CHAPTER 4

# Type II Singularities

We assume now that we are in the type II singularity case, that is,

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t}=+\infty$$

for the mean curvature flow of a compact hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ in its maximal interval of existence.

A good question is actually whether type II singularities there exist.
An example is given by a closed, symmetric, self–intersecting curve with the shape of a symmetric "eight" figure in the plane, which has zero *rotation number*. Pushing a little the analysis of the previous chapter and keeping into account the symmetries of the curve, if the curve develops a type I singularity, we can produce a nonflat blow up limit which is homothetic and nonflat. Then such a limit must be a circle or one of Abresch–Langer curves. In both cases, the limit would be a compact closed curve and by the smooth convergence, the rotation number would still be zero. Hence, the circle has to be excluded and the contradiction is given by the fact that there are no Abresch–Langer curves with zero rotation number. Hence, type I singularities do not describe all the possible ones.
Another example is given by a cardioid–like curve in the plane with a very small loop, hence high curvature: one can right guess that at some time the loop shrinks while the rest of the curve remains smooth and a cusp develops. Such a singularity is of type II, since if we have a Type I singularity we would get an Abresch–Langer curve as a blow up limit and this implies, as these latter are compact, that the entire curve has vanished in a single point (see the analysis in [**15**] and also [**14, 16**]).
As we will see in Theorem 4.5.5 that embedded curves do not develop type II singularities, one could reasonably conjecture that also for embedded hypersurfaces (at least in low dimension) all the singularities are of type I. Unfortunately, this is not true even if the dimension is only two, indeed, the following example excludes such a good behavior.

EXAMPLE (The Degenerate Neckpinch). For a given $\lambda > 0$, let us set

$$\phi_\lambda(x) = \sqrt{(1-x^2)(x^2+\lambda)}, \qquad -1 \leq x \leq 1.$$

For any $n \geq 2$, let $M^\lambda$ be the $n$–dimensional hypersurface in $\mathbb{R}^{n+1}$ obtained by rotation of the graph of $\phi_\lambda$ in $\mathbb{R}^2$. The hypersurface $M^\lambda$ looks like a dumbbell, where the parameter $\lambda$ measures the width of the central part. Then, it is possible to prove the following properties (see [**4**]):

(1) if $\lambda$ is large enough, the hypersurface $M^\lambda_t$ eventually becomes convex and shrinks to a point in finite time;
(2) if $\lambda$ is small enough, $M^\lambda_t$ exhibits a neckpinch singularity as in the case of the *standard neckpinch* (see Section 1.4);
(3) there exists at least one intermediate value of $\lambda > 0$ such that $M^\lambda_t$ shrinks to a point in finite time, has positive mean curvature up to the singular time, but never becomes convex. The maximum of the curvature is attained at the two points where the surface meets the axis of rotation;
(4) in this latter case the singularity is of type II, otherwise the blow up at the singular time would give a sphere (for all $p \in M$ we would have $\widehat{p} = O \in \mathbb{R}^{n+1}$ hence, by estimate (3.2.2), any limit hypersurface is bounded). This is impossible as it would imply that the surface would have been convex at some time.

The flowing hypersurface at point (3) is called the *degenerate neckpinch* and was first conjectured by Hamilton for the Ricci flow [**61**, Section 3]. Intuitively speaking, it is a limiting case of the neckpinch where the cylinder in the middle and the two spheres on the sides shrink at the same time. One can also build the example in an asymmetric way, with only one of the two spheres shrinking simultaneously with the neck, while the other one remains nonsingular.
A sharp analysis of the singular behavior for a class of rotationally symmetric surfaces exhibiting a degenerate neckpinch has been done by Angenent and Velázquez in [**19**].
Another interesting example of singularity formation (a family of evolving tori, proposed by De Giorgi) was carefully studied by Soner and Souganidis in [**110**, Proposition 3] (see also the numerical analysis performed by Paolini and Verdi in [**101**, Section 7.5]).

## 4.1. Hamilton's Blow Up

In order to deal with the blow up around type II singularities we need a new set of estimates which are actually independent of the type II hypothesis and scaling invariant (see [**3**] and [**104**]).

PROPOSITION 4.1.1. *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p,0)| \leq \Lambda < +\infty$. Then, there exists a time $\tau = \tau(\Lambda) > 0$ and constants $C_m = C_m(\Lambda)$, for every $m \in \mathbb{N}$ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m/t^m$ for every $p \in M$ and $t \in (0,\tau)$.*

PROOF. We prove the claim by induction. By the evolution equation for $|\mathrm{A}|^2$,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

we get

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,,$$

hence, there exists a time $\tau = \tau(\Lambda) > 0$ and a constant $C_0 = C_0(\Lambda)$ such that $|\mathrm{A}(p,t)|^2 \leq C_0$ for every $p \in M$ and $t \in [0,\tau)$. This is the case $m = 0$.
Recalling equation (2.3.5), setting $f = \sum_{k=0}^m |\nabla^k \mathrm{A}|^2 \lambda_k t^k$ for some positive constants $\lambda_0, \dots, \lambda_m$ and assuming the inductive hypothesis $|\nabla^k \mathrm{A}(p,t)|^2 \leq C_k(\Lambda)/t^k$ for any $k \in \{0, \dots, m-1\}$, $p \in M$ and $t \in (0,\tau)$, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} f = &\frac{\partial}{\partial t} \sum_{k=0}^m |\nabla^k \mathrm{A}|^2 \lambda_k t^k \\
= &\sum_{k=1}^m |\nabla^k \mathrm{A}|^2 k \lambda_k t^{k-1} \\
&+ \sum_{k=0}^m \lambda_k t^k \Big( \Delta |\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \Big) \\
\leq &\,\Delta f + \sum_{k=1}^m |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} - 2|\nabla^{m+1}\mathrm{A}|^2 \lambda_m t^m \\
&+ \sum_{k=0}^m \lambda_k t^k C(k) \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} |\nabla^p \mathrm{A}||\nabla^q \mathrm{A}||\nabla^r \mathrm{A}||\nabla^k \mathrm{A}| \\
\leq &\,\Delta f + \sum_{k=1}^m |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + \sum_{k=0}^{m-1} \lambda_k C(k) \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} C_p C_q C_r C_k \\
&+ \lambda_m t^{m/2} C(m) \Big( \sum_{p+q+r=m \,|\, p,q,r<m} C_p C_q C_r \Big) |\nabla^m \mathrm{A}| + \lambda_m t^m C(m) |\mathrm{A}|^2 |\nabla^m \mathrm{A}|^2 \\
\leq &\,\Delta f + \sum_{k=1}^m |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D
\end{aligned}$$