{aligned}$$

and let $LL_a(M)$, $LW_a(M)$ and $WW_a(M)$ be the Hilbert spaces obtained by completion with respect to the relative norms.

Suppose to have a smooth solution $u \in C^\infty(M \times [0, +\infty))$ of problem (A.1.1), then if $\phi \in C^\infty_c(M \times (0, +\infty))$ we have,

$$\begin{aligned} 0 &= \int_0^{+\infty} e^{-2at}\int_M \phi(u_t - L(u))\,d\mu\,dt \\ &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\ &\quad - \int_0^{+\infty} e^{-2at}\int_M \phi L(u)\,d\mu\,dt \\ &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\ &\quad - \int_0^{+\infty} e^{-2at}\int_M \phi(Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b)\,d\mu\,dt \\ &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\ &\quad + \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,d\mu\,dt \\ &\quad - \int_0^{+\infty} e^{-2at}\int_M \phi\,b\,d\mu\,dt\,. \end{aligned}$$

Notice that, by integration by parts, this equation has a meaning also if u is merely in $LW_a(M)$, so we can use it to define a weak solution of the linear problem.

DEFINITION A.1.2. We say that $u \in LW_a(M)$ is a *weak* solution of

$$u_t = Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b\,, \tag{A.1.4}$$

if the following equality holds for every $\phi \in C^\infty_c(M \times (0, +\infty))$

$$\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\
&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,d\mu\,dt \\
&= \int_0^{+\infty} e^{-2at}\int_M \phi\,b\,d\mu\,dt\,.
\end{aligned} \tag{A.1.5}$$

If a weak solution u is smooth it is easy to see that u is a *classical* solution of the parabolic equation in (A.1.1).

We let Φ be the space of functions $C^\infty_c(M \times (0, +\infty))$, which are clearly zero for small time.

If $WW_{a,0}(M)$ is the completion of Φ with respect to the norm of $WW_a(M)$, asking that a function u belongs to this space is a *weak way* to express the condition $u(\,\cdot\,, 0) = 0$.

By simplicity, we define the bilinear form

$$B(u, \phi) = Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,,$$

and we define $P : WW_{a,0}(M) \times \Phi \to \mathbb{R}$ and $K : \Phi \to \mathbb{R}$ as follows,

$$\begin{aligned}
P(u, \phi) &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_{tt} u\,d\mu\,dt \\
&\quad+\int_0^{+\infty} e^{-2at}\int_M B(u, \phi_t)\,d\mu\,dt\,, \\
K(\phi) &= \int_0^{+\infty} e^{-2at}\int_M \phi_t\,b\,d\mu\,dt\,,
\end{aligned}$$

then a weak solution $u \in WW_{a,0}(M)$ satisfies $P(u, \phi) = K(\phi)$ for every $\phi \in \Phi$.
Notice that K is a continuous functional on Φ (with the norm of $WW_a(M)$) when $b \in LL_a(M)$.

We need now the following variation of Lax–Milgram whose proof can be found in [**45**, Chapter 10, Theorem 16].

LEMMA A.1.3. *Let H be a Hilbert space and Φ a space with a scalar product (not necessarily complete) continuously embedded in H. Moreover, let $P : H \times \Phi \to \mathbb{R}$ be a bilinear form such that*

- *$h \mapsto P(h, \phi)$ is continuous for every fixed $\phi \in \Phi$,*
- *$P|_\Phi$ is coercive, that is, there exists a positive constant C such that $P(\phi, \phi) \geq C\|\phi\|^2$, for every $\phi \in \Phi$.*

Then, for every $K \in \Phi^*$ there exists $h \in H$ such that $K(\phi) = P(h, \phi)$ for every $\phi \in \Phi$.

PROPOSITION A.1.4. *If $b \in LL_a(M)$ and u_0 is smooth, the problem* (A.1.1) *has a weak solution $u \in WW_a(M)$, for $a > 0$ large enough, that is, $u - u_0 \in WW_{a,0}(M)$ and u is a weak solution of equation* (A.1.4).

PROOF. First we assume that $u_0 = 0$.
We check the hypotheses of Lemma A.1.3. This would imply that there exists a function $u \in WW_{a,0}(M)$ such that it satisfies equation (A.1.5) for every function $\phi = \psi_t$, where $\psi \in \Phi$.

Fixing $\phi \in \Phi$, a repeated application of Hölder's inequality shows easily that $P(\cdot, \phi)$ is continuous with respect to the norm of $WW_a(M)$ and we already noticed that K is continuous on Φ, considering the $WW_a(M)$–norm on it. We only need to show the coerciveness of P on Φ.
Keeping in mind that ϕ is regular and has compact support in $M \times (0, +\infty)$, integrating by parts,

we get

$$\begin{aligned}P(\phi,\phi) &= \int_0^{+\infty} e^{-2at}\int_M \phi_t^2\,d\mu\,dt + \int_0^{+\infty} e^{-2at}\int_M B(\phi,\phi_t)\,d\mu\,dt\\ &= \int_0^{+\infty} e^{-2at}\int_M \phi_t^2\,d\mu\,dt + \int_0^{+\infty} +a\int_0^{+\infty} e^{-2at}\int_M B(\phi,\phi)\,d\mu\,dt\\ &\quad - \int_0^{+\infty} e^{-2at}B'(\phi,\phi)\,d\mu\,dt\,,\end{aligned}$$

where

$$B'(\phi,\phi) = Q^{ij}_t\nabla_i\phi\nabla_j\phi + \phi\nabla_j\phi\nabla_i Q^{ij}_t - \phi R^k_t\nabla_k\phi - \phi S_t\phi\,.$$

Hence, using Gårding's inequality (A.1.3) and estimating $B'(\phi,\phi)\leq C\|\phi\|^2_{LW_a(M)}$, for some constant C, we get

$$P(\phi,\phi)\geq \|\phi\|^2_{LL_a(M)} + \Big(\frac{a\lambda}{2} - C\Big)\|\phi\|^2_{LW_a(M)}\,.$$

Choosing any $a > 2C/\lambda$, the coerciveness of P on Φ with the $WW_a(M)$–norm follows.

We want to show now that for such $u\in WW_{a,0}(M)$, actually equation (A.1.5) holds for *every* $\phi\in\Phi$, not only $\phi=\psi_t$ for some $\psi\in\Phi$. Considering any $\phi\in\Phi$, we set

$$\widetilde{\phi}(x,t) = \phi(x,t) - \phi(x,t-C)$$

(setting $\phi(x,t)=0$ when $t-C\leq 0$) and we notice that if C is large enough $\widetilde{\phi}\in\Phi$ and it is the time–derivative of the function $\psi(x,t)=\int_{t-C}^t\phi(s,x)\,ds$ which is also in Φ. Hence, equation (A.1.5) holds with $\widetilde{\phi}$ instead of ϕ, then, sending $C\to+\infty$, it is easy to see that the contributions given by the function $\phi(x,t-C)$ go to zero because of the exponential factor e^{-2at}. Thus, we can conclude that we have a weak solution $u\in WW_{a,0}(M)$ of the problem (A.1.1) with $u_0=0$.

Suppose now that the smooth initial datum u_0 is not identically zero. We consider the equation satisfied by the function $v=u-u_0$, with a null initial datum and we solve it with the previous method.

The last term of this new problem is $L(u_0)=\widetilde{L}(u_0)+b$ (the other parts of the operator are the same, by linearity), then the regularity of u_0 implies that it satisfies the hypotheses for the existence of a weak solution $v\in WW_{a,0}(M)$, hence of $u\in WW_a(M)$ as in the statement of the proposition. □

LEMMA A.1.5. *If $u\in WW_a(M)$ is a weak solution of problem* (A.1.1), *then for every $\phi\in C^\infty_c(M\times[0,+\infty))$, the following equation holds.*

$$\begin{aligned}&2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt - \int_M \phi u_0\,d\mu\,, \qquad\text{(A.1.6)}\\ &+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi S u\,d\mu\,dt\\ &=\int_0^{+\infty} e^{-2at}\int_M \phi\, b\,d\mu\,dt\,.\end{aligned}$$

PROOF. Let $v=u-u_0\in WW_{a,0}(M)$ satisfying the relative equation (A.1.5) in the modified system, that is,

$$\begin{aligned}&2a\int_0^{+\infty} e^{-2at}\int_M \phi v\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t v\,d\mu\,dt \qquad\text{(A.1.7)}\\ &+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j v + \phi\nabla_j v\nabla_i Q^{ij} - \phi R^k\nabla_k v - \phi S v\,d\mu\,dt\\ &=\int_0^{+\infty} e^{-2at}\int_M \phi\, L(u_0) + b\,d\mu\,dt\,,\end{aligned}$$

for every function $\phi\in C^\infty_c(M\times(0,+\infty))$.

