Mathematical Methods 2017/06/13

Solve the following exercises in a fully detailed way, explaining and justifying any step.

(1) (4 points) Compute $\displaystyle\int_{\mathbb{R}} \frac{\sin(-x)}{x^4+4}$.

(2) (7 points) Let $f : [0,\infty) \to \mathbb{R}$ be a continuous, monotone, and summable function. Let $f_n(x) = f(nx)$. Discuss the convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ on $[0,\infty)$ with respect to the sup-norm, the L^1-norm and the point-wise convergence.

(3) (5 points) Give the definition of Hilbert basis of a separable Hilbert space.

SOLUTIONS

(1) The function $f(x) = \frac{\sin(-x)}{x^4+4}$ is summable on $\mathbb{R}$ because

$$|\frac{\sin(-x)}{x^4+4}| \leq \frac{1}{x^4+4}$$

which is summable on $\mathbb{R}$. Moreover $f(x) = -f(-x)$. Thus

$$\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = -\int_{0}^{\infty} f(x)dx + \int_{0}^{\infty} f(x)dx = 0$$

□

(2) Since f is monotone and summable, then $\lim_{x\to\infty} f(x) = 0$. Therefore, for any $x \neq 0$ $f_n(x)$ point-wise converges to 0. $f_n(0) = f(0)$ is a constant sequence, hence f_n converges point-wise to

$$f_\infty(x) = \begin{cases} f(0) & x = 0 \\ 0 & x \neq 0 \end{cases}$$

In particular, since the set $\{0\}$ has measure zero, f_∞ is equivalent to the zero function.

Since f is monotone, so is f_n. This implies that the essential sup of f_n is $\lim_{x\to 0} f_n(x)$, and by continuity we get $||f_n||_\infty = f_n(0) = f(0) = ||f||_\infty$. Therefore, either $||f||_\infty = 0$, hence $f = 0$ because f is monotone, or there is no uniform convergence.

For the L^1 convergence, it suffices to note that since f is monotone, then the sequence of function f_n is monotone. Therefore we can apply the monotone convergence theorem and

$$\lim \int |f_n| = \int \lim |f_n| = \int |f_\infty| = \int 0 = 0$$

so f_n converges to f_∞ in $L^1([0,\infty))$. □

(3) An Hilbert space is separable if it has a countable dense set. An Hilbert basis of a separable Hilbert space H is an ordered set $(v_n)_{n\in\mathbb{N}}$ of vectors of H which are linearly independent, unitary, pairwise orthogonal, and that span a dense subspace of H.