Mathematical Methods 2017/07/11

Solve the following exercises in a fully detailed way, explaining and justifying any step.

(1) (10 points) For $y \in (1, \infty)$ let $f(y) = \displaystyle\int_0^{2\pi} \frac{1}{\sin(x) + y} dx$.
  (a) Compute $f(y)$.
  (b) Say if f is summable in $(1, \infty)$.
  (c) Say if f is summable in $(2, \infty)$.
  (d) Say if f is summable in $(1, 2)$.
  (e) Say if f is summable in $(2, 3)$.
  (f) Say if f^2 is summable in $(2, \infty)$.
  (g) Say if f^2 is summable in $(1, 2)$.

(2) (6 points) Let V be an Hilbert space. Let $v_1, v_2 \in V$ and let W be the space generated by v_1, v_2. Prove, or disprove with a counterexample, that $V = W \oplus W^\perp$. Is is true that for any subspace $U < V$ we have $V = U \oplus U^\perp$? (prove it or disprove with a counterexample.)

SOLUZIONI

(1) By setting $\sin x = (e^{ix} - e^{-ix})/2i$ and by changing variable $z = e^{ix}$, $dz = ie^{ix}dx = izdx$, the integral becomes an integral over the unit circle γ.

$$\int_0^{2\pi} \frac{1}{\sin(x) + y} dx = \int_0^{2\pi} \frac{1}{\frac{e^{ix} - e^{ix}}{2i} + y} dx = \int_\gamma \frac{1}{\frac{z - z^{-1}}{2i} + y} \frac{1}{iz} dz = \int_\gamma \frac{1}{\frac{z^2 - 1}{2iz} + y} \frac{1}{iz} dz$$

$$= \int_\gamma \frac{2}{z^2 - 1 + 2izy} dz$$

The function $\frac{2}{z^2 - 1 + 2izy}$ has two simple poles at

$$z_\pm = -iy \pm \sqrt{-y^2 + 1}$$

which, since $y > 1$, equals

$$z_\pm = i(-y \pm \sqrt{y^2 - 1})$$

Again because $y > 1$, only $z_+ = i(-y + \sqrt{y^2 - 1})$ sits in the interior of the unit disk. The residue of $\frac{2}{z^2 - 1 + 2izy}$ at z_+ is

$$\frac{2}{z_+ - z_-} = \frac{2}{i(-y + \sqrt{y^2 - 1} - (-y - \sqrt{y^2 - 1}))} = \frac{1}{i\sqrt{y^2 - 1}}$$

The index of γ around z_+ is 1. By Residue Theorem it follows that

$$f(y) = 2\pi i \frac{1}{i\sqrt{y^2 - 1}} = \frac{2\pi}{\sqrt{y^2 - 1}} = \frac{2\pi}{\sqrt{(y - 1)(y + 1)}}$$

and

$$f^2(y) = \frac{4\pi^2}{y^2-1} = \frac{4\pi^2}{(y-1)(y+1)}$$

In particular, $f(y)$ behaves like $1/y$ for $y \to \infty$, hence it is not summable in $(1,\infty)$ nor in $(2,\infty)$. On the other hand, near 1, $f(y)$ behaves like $1/\sqrt{y-1}$ and since $1/\sqrt{x}$ is summable near zero, f is summable on $(1,2)$. On $[2,3]$ the function f is continuous hence summable because $(2,3)$ is a bounded interval.

As for the summability of f^2 we have: f^2 behaves like $1/y^2$ at infinity, so it is summable on $(2,\infty)$. On the other hand, f behaves like $1/(y-1)$ near 1 so it is not summable in $(1,2)$. $\square$

(2) If both v_1 and v_2 are zero then there is nothing to prove. W.l.o.g. we can suppose $v_1 \neq 0$. We apply the Gram-Schmidt process: Let $w_1 = v_1$ and $w_2 = v_2 - \frac{\langle v_1,v_2\rangle}{\langle v_1,v_1\rangle}v_1$. Then w_2 is orthogonal to w_1 and, if w_2 is not zero, then w_1, w_2 is a basis of W. For any $v \in V$ we have

$$v = v - \frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle}w_1 - \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle}w_2 + (\frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle}w_1 + \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle}w_2)$$

where we agree that if $w_2 = 0$, the we just omit it. Clearly

$$w = \frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle}w_1 + \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle}w_2$$

belongs to W. We claim that

$$u = v - \frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle}w_1 - \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle}w_2$$

is orthogonal to both w_1 and w_2, hence to W. For $i = 1,2$ we have

$$\langle u,w_i\rangle = \langle v - \frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle}w_1 - \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle}w_2, w_i\rangle = \langle v,w_i\rangle - \langle\frac{\langle v,w_i\rangle}{\langle w_i,w_i\rangle}w_i, w_i\rangle = 0.$$

It follows that $V = W + W^\perp$. If $u \in W \cap W^\perp$ then we have $u = a_1w_1 + a_2w_2$ because $u \in W$ and for $i = 1,2$

$$0 = \langle u,w_i\rangle = \langle a_1w_1 + a_2w_2, w_i\rangle = a_i\langle w_i,w_i\rangle.$$

Thus $u = 0$. So $W \cap W^\perp = 0$ and $V = W \oplus W^\perp$.

For the second claim consider the space U of simple functions $[0,1]$ as a subspace of $L^2([0,1])$. It's orthogonal is the zero space because if $f \neq 0$ in L^2 then, up to changing f with $-f$, we may suppose that $f_+ \neq 0$, hence there is a set $A \subset [0,1]$ of positive measure where $f > 0$, and letting χ_A be the characteristic function of A, we have $\langle \chi_A, f\rangle = \int_A f > 0$.

But $U \neq L^2$, hence $L^2 \neq U \oplus U^\perp$. $\square$