Let now $\phi=\sigma\varphi$ where $\varphi\in C^\infty_c(M\times[0,+\infty))$ and $\sigma:[0+\infty)\to\mathbb{R}$ is a function which is zero

on $[0,\varepsilon]$, one on $[2\varepsilon,+\infty)$ and linear in the middle (we can "put" such a function σ in the formula above by approximation).
We compute,

$$
\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\sigma\int_M \varphi v\,d\mu\,dt-\int_0^{+\infty} e^{-2at}\sigma\int_M \varphi_t v\,d\mu\,dt\\
&-\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} e^{-2at}\int_M \varphi v\,d\mu\,dt\\
&+\int_0^{+\infty} e^{-2at}\sigma\int_M Q^{ij}\nabla_i\varphi\nabla_j v+\phi\nabla_j v\nabla_i Q^{ij}-\phi R^k\nabla_k v-\phi Sv\,d\mu\,dt\\
&=\int_0^{+\infty} e^{-2at}\sigma\int_M \varphi\,L(u_0)+b\,d\mu\,dt\,.
\end{aligned}
$$

As $\varepsilon\to 0$, the term $\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} e^{-2at}\int_M \varphi v\,d\mu\,dt$ converges to zero as $v\in WW_{a,0}(M)$ (since $u_t\in LL_a(M)$) and the other terms converge to the corresponding ones without σ inside, as $\sigma\to\chi_{[0,+\infty)}$ when $\varepsilon\to 0$. Hence, v satisfies relation (A.1.7) for every $\phi\in C_c^\infty(M\times[0,+\infty))$.
Substituting $v=u-u_0$ we get

$$
\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt-\int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt\\
&-2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt+\int_0^{+\infty} e^{-2at}\int_M \phi_t u_0\,d\mu\,dt\\
&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u+\phi\nabla_j u\nabla_i Q^{ij}-\phi R^k\nabla_k u-\phi Su\,d\mu\,dt\\
&=\int_0^{+\infty} e^{-2at}\int_M \phi\,b\,d\mu\,dt\,,
\end{aligned}
$$

and the second line is equal to

$$
\begin{aligned}
-2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt+\int_0^{+\infty} e^{-2at}\int_M \phi_t u_0\,d\mu\,dt
&=-2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt\\
&\quad+\int_0^{+\infty} e^{-2at}\frac{d}{dt}\int_M \phi u_0\,d\mu\,dt\\
&=\int_0^{+\infty}\frac{d}{dt}\Big[e^{-2at}\int_M \phi u_0\,d\mu\Big]\,dt\\
&=-\int_M \phi u_0\,d\mu\,,
\end{aligned}
$$

which gives the thesis of the lemma. □

A.2. Regularity in the Linear Case

DEFINITION A.2.1. Let

$$
LW_a^s(M)=\left\{f:M\times[0,+\infty)\to\mathbb{R}\;\middle|\;\int_0^{+\infty} e^{-2at}\|f(\,\cdot\,,t)\|^2_{W^{s,2}(M)}\,dt<+\infty\right\}
$$

with the scalar product

$$
\langle f,g\rangle_{LW_a^s(M)}=\int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t),g(\,\cdot\,,t)\rangle_{W^{s,2}(M)}\,dt\,.
$$

Moreover,

$$
P_a^l(M)=\left\{f:M\times[0,+\infty)\to\mathbb{R}\;\middle|\;\frac{\partial^j f}{\partial t^j}\in LW_a^{2(l-j)}(M)\;\forall j\le l\right\},
$$

where $\frac{\partial^j f}{\partial t^j}$ is in distributional sense. Clearly $P^l_a(M) \subset LW^{2l}_a(M)$.
On this space we have the following scalar product

$$\langle f, g\rangle_{P^l_a(M)} = \sum_{j\leq l}\left\langle \frac{\partial^j f}{\partial t^j}\,, \frac{\partial^j g}{\partial t^j}\right\rangle_{LW_a^{2(l-j)}(M)}.$$

LEMMA A.2.2. *The trace f_0 of a function $f \in P^l_a(M)$ on the parabolic boundary $M \times \{0\}$ belongs to the space $W^{2l-1,2}(M)$.*

PROOF. It is easy to see that $C^\infty_c(M\times[0,+\infty))$ is dense in $P^l_a(M)$. If f is smooth with compact support, we have

$$\begin{aligned}\int_0^{+\infty} e^{-2at}\int_M g\Big(\nabla^{2l-1}f, \nabla^{2l-1}\frac{\partial f}{\partial t}\Big)\,d\mu\,dt &= \frac{1}{2}\int_0^{+\infty} e^{-2at}\int_M \frac{\partial}{\partial t}|\nabla^{2l-1}f|^2\,d\mu\,dt\\ &= a\int_0^{+\infty} e^{-2at}\int_M |\nabla^{2l-1}f|^2\,d\mu\,dt\\ &\quad -\frac{1}{2}\int_M |\nabla^{2l-1}f_0|^2\,d\mu\,.\end{aligned}$$

Hence, keeping in mind the definition of the space $P^l_a(M)$,

$$\begin{aligned}\int_M |\nabla^{2l-1}f_0|^2\,d\mu &= 2a\int_0^{+\infty} e^{-2at}\int_M |\nabla^{2l-1}f|^2\,d\mu\,dt\\ &\quad -2\int_0^{+\infty} e^{-2at}\int_M g\Big(\nabla^{2l-1}f, \nabla^{2l-1}\frac{\partial f}{\partial t}\Big)\,d\mu\,dt\\ &= 2a\int_0^{+\infty} e^{-2at}\int_M |\nabla^{2l-1}f|^2\,d\mu\,dt\\ &\quad +2\int_0^{+\infty} e^{-2at}\int_M g\Big(\Delta\nabla^{2l-2}f, \nabla^{2l-2}\frac{\partial f}{\partial t}\Big)\,d\mu\,dt\\ &\leq 2a\|f\|^2_{LW^{2l-1}_a(M)} + 2\|f\|_{LW^{2l}_a(M)}\|f\|_{P^l_a(M)}\\ &\leq 3a\|f\|^2_{P^l_a(M)}\,.\end{aligned}$$

Then, the conclusion follows by approximation. □

We are now ready to state the main result of this section.

PROPOSITION A.2.3. *For every $l \in \mathbb{N}$ the linear map*

$$F(u) = \Big(u_0, u_t - \widetilde{L}(u)\Big) \tag{A.2.1}$$

is an isomorphism of $P^l_a(M)$ onto $W^{2l-1,2}(M) \times P^{l-1}_a(M)$, for $a > 0$ large enough.

By the previous lemma the map F is well defined and bounded, we only need to show that it is a bijection, by the open mapping theorem.

LEMMA A.2.4. *If $u \in WW_a(M)$ satisfies equation* (A.1.6) *for every $\phi \in C^\infty_c(M \times [0,+\infty))$ then, the following estimate holds,*

$$\|u\|^2_{LW_a(M)} \leq C\left(\|u_0\|^2_{L^2(M)} + \|b\|^2_{LL_a(M)}\right)\,.$$

PROOF. Let $\phi \in C_c^\infty(M \times [0,+\infty))$, as $u_t \in LL_a(M)$, by the density of $C_c^\infty(M \times [0,+\infty))$ in $LW_a(M))$, we can substitute ϕ with u, obtaining

$$\begin{aligned}&2a\int_0^{+\infty} e^{-2at}\int_M u^2\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M u_t u\,d\mu\,dt - \int_M u_0^2\,d\mu\\&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i u\nabla_j u + u\nabla_j u\nabla_i Q^{ij} - uR^k\nabla_k u - Su^2\,d\mu\,dt\\&=\int_0^{+\infty} e^{-2at}\int_M u\,b\,d\mu\,dt\,.\end{aligned}$$

Taking the time derivative outside the inner integral of the second term and integrating by parts, we get

$$\begin{aligned}&a\int_0^{+\infty} e^{-2at}\varphi\int_M u^2\,d\mu\,dt - \frac{1}{2}\int_M u_0^2\,d\mu\\&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i u\nabla_j u + u\nabla_j u\nabla_i Q^{ij} - uR^k\nabla_k u - Su^2\,d\mu\,dt\\&=\int_0^{+\infty} e^{-2at}\int_M u\,b\,d\mu\,dt\,.\end{aligned}$$

By Gårding's inequality (A.1.3) this formula implies

$$\frac{\lambda}{2}\|u\|^2_{LW_a(M)} \leq -a\|u\|^2_{LL_a(M)} + \frac{1}{2}\|u_0\|^2_{L^2(M)} + \|u\|_{LL_a(M)}\|b\|_{LL_a(M)}$$

and using Peter–Paul inequality on the last term,

$$\|b\|_{LL_a(M)}\|u\|_{LL_a(M)} \leq \varepsilon\|u\|^2_{LL_a(M)} + C\|b\|^2_{LL_a(M)} \leq \varepsilon\|u\|^2_{LW_a(M)} + C\|b\|^2_{LL_a(M)}\,,$$

the lemma follows by choosing $\varepsilon < \lambda/4$. ☐

In order to get estimates on the higher derivatives of u we work with the incremental ratios, being the ambient space M a manifold we need to use local charts.
Given $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h \neq 0$, fixing $v \in R^n$, let

$$(\Theta_h f)(x) = h^{-1}(f(x+hv) - f(x))\,.$$

The following properties of the operators Θ_h are easily checked.

- If $f, g : \mathbb{R}^n \to \mathbb{R}^n$ then
$$(\Theta_h(fg))(x) = (\Theta_h f)(x)g(x+hv) + (\Theta_h g)(x)f(x)\,.$$
- If $f, g \in L^1(\mathbb{R}^n)$ with compact support contained in an open set $\Omega \subset \mathbb{R}^n$, then for h small enough we have
$$\int_\Omega \Theta_h f\,dx = 0 \qquad \text{and} \qquad \int_\Omega f\,\Theta_h g\,dx = -\int_\Omega g\,\Theta_{-h} f\,dx\,.$$

Let $\psi_l : \mathbb{R}^n \supset B_1^n \to M$, for $l = 1,\dots,m$ be a family of local charts such that the union of $\psi_l(B^n_{1/2})$ covers M (B^n_r is the n–dimensional ball of radius r).
Moreover, let $\rho : \mathbb{R}^n \to [0,1]$ be a smooth function which is 0 outside $B^n_{3/4}$ and 1 in $B^n_{1/2}$. We define $U_l = \psi_l(B^n_1)$ and $V_l = \psi_l(B^n_{1/2})$.

We "lift" now Θ and ρ on M via the coordinate charts ψ_l, still using the same notation. Notice that the relations above still hold for functions on M whose support is contained in a single chart.

LEMMA A.2.5. *If $u \in WW_a(M)$ is satisfies equation* (A.1.6) *with a smooth u_0, then $u \in LW^2_a(M)$ and the following estimate holds,*

$$\|u\|^2_{LW^2_a(M)} \leq C\left(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\right)\,. \tag{A.2.2}$$

PROOF. We prove the estimate

$$\|\nabla^2 u\|^2_{LL_a(M)} \leq C\left(\|u_0\|^2_{W^{1,2}(M)} + \|u\|^2_{LW_a(M)} + \|b\|^2_{LL_a(M)}\right), \tag{A.2.3}$$

then the conclusion follows by means of Lemma A.2.4.
We fix a chart ψ_l and we consider the test function $\phi = \Theta_{-h}(\rho^2\Theta_h u)$ extended to 0 outside U_l in equation (A.1.6), integrating by parts, as $u_t \in LL_a(M)$, we get

$$\Big\langle u_t, \Theta_{-h}(\rho^2\Theta_h u)\Big\rangle_{LL_a(M)} + \int_0^{+\infty} e^{-2at}\int_M B(u, \Theta_{-h}(\rho^2\Theta_h u))\,d\mu\,dt = \langle b, \Theta_{-h}(\rho^2\Theta_h u)\rangle_{LL_a(M)},$$

recalling that we defined the form B as

$$B(u,\phi) = Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi S u\,.$$

Now "moving" the incremental ratios and integrating by parts, we obtain

$$\begin{aligned}
-\langle b, \Theta_{-h}(\rho^2\Theta_h u)\rangle_{LL_a(M)} &= \Big\langle \frac{\partial(\Theta_h u)}{\partial t}, \rho^2\Theta_h u\Big\rangle_{LL_a(M)} \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&= \Big\langle \frac{\partial(\rho\Theta_h u)}{\partial t}, \rho\Theta_h u\Big\rangle_{LL_a(M)} \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&= a\int_0^{+\infty} e^{-2at}\int_M \rho^2|\Theta_h u|^2\,d\mu\,dt - \frac{1}{2}\int_M \rho^2|\Theta_h u_0|^2\,d\mu \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt
\end{aligned}$$

where the term $(\Theta_h B)(u, \rho^2\Theta_h u)$ is given by the application of the Leibniz rule for the incremental ratios. Anyway, this term is not a problem as all the coefficients of the form B are bounded in C^∞.
Hence, we have,

$$\begin{aligned}
\int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt &+ \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\leq C\|b\|_{LL_a(M)}\|\Theta_{-h}(\rho^2\Theta_h u)\|_{LL_a(M)} + C\|\Theta_h u\|^2_{LL_a(U_l)} + \frac{1}{2}\int_M \rho^2|\Theta_h u_0|^2\,d\mu \\
&\leq C\|b\|_{LL_a(M)}\|\rho^2\Theta_h u\|_{LW_a(M)} + C\|u\|^2_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)},
\end{aligned}$$

by the standard integral estimates on the incremental ratios.

We deal now with the two integrals in this formula. For the first one that we call $\mathcal{A}$, after some manipulations with the Leibniz rule, we have

$$
\begin{aligned}
\mathcal{A} &= \int_0^{+\infty} e^{-2at} \int_M B(\Theta_h u, \rho^2 \Theta_h u)\, d\mu\, dt \\
&\geq \int_0^{+\infty} e^{-2at} \int_M \rho Q^{ij} \nabla_i \Theta_h u \nabla_j (\rho \Theta_h u)\, d\mu\, dt \\
&\quad - C \int_0^{+\infty} e^{-2at} \int_M \rho |\nabla \Theta_h u|\, |\Theta_h u|\, |\nabla \rho|\, d\mu\, dt \\
&\quad - C \int_0^{+\infty} e^{-2at} \int_M \rho^2 |\Theta_h u| (|\nabla \Theta_h u| + |\Theta_h u|)\, d\mu\, dt \\
&\geq \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i (\rho \Theta_h u) \nabla_j (\rho \Theta_h u)\, d\mu\, dt \\
&\quad - C \int_0^{+\infty} e^{-2at} \int_M |\nabla(\rho \Theta_h u)|\, |\Theta_h u|\, (\rho + |\nabla \rho|)\, d\mu\, dt \\
&\quad - C \int_0^{+\infty} e^{-2at} \int_M |\Theta_h u|^2 (\rho^2 + \rho |\nabla \rho| + |\nabla \rho|^2)\, d\mu\, dt \\
&\geq \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i (\rho \Theta_h u) \nabla_j (\rho \Theta_h u)\, d\mu\, dt \\
&\quad - \varepsilon_l \int_0^{+\infty} e^{-2at} \int_M |\nabla(\rho \Theta_h u)|^2\, d\mu\, dt \\
&\quad - C_l \int_0^{+\infty} e^{-2at} \int_{U_l} |\Theta_h u|^2\, d\mu\, dt \\
&\geq \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i (\rho \Theta_h u) \nabla_j (\rho \Theta_h u)\, d\mu\, dt \\
&\quad - \varepsilon_l \|\nabla(\rho \Theta_h u)\|^2_{LL_a(M)} - C_l \|\Theta_h u\|^2_{LL_a(U_l)}\,,
\end{aligned}
$$

for some small $\varepsilon_l > 0$ and constants C_l that we obtained by means of the use of Peter–Paul inequality. Again, by standard estimates, we conclude

$$
\begin{aligned}
\mathcal{A} \geq \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i (\rho \Theta_h u) \nabla_j (\rho \Theta_h u)\, d\mu\, dt \\
- \varepsilon_l \|\rho \Theta_h u\|^2_{LW_a(M)} - C_l \|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

where the constants ε_l and C_l are independent of u and h (small enough).
By Gårding's inequality (A.1.3) it follows easily that

$$
\begin{aligned}
\int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i (\rho \Theta_h u) \nabla_j (\rho \Theta_h u)\, d\mu\, dt &\geq \frac{\lambda}{2} \|\rho \Theta_h u\|^2_{LW_a(M)} - C \|\rho \Theta_h u\|^2_{LL_a(M)} \\
&\geq \frac{\lambda}{2} \|\rho \Theta_h u\|^2_{LW_a(M)} - C \|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

hence,

$$
\mathcal{A} \geq \Big(\frac{\lambda}{2} - \varepsilon_l \Big) \|\rho \Theta_h u\|^2_{LW_a(M)} - C_l \|u\|^2_{LW_a(M)}\,.
$$

For the second integral that we call $\mathcal{B}$, we have

$$\begin{aligned}\mathcal{B} =& \int_0^{+\infty} e^{-2at} \int_M (\Theta_h B)(u, \rho^2\Theta_h u)\, d\mu\, dt \\ \geq& \int_0^{+\infty} e^{-2at} \int_M (\Theta_h Q^{ij})\nabla_i u \nabla_j (\rho^2 \Theta_h u)\, d\mu\, dt \\ &- C\int_0^{+\infty} e^{-2at} \int_M \rho^2|\Theta_h u|\,(|\nabla u| + |u|)\, d\mu\, dt \\ \geq& - C\int_0^{+\infty} e^{-2at} \int_M \rho|\nabla u|\,|\nabla(\rho\Theta_h u)|\, d\mu\, dt \\ &- C\int_0^{+\infty} e^{-2at} \int_M \rho|\Theta_h u|\,|\nabla u|\,|\nabla\rho|\, d\mu\, dt \\ &- C\int_0^{+\infty} e^{-2at} \int_M \rho^2|\Theta_h u|\,(|\nabla u| + |u|)\, d\mu\, dt \\ \geq& - \varepsilon_l \|\rho\Theta_h u\|^2_{LW_a(M)} - C_l\|u\|^2_{LW_a(M)}\,,\end{aligned}$$

by repeated use of Peter–Paul inequality.
Finally we obtain,

$$\begin{aligned}\Big(\frac{\lambda}{2} - \varepsilon_l\Big)\|\rho\Theta_h u\|^2_{LW_a(M)} \leq& \mathcal{A} + C_l\|u\|^2_{LW_a(M)} \\ \leq& C\|b\|_{LL_a(M)}\|\rho^2\Theta_h u\|_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\ &- \mathcal{B} + C_l\|u\|^2_{LW_a(M)} \\ \leq& C\|b\|_{LL_a(M)}\|\rho^2\Theta_h u\|_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\ &+ \varepsilon_l\|\rho\Theta_h u\|^2_{LW_a(M)} + C_l\|u\|^2_{LW_a(M)} \\ \leq& C_l\|b\|^2_{LL_a(M)} + \varepsilon_l\|\rho^2\Theta_h u\|^2_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\ &+ \varepsilon_l\|\rho\Theta_h u\|^2_{LW_a(M)} + C_l\|u\|^2_{LW_a(M)}\,.\end{aligned}$$

It can be seen easily that

$$\begin{aligned}\|\rho^2\Theta_h u\|^2_{LW_a(M)} \leq& \|\rho\Theta_h u\|^2_{LW_a(M)} + C\|\Theta_h u\|^2_{LL_a(U_l)} \\ \leq& \|\rho\Theta_h u\|^2_{LW_a(M)} + C\|u\|^2_{LW_a(M)}\,,\end{aligned}$$

as $\rho \leq 1$, hence,

$$\Big(\frac{\lambda}{2} - 3\varepsilon_l\Big)\|\rho\Theta_h u\|^2_{LW_a(M)} \leq C_l\|b\|^2_{LL_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} + C_l\|u\|^2_{LW_a(M)}\,.$$

By the arbitrariness of h, after choosing ε_l small enough, this estimate implies that

$$\|\nabla^2 u\|^2_{LL_a(V_l)} \leq C_l\Big(\|b\|^2_{LL_a(M)} + \|u_0\|^2_{W^{1,2}(M)} + \|u\|^2_{LW_a(M)}\Big)\,.$$

As the coordinate charts are finite (M is compact) we get inequality (A.2.3), concluding the proof. □

An immediate consequence of the fact that $u \in LW^2_a(M)$ is the estimate

$$\begin{aligned}\|u_t\|^2_{LL_a(M)} \leq& C\Big(\|u\|^2_{LW^2_a(M)} + \|u_0\|^2_{LL_a(M)} + \|b\|^2_{LL_a(M)}\Big) \\ \leq& C\Big(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\Big)\,,\end{aligned}$$

that follows from equation (A.1.6).
Combining it with estimate (A.2.2) we have the following proposition.

PROPOSITION A.2.6. *If $u_0 \in W^{1,2}(M)$ and $b \in LL_a(M)$ then $u \in P^1_a(M)$, with the estimate*

$$\|u\|^2_{P^1_a} \leq C\left(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\right)\,.$$

REMARK A.2.7. By means of approximation with smooth functions, this proposition implies the existence and uniqueness of a weak solution of problem (A.1.1) with an initial datum $u_0 \in W^{1,2}(M)$.

Suppose now that $b \in P_a^{l-1}(M)$ (hence, $b \in LW_a^{2l-2}(M)$) and $u_0 \in W^{2l-1,2}(M)$. We consider the test function $\Theta_{-h}^{2l-1}\rho^2\Theta_h^{2l-1}u$ and we work as in the proof of Lemma A.2.5.

$$\begin{aligned}
&-\langle b, \Theta_{-h}^{2l-1}(\rho^2\Theta_h^{2l-1}u)\rangle_{LL_a(M)} \\
&\quad= \left\langle \frac{\partial(\Theta_h^{2l-1}u)}{\partial_t}, \rho^2\Theta_h^{2l-1}u \right\rangle_{LL_a(M)} \\
&\quad\quad+ \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h^{2l-1}u, \rho^2\Theta_h^{2l-1}u)\,dt \\
&\quad\quad+ \sum_{j=1}^{2l-1}\binom{2l-1}{j}(-1)^j\int_0^{+\infty} e^{-2at}\int_M (\Theta_h^j B)(\Theta_h^{2l-1-j}u, \rho^2\Theta_h^{2l-1}u)\,dt\,.
\end{aligned}$$

Proceeding analogously, with the only difference that we deal with the term containing b as follows,

$$\begin{aligned}
-\langle b, \Theta_{-h}^{2l-1}(\rho^2\Theta_h^{2l-1}u)\rangle_{LL_a(M)} &\leq |\langle \Theta_h^{2l-2}b, \Theta_{-h}(\rho^2\Theta_h^{2l-1}u)\rangle_{LL_a(M)}| \\
&\leq \|b\|_{LW_a^{2l-2}(M)}\|\rho^2\Theta_h^{2l-1}u\|_{LW_a(M)}\,,
\end{aligned}$$

we obtain

$$\|\nabla(\rho\Theta_h^{2l-1}u)\|^2_{LL_a(M)} \leq C\Big(\|u_0\|^2_{W^{2l-1,2}(M)} + \|u\|^2_{LW_a^{2l-1}(M)} + \|b\|^2_{LW_a^{2l-2}(M)}\Big)\,.$$

By means of Proposition A.2.6 and iteration, we conclude

$$\|u\|^2_{LW_a^{2(l-1)}(M)} \leq C\Big(\|u_0\|^2_{W^{2l-1,2}(M)} + \|b\|^2_{LW_a^{2l-2}(M)}\Big)\,.$$

Suppose now, by induction, that for every $j < m \leq l$ we have $\frac{\partial^j u}{\partial t^j} \in LW_a^{2(l-1-j)}(M)$, the case $j = 0$ being the previous estimate. Putting in equation (A.1.6) a smooth test function $\phi = \frac{\partial^{m-1}}{\partial t^{m-1}}\varphi$ in $C_c^\infty(M \times (0, +\infty))$, integrating by parts and estimating, we see that u satisfies the estimates

$$\begin{aligned}
\left\|\frac{\partial^m u}{\partial t^m}\right\|^2_{LW_a^{2(l-1-m)}(M)} &\leq C\Big(\left\|\frac{\partial^{m-1}b}{\partial t^{m-1}}\right\|^2_{LW_a^{2(l-1-m)}(M)} + \sum_{j<m}\left\|\frac{\partial^j u}{\partial t^j}\right\|^2_{LW_a^{2(l-1-j)}(M)}\Big) \\
&\leq C\Big(\|b\|^2_{P_a^{l-1}(M)} + \sum_{j<m}\left\|\frac{\partial^j u}{\partial t^j}\right\|^2_{LW_a^{2(l-1-j)}(M)}\Big)\,.
\end{aligned}$$

Hence, we conclude that $\frac{\partial^j u}{\partial t^j} \in LW_a^{2(l-1-j)}(M)$, for every $j \in \{0, \dots, l\}$.
We summarize all this argument in the following proposition.

PROPOSITION A.2.8. *For every $l \in \mathbb{N}$, if $u_0 \in W^{2l-1,2}(M)$ and $b \in P_a^{l-1}(M)$ then $u \in P_a^l(M)$ with the estimate*

$$\|u\|^2_{P_a^l(M)} \leq C\left(\|u_0\|^2_{W^{2l-1,2}(M)} + \|b\|^2_{P_a^{l-1}(M)}\right)\,.$$

We can now show Proposition A.2.3.

PROOF OF PROPOSITION A.2.3. As we already said, the map F is well defined and continuous and in order to conclude that F is an isomorphism it is sufficient to show that there exists a unique weak solution in $u \in P_a^l(M)$ of problem (A.1.1) with any initial datum $u_0 \in W^{2l-1,2}(M)$ and $b \in P_a^{l-1}(M)$.
As $W^{2l-1,2}(M)$ is a subspace of $W^{1,2}(M)$, by Remark A.2.7 we have a unique weak solution $u \in WW_a(M)$, then Proposition A.2.8 implies that $u \in P_a^l(M)$, as we wanted. □

A.3. The General Case

THEOREM A.3.1. *Problem* (PDE) *has a unique smooth solution defined in a time interval* $[0,T)$ *and such solution depends continuously in* C^∞ *on the smooth initial datum* u_0.

As M is compact there exists a constant $C>0$ such that the initial datum satisfies $|u_0| + |\nabla u_0|_g \leq C$. Since we are looking for a short time solution, possibly modifying the functions Q^{ij} and b with some "cut–off" functions we can assume that if $|u| + |\nabla u|_g + t \geq 2C$, the matrix $Q^{ij}(p,t,u,\nabla u)$ coincides with $g^{ij}(p)$ and $b(p,t,u,\nabla u)$ is zero. It follows that the operator $Q^{ij}(\,\cdot\,)\nabla^2_{ij}$ has an ellipticity constant $\lambda>0$ uniformly bounded from below away from zero and that for large time there holds $L(u)=\Delta u$.

For any $l\in\mathbb{N}$ we define $P^l(M,T)$ as the completion of $C^\infty(M\times[0,T])$ under the norm

$$\|f\|^2_{P^l(M,T)} = \sum_{j,\,k\,\in\,\mathbb{N}\text{ and }2j\,+\,k\,\leq\,2l} \int_{M\times[0,T]} |\partial_t^j\nabla^k f|^2\,d\mu\,dt\,,$$

for every $T\in\mathbb{R}^+$.
Clearly, there is a natural continuous embedding $P^l_a(M)\hookrightarrow P^l(M,T)$. In the following it will be easier (though conceptually equivalent) to use the spaces $P^l(M,T)$ instead of Polden's weighted spaces $P^l_a(M)$. For this reason we translate Proposition A.2.3 into the setting of $P^l(M,T)$ spaces.

PROPOSITION A.3.2. *For every* $T>0$ *and* $l\in\mathbb{N}$ *the map* F *defined by formula* (A.2.1) *is an isomorphism of* $P^l(M,T)$ *onto* $W^{2l-1,2}(M)\times P^{l-1}(M,T)$.

PROOF. The continuity of the second component of F is obvious while the continuity of the first component follows arguing like in the trace Lemma A.2.2. Hence, the map F is continuous, now we show that it is an isomorphism.
Given any $b\in P^{l-1}(M,T)$ we consider an extension $\widetilde{b}\in P^{l-1}_a(M)$ of the function b and we let $\widetilde{u}\in P^l_a(M)$ be the solution of problem (A.1.1) for $\widetilde{b}$. Clearly, $u=\widetilde{u}|_{M\times[0,T]}$ belongs to $P^l(M,T)$ and satisfies $F(u)=(u_0,b)$ in $M\times[0,T]$. Suppose that $v\in P^m(M,T)$ is another function such that $F(v)=(u_0,b)$ in $M\times[0,T]$, then by the maximum principle the functions u and v must coincide in all $M\times[0,T]$.

Since the map $F:P^l(M,T)\to W^{2l-1,2}(M)\times P^{l-1}(M,T)$ is continuous, one–to–one and onto, it is an isomorphism by the open mapping theorem. ☐

REMARK A.3.3. When u_0 and b are smooth the unique solution u of problem (A.1.1) belongs to all the spaces $P^l(M,T)$ for every $l\in\mathbb{N}$. As by Sobolev embeddings for any $k\in\mathbb{N}$ we can find a large $l\in\mathbb{N}$ so that $P^l(M,T)$ continuously embeds into $C^k(M\times[0,T])$, we can conclude that u actually belongs to $C^\infty(M\times[0,T])$.

Fixing $l\in\mathbb{N}$, we consider now the map

$$\mathcal{F}(u) = (u_0, u_t - L(u)) = \Big(u(\,\cdot\,,0), u_t - Q^{ij}(u)\nabla^2_{ij}u - b(u)\Big)\,,$$

defined on $P^l(M,T)$, where $Q^{ij}(u)=Q^{ij}(p,t,u,\nabla u)$ and $b(u)=b(p,t,u,\nabla u)$.
The image of the map $\mathcal{F}$ does not belong in general to $W^{2l-1,2}(M)\times P^{l-1}(M)$, this holds when $l\in\mathbb{N}$ is large enough and in this case $\mathcal{F}$ is actually C^1.

LEMMA A.3.4. *Assume that* $l>n/4+1$, *then* $u\in P^l(M,T)$ *implies that* ∇u *belongs to* $C^0(M\times[0,T])$. *Moreover,* $\mathcal{F}$ *is a well defined* C^1 *map from* $P^l(M,T)$ *to* $W^{2l-1,2}(M)\times P^{l-1}(M,T)$.

We postpone the proof of this lemma to the end of the section.

We fix $l\in\mathbb{N}$ such that the hypothesis of Lemma A.3.4 holds and we set $\widetilde{u}_0(p,t)=\sum_{m=0}^{l-1}a_m(p)\,t^m/m!$ for some functions $a_0,\dots,a_{l-1}\in C^\infty(M)$ to be determined later. Let $w\in P^l(M,T)$ be the unique solution of the linear problem

$$\begin{cases} w_t = Q^{ij}(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0)\nabla^2_{ij}w + b(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0) \\ w(\,\cdot\,,0)=u_0\,. \end{cases}$$

Such solution exists by Proposition A.2.3 and it is smooth, as u_0 and $\widetilde{u}_0$ are smooth (see Remark A.3.3), hence we have

$$\mathcal{F}(w) = (w_0, w_t - L(w)) = \Big(u_0, (Q^{ij}(\widetilde{u}_0) - Q^{ij}(w))\nabla^2_{ij}w + b(\widetilde{u}_0) - b(w)\Big) = (u_0, f)\,,$$

where we set $f = (Q^{ij}(\widetilde{u}_0) - Q^{ij}(w))\nabla^2_{ij}w + b(\widetilde{u}_0) - b(w)$ which is a smooth function.

If we compute the differential $d\mathcal{F}_u$ of the map $\mathcal{F}$ at the "point" $w \in C^\infty(M \times [0,T]) \subset P^l(M,T)$, acting on $v \in P^l(M,T)$, we obtain

$$d\mathcal{F}_w(v) = \Big(v_0, v_t - Q^{ij}(w)\nabla^2_{ij}v - \partial_w Q^{ij}(w)v\nabla^2_{ij}w - \partial_{w_k}Q^{ij}(w)\nabla_k v\nabla^2_{ij}w - \partial_w b(w)v - \partial_{w_k}b(w)\nabla_k v\Big)\,, \tag{A.3.1}$$

where $v_0 = v(\,\cdot\,,0)$.
Then, we can see that $d\mathcal{F}_w(v) = (z,h) \in W^{2l-1,2}(M) \times P^{l-1}(M,T)$ implies that v is a solution of the linear system

$$\begin{cases} v_t - \widetilde{Q}^{ij}\nabla^2_{ij}v - \widetilde{R}^k\nabla_k v - \widetilde{S}v = h \\ v(\,\cdot\,,0) = z\,, \end{cases}$$

where $\widetilde{Q}^{ij} = Q^{ij}(w)$, $\widetilde{R}^k = \partial_{w_k}Q^{ij}(w)\nabla^2_{ij}w + \partial_{w_k}b(w)$ and $\widetilde{S} = \partial_w Q^{ij}(w)\nabla^2_{ij}w + \partial_w b(w)$ are smooth functions independent of v.
By Proposition A.2.3 for every $(z,h) \in W^{2l-1,2}(M) \times P^{l-1}(M,T)$ there exists a unique solution $v \in P^l(M,T)$ of this system, hence $d\mathcal{F}_w$ is a Hilbert space isomorphism and the inverse function theorem can be applied, as the map $\mathcal{F}$ is C^1 by Lemma A.3.4. Hence, the map $\mathcal{F}$ is a diffeomorphism of a neighborhood $U \subset P^l(M,T)$ of w onto a neighborhood $V \subset W^{2l-1,2}(M)\times P^{l-1}(M,T)$ of (u_0, f).

Getting back to the functions a_m, we claim that we can choose them such that $a_m = \partial_t^m w|_{t=0} \in C^\infty(M)$ for every $m = 0, \dots, l-1$.
We apply the following recurrence procedure. We set $a_0 = u_0 \in C^\infty(M)$ and, assuming to have defined $a_0, \dots, a_m$, we consider the derivative

$$\partial_t^{m+1}w|_{t=0} = \partial_t^\ell[Q^{ij}(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0)\nabla^2_{ij}w + b(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0)]\Big|_{t=0}$$

and we see that the right–hand side contains time-derivatives at time $t = 0$ of $\widetilde{u}_0$, $\nabla\widetilde{u}_0$ and $\nabla^2_{ij}w$ only up to the order m, hence it only depends on the functions $a_0, \dots, a_m$. Then, we define a_{m+1} to be equal to such expression. Iterating up to $l-1$, the set of functions $a_0, \dots, a_{l-1}$ satisfies the claim.

Then, $a_m = \partial_t^m\widetilde{u}_0|_{t=0} = \partial_t^m w|_{t=0}$ and it easily follows by the "structure" of the function $f \in C^\infty(M \times [0,T])$, that we have $\partial_t^m f|_{t=0} = 0$ and $\nabla^j\partial_t^m f|_{t=0} = 0$ for any $0 \le \ell \le l-1$ and $j \in \mathbb{N}$.

We consider now for any $k \in \mathbb{N}$ the "translated" functions $f_k : M \times [0,T] \to \mathbb{R}$ given by

$$f_k(p,t) = \begin{cases} 0 & \text{if } t \in [0, 1/k] \\ f(p, t-1/k) & \text{if } t \in (1/k, T]\,. \end{cases}$$

Since $f \in C^\infty(M\times[0,T])$ and $\nabla^j\partial_t^m f|_{t=0} = 0$ for every $0 \le m \le l-1$ and $j \in \mathbb{N}$, all the functions $\nabla^j\partial_t^m f_k \in C^0(M \times [0,T])$ for every $0 \le m \le l-1$ and $j \ge 0$, it follows easily that

$$\nabla^j\partial_t^m f_k \to \nabla^j\partial_t^m f \quad \text{in } L^2(M\times[0,T]) \text{ for } 0 \le m \le l-1,\, j \ge 0\,,$$

hence $f_k \to f$ in $P^l(M,T)$.

Hence, there exists a function $\widetilde{f} \in P^{l-1}(M,T)$ such that $(u_0, \widetilde{f})$ belongs to the neighborhood V of $\mathcal{F}(w)$, defined above and $\widetilde{f} = 0$ in $M \times [0,T']$ for some $T' \in (0,T]$. Since $\mathcal{F}|_U$ is a diffeomorphism between U and V, we can find a function $u \in U$ such that $\mathcal{F}(u) = (u_0, \widetilde{f})$. Clearly such $u \in P^l(M,T)$ is a solution of problem (PDE) in $M \times [0,T']$. Since $u \in P^l(M,T')$ implies that $\nabla u \in C^0(M \times [0,T'])$, parabolic regularity implies that actually $u \in C^\infty(M \times [0,T'])$. We briefly sketch the argument (see [**82**, Chapter 4] for details): being its gradient continuous, the function u is a solution of a linear equation with continuous coefficients. Then, by parabolic Calderon–Zygmund theory it belongs to every parabolic Sobolev space $W^{2,1}_p(M \times [0,T'])$, for any $p > 1$, which actually implies that ∇u is Hölder continuous (see [**82**, Chapter 4, Theorem 9.1 and Corollary]). So, the function u is a solution of a linear equation with Hölder continuous coefficients, hence, it is in $C^{2+\alpha,1+\alpha}(M \times [0,T'])$, by parabolic Schauder estimates (see [**82**, Chapter 4, Theorem 5.2]). Then, an easy bootstrap argument along the same line shows that actually $u \in C^\infty(M \times [0,T'])$.

Now we get back to the original operator L, that we modified far from the initial time and initial datum in order to make it uniformly elliptic. The above argument shows that we can find a smooth solution u in $M \times [0,T]$ for some $T > 0$ (we relabeled T the time T' found above). Such solution is unique in every $P^l(M,T)$ for $l \in \mathbb{N}$ large enough (depending on the dimension of M). Indeed, by the Sobolev embeddings $P^l(M,T)$ is a subspace of C^k, with k growing with l, in particular for every $l \in \mathbb{N}$ large enough the solution we found is C^2 at least, this allows the use of the maximum principle in order to show that such solution is unique.

We finally prove the continuous dependence of a solution $u \in C^\infty(M \times [0,T])$ on its initial datum $u_0 = u(\,\cdot\,,0) \in C^\infty(M)$.
Fix any $l \in \mathbb{N}$ such that $l > n/4 + 1$, then by Lemma A.3.4 $u \in P^l(M,T)$ implies $\nabla u \in C^0(M \times [0,T])$. By the above argument, $u = (\mathcal{F}|_U)^{-1}(u_0,0) \in P^l(M,T)$ where $\mathcal{F}|_U$ is a diffeomorphism of an open set $U \subset P^l(M,T)$ onto $V \subset W^{2l-1,2}(M) \times P^{l-1}(M,T)$, with $(u_0,0) \in V$. Then, assuming that $u_{k,0} \to u_0$ in $C^\infty(M)$ as $k \to \infty$, we also have $u_{k,0} \to u_0$ in $W^{2l-1,2}(M)$, hence for k large enough $(u_{k,0},0) \in V$ and there exists $u_k \in U$ such that $\mathcal{F}(u_k) = (u_{k,0},0)$. This is the unique solution in $P^l(M,T)$ (hence in $C^\infty(M \times [0,T])$ by parabolic bootstrap) with initial datum $u_{k,0}$. Moreover, since $\mathcal{F}|_U$ is a diffeomorphism, we have $u_k \to u$ in $P^l(M,T)$.
By uniqueness, we can repeat the same procedure for any $l \in \mathbb{N}$ satisfying the condition in Lemma A.3.4 concluding that $u_k \to u$ in $P^l(M,T)$ for every such $l \in \mathbb{N}$, hence in $C^\infty(M \times [0,T])$.

REMARK A.3.5. Uniqueness can also be obtained by means of energy estimates based on Gårding's inequality for the operator $Q^{ij}(\,\cdot\,)\nabla^2_{ij}$, computing the ODE for the quantity $\int_M (w^2 + |\nabla w|^2)\,d\mu$ where $w = u - v$ and $u, v \in C^\infty(M \times [0,T])$ are a pair of solutions of problem (PDE).

We now prove Lemma A.3.4.
We need the following proposition which follows from standard arguments of parabolic interpolation theory, see [**88**, Theorem 2.3] and [**91**] for details.

PROPOSITION A.3.6. *Let $u \in P^l(M,T)$ where M is compact and n–dimensional. Then for $T > 0$ and $p, q \in \mathbb{N}$ with $p + 2q \leq 2l$, we have*

$$\|\partial_t^q \nabla^p u\|_{L^r(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \text{ if } \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} = \frac{1}{r} > 0\,, \tag{A.3.2}$$

$$\|\partial_t^q \nabla^p u\|_{L^r(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \text{ if } \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} = 0\,, \tag{A.3.3}$$

for every $r \geq 1$.
Finally, $\partial_t^q \nabla^p u$ is continuous and

$$\|\partial_t^q \nabla^p u\|_{C^0(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \text{ if } \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} < 0\,, \tag{A.3.4}$$

where C is a constant independent of $u \in P^l(M,T)$.

PROOF OF LEMMA A.3.4. The first claim follows immediately by the above proposition as the condition $l > n/4 + 1$ implies, choosing $p = 1$ and $q = 0$,

$$\frac{1}{2} - \frac{2l-p-2q}{n+2} = \frac{n+4-4l}{2(n+2)} < 0\,,$$

hence, ∇u is continuous.
We deal with the second claim, by simplicity we shall write $P^l = P^l(M,T)$, $L^q = L^q(M \times [0,T])$, $C^0 = C^0(M \times [0,T])$ etc..., so that for instance $C^0(P^l; C^1)$ will denote the space of continuous maps from $P^l(M,T)$ to $C^1(M \times [0,T])$.
First we show that actually $\mathcal{F}(u) \in W^{2l-1,2} \times P^{l-1}$ when $u \in P^l$, hence the map $\mathcal{F}$ is well defined.

The regularity of the first component follows from Lemma A.2.2 and linearity, the same holds for the term u_t in the second component. Then we only need to prove that $\partial_t^m \nabla^k (Q^{ij}(u)\nabla^2_{ij}u + b(u))$ belongs to L^2 when $k + 2m = 2l - 2$ (actually, by looking at Definition A.2.1 of P^{l-1}, we should also check it when $2m + k < 2l - 2$ but this latter task is obviously easier).
Clearly, the "most difficult" term is $\partial_t^m \nabla^k (Q^{ij}(u)\nabla^2_{ij}u)$, moreover, when at least one time or space derivative apply to the t, p and u variables in $Q^{ij}(u)$ the resulting term will have a higher integrability than when all the derivatives go on the gradient term ∇u in $Q^{ij}(u)$.
Hence, by simplicity, we assume that $b(u) = 0$ and $Q^{ij}(u)\nabla^2_{ij}u = A^{ij}(\nabla u)\nabla^2_{ij}u$ that we will denote by $A(\nabla u)\nabla^2 u$, for some smooth tensor A. It will be clear by the following estimates that the other possible terms, when $b(u)$ is not zero and $Q^{ij}(u)$ depends also by the other variables, can be bounded analogously (actually more easily).
We underline that if all the derivatives go on the factor $\nabla^2 u$ of $A(\nabla u)\nabla^2 u$, the resulting term $A(\nabla u)\partial_t^m \nabla^k \nabla^2 u$ clearly belongs to L^2 as $A(\nabla u)$ is uniformly bounded (the gradient ∇u is continuous and A is smooth), $k + 2m = 2l - 2$ and $u \in P^l$.

It can be easily proved by Leibniz formula and induction that for every pair of integers m and k with $k + 2m = 2l - 2$, the derivative $\partial_t^m \nabla^k (A(\nabla u)\nabla^2 u)$ is a finite sum of terms (the total number is bounded by a function of l) each one of the form

$$B(\nabla u) \prod_{p=1}^{2l} \prod_{q=0}^{l-1} \prod_{|\alpha|=p} (\partial_t^q \nabla_\alpha^p u)^{\sigma_{pq\alpha}} \tag{A.3.5}$$

where α is a multiindex of order $|\alpha|$, the exponents $\sigma_{pq\alpha}$ are nonnegative integers and B stands for some smooth and bounded tensor. Hence, since ∇u is continuous, we estimate any of such terms as

$$|B(\nabla u)| \prod_{p=1}^{2l} \prod_{q=0}^{l-1} \prod_{|\alpha|=p} |\partial_t^q \nabla_\alpha^p u|^{\sigma_{pq\alpha}} \leq C \prod_{p=1}^{2l} \prod_{q=0}^{l-1} |\partial_t^q \nabla^p u|^{b_{pq}}$$

with $b_{pq} = \sum_{|\alpha|=p} \sigma_{pq\alpha} \in \mathbb{N}$ and nonnegative.
Moreover, it can be seen by induction on $l \in \mathbb{N}$ that the following formula holds

$$\sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1}} b_{pq}(p + 2q - 1) = 2l - 1\,. \tag{A.3.6}$$

Then,

$$\begin{aligned}
\|\partial_t^m \nabla^k (A(\nabla u)\nabla^2 u)\|_{L^2} &\leq \sum C \Big\| \prod_{p=1}^{2l} \prod_{q=0}^{l-1} |\partial_t^q \nabla^p u|^{b_{pq}} \Big\|_{L^2} \\
&\leq \sum C \Big(\int_{M \times [0,T]} \prod_{p=1}^{2l} \prod_{q=0}^{l-1} |\partial_t^q \nabla^p u|^{2b_{pq}}\, d\mu\, dt \Big)^{1/2},
\end{aligned}$$

where the symbol of sum means that we are adding all the terms described by formula (A.3.5) above.
We now apply Proposition A.3.6 noticing that $(p + 2q - 1)$ is always positive, otherwise we must have $p = 1$ and $q = 0$ but the simple gradient ∇u cannot appear as a factor in the product formula (A.3.5), by the structure of $\partial_t^m \nabla^k (A(\nabla u)\nabla^2 u)$.
If for at least one pair (p, q) with $b_{pq} \neq 0$ the derivative $\partial_t^q \nabla^p u$ is continuous by the embedding (A.3.4), we simply bound the relative factor with a constant and we modify the relative integer exponent b_{pq} to be zero. It follows that we have also to modify the formula (A.3.6) to the inequality $\sum_{p=1}^{2l} \sum_{q=0}^{l-1} b_{pq}(p + 2q - 1) < 2l - 1$.
If at least one pair (p, q) with $b_{pq} \neq 0$ satisfies $2(2l - p - 2q) = n + 2$, that is, we are in a critical

case (A.3.3) of the embeddings, formula (A.3.6) gives

$$\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} b_{pq}(p+2q-1) < 2l-1\,, \tag{A.3.7}$$

as we "dropped" at least one nonzero term $b_{pq}(p+2q-1)$.
Hence, either formula (A.3.7) holds or we did not "set to zero" any of the integers b_{pq} and there are no factors in the critical cases of the embeddings. In this latter situation, either all the derivatives went on $\nabla^2 u$ and the resulting term $A(\nabla u)\partial_t^m \nabla^{k+2} u$ is bounded in L^2, or clearly there are at least two integers b_{pq} which are nonzero.

We now estimate as follows the previous integral with Hölder's inequality and the embeddings (A.3.2) for the factors with $2(2l-p-2q)\neq n+2$ and we choose a large r_{pq} for any factor such that $2(2l-p-2q)=n+2$ (the critical cases), by the embeddings (A.3.3),

$$\begin{aligned}
\int_{M\times[0,T]} \prod_{p=1}^{2l}\prod_{q=0}^{l-1} |\partial_t^q \nabla^p u|^{2b_{pq}}\,d\mu\,dt &\leq C \prod_{p=1}^{2l}\prod_{q=0}^{l-1} \left(\int_{M\times[0,T]} |\partial_t^q \nabla^p u|^{2b_{pq}/d_{pq}}\,d\mu\,dt\right)^{d_{pq}}\\
&= C \prod_{p=1}^{2l}\prod_{q=0}^{l-1} \left(\int_{M\times[0,T]} |\partial_t^q \nabla^p u|^{r_{pq}}\,d\mu\,dt\right)^{d_{pq}}\\
&\leq C\|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} d_{pq}r_{pq}}\\
&= C\|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} 2b_{pq}}
\end{aligned}$$

where $\frac{1}{r_{pq}}=\frac{1}{2}-\frac{2l-p-2q}{n+2}>0$ and $d_{pq}=2b_{pq}/r_{pq}$.
This application of Hölder's inequality is justified if the sum of all the exponents d_{pq} with $2(2l-p-2q)\neq n+2$ is less than 1, as we can choose the other d_{pq} (associated to the critical cases) arbitrarily small.
In such case we conclude that

$$\|\partial_t^m \nabla^k (A(\nabla u)\nabla^2 u)\|_{L^2} \leq \sum C\|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} b_{pq}}$$

and we are done.

Hence, we now check such condition on the exponents d_{pq} assuming that at least one of the integers b_{pq} with $2(2l-p-2q)\neq n+2$ is nonzero, otherwise the conclusion is trivial.

$$\begin{aligned}
\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} d_{pq} &= \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} \frac{2b_{pq}}{r_{pq}}\\
&= \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-p-2q}{n+2}\right)\\
&= \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}+\frac{p+2q-1}{n+2}\right)\\
&= \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right) + \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\frac{p+2q-1}{n+2}\,.
\end{aligned}$$

We now separate the two cases by the discussion above. If the strict inequality (A.3.7) holds we have

$$\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} d_{pq} < \Bigg[\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)\Bigg] + 2\frac{2l-1}{n+2} < 1\,,$$

as at least one of the integers b_{pq} is not zero and since $\frac{1}{2}-\frac{2l-1}{n+2}<0$, by the hypothesis $l > n/4+1$. If instead equality (A.3.6) holds, we have seen that at least two of the integers b_{pq} are nonzero otherwise the conclusion is trivial, then for all the pairs (p,q) with $b_{pq}>0$ there holds $2(2l-p-2q)\neq n+2$, hence,

$$\begin{aligned}\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1}} d_{pq} &= \sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)+2\frac{2l-1}{n+2}\\ &\leq 4\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)+2\frac{2l-1}{n+2}\\ &= 2-2\frac{2l-1}{n+2}\,,\end{aligned}$$

which is less than 1 again since $l > n/4+1$.

It remains to prove that $d\mathcal{F}\in C^0(P^l; L(P^l;P^{l-1}))$, where $L(P^l;P^{l-1})$ denotes the Banach space of bounded linear maps from P^l into P^{l-1}. Again we assume by simplicity $b(u)=0$ and $Q(u)\nabla^2 u = A(\nabla u)\nabla^2 u$, for some smooth tensor A and we define $\mathcal{F}_A : P^l \to P^{l-1}$ given by $u\mapsto A(\nabla u)\nabla^2 u$.

We first claim that the Gateaux derivative

$$(u,v)\mapsto d\mathcal{F}_A(u)(v) = \frac{d}{dt}\mathcal{F}_A(u+\varepsilon v)\Big|_{\varepsilon=0}$$

belongs to $C^0(P^l\times P^l;P^{l-1})$. Indeed, $d\mathcal{F}_A(u)(v)$ is given by (see formula (A.3.1))

$$d\mathcal{F}_A(u)(v) = D(\nabla u)\nabla v\nabla^2 u + A(\nabla u)\nabla^2 v\,,$$

where D is a smooth tensor and the procedure previously used to estimate $\mathcal{F}(u)$ can also be applied to any term $\partial_t^m\nabla^k(D(\nabla u)\nabla v\nabla^2 u)$ or $\partial_t^m\nabla^k(A(\nabla u)\nabla^2 v)$, since they can be expressed as a sum of terms similar to the ones of formula (A.3.5) with the only difference being that now in every term exactly one linear occurrence of u is replaced by v.

It is then easy to conclude, since $v\in P^l$ like u, that we obtain the continuity of $(u,v)\mapsto d\mathcal{F}_A(u)(v)$ in the same way. This proves in particular that $d\mathcal{F}_A(u)\in L(P^l;P^{l-1})$.

In order now to show that $d\mathcal{F}_A\in C^0(P^l;L(P^l;P^{l-1}))$ we need to see that

$$\sup_{\|v\|_{P^l}\leq 1}\|d\mathcal{F}_A(\widetilde{u})(v)-d\mathcal{F}_A(u)(v)\|_{P^{l-1}}\to 0 \quad \text{as } \widetilde{u}\to u \text{ in } P^l\,.$$

Again, this estimate is similar to what we have already done. Indeed, by what we said above about the structure of the terms $\partial_t^m\nabla^k(D(\nabla u)\nabla v\nabla^2 u)$ and $\partial_t^m\nabla^k(A(\nabla u)\nabla^2 v)$, assuming that there are no time derivatives for the sake of simplicity, we have to show that, as $\widetilde{u}\to u$ in P^l,

$$\sup_{\|v\|_{P^l}\leq 1}\|B(\widetilde{u})\nabla^{i_1}\widetilde{u}\cdots\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v - B(u)\nabla^{i_1}u\cdots\nabla^{i_j}u\,\nabla^{i_{j+1}}v\|_{L^2}\to 0\,, \tag{A.3.8}$$

where $i_1+\dots+i_{j+1}=2l+j$ (see formula (A.3.5) and condition (A.3.6)).

Adding and subtracting terms, one gets

$$\begin{aligned}&\Big|\,B(\widetilde{u})\nabla^{i_1}\widetilde{u}\cdots\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v - B(u)\nabla^{i_1}u\cdots\nabla^{i_j}u\,\nabla^{i_{j+1}}v\,\Big|\\ &\qquad\leq\Big\{\,|B(\widetilde{u})-B(u)|\,|\nabla^{i_1}\widetilde{u}|\,\cdots\,|\nabla^{i_j}\widetilde{u}|\\ &\qquad\quad + |B(u)|\,|\nabla^{i_1}(\widetilde{u}-u)|\,|\nabla^{i_2}\widetilde{u}|\,\cdots\,|\nabla^{i_j}\widetilde{u}|\\ &\qquad\quad + \cdots + |B(u)|\,|\nabla^{i_1}u|\,\cdots\,|\nabla^{i_j}(\widetilde{u}-u)|\Big\}\,|\nabla^{i_{j+1}}v|\,.\end{aligned}$$

Studying now the L^2 norm of this sum, the first term can be bounded as before and it goes to zero as $B(u)$ is continuous from P^l to L^∞. The L^2 norm of all the other terms, repeating step by step the previous estimates, using Hölder's inequality and embeddings (A.3.2)–(A.3.4), will be estimated by some product

$$C\|u\|^\alpha_{P^l}\|\widetilde{u}\|^\beta_{P^l}\|v\|^\gamma_{P^l}\|\widetilde{u}-u\|^\sigma_{P^l} \leq C\|u\|^\alpha_{P^l}\|\widetilde{u}\|^\beta_{P^l}\|\widetilde{u}-u\|^\sigma_{P^l}$$

for a constant C and some nonnegative exponents $\alpha, \beta, \gamma, \sigma$ satisfying $\alpha+\beta+\gamma+\sigma \leq 1$ and $\sigma > 0$. Here we we used the fact that $\|v\|_{P^l} \leq 1$.
As $\widetilde{u}-u \to 0$ in P^l, this last product goes to zero in L^2, hence uniformly for $\|v\|_{P^l} \leq 1$ and inequality (A.3.8) follows, as claimed. The analysis of the estimates with mixed time/space derivatives is analogous.
Then, the Gateaux differential $d\mathcal{F}_A$ is continuous, which implies that it coincides with the Frechét differential, hence $\mathcal{F}_A \in C^1(P^l; P^{l-1})$. It follows that the map $\mathcal{F}$ is C^1 and we are done. □