# KNOT THEORY AND APPLICATIONS: LECTURE NOTES

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## 1. Introduction

The object we are interested in are knots and, their multicomponent counterpart called links. Everyone knows what a knot is in real life, but it is the same think as a knot in topology? The answer is yes and no: as in real life we want to study the knotting of curves in the space, but while a knot in real life could be modeled as a compact but not closed curve (i.e. it has boundary), a topological knot is closed [1]. The reason is essentially that every knot in real life can be untied (even if sometimes it is really hard!), while in topological knots only the trivial knot could be untied. Nevertheless, as we will see, deciding whether a given knot is the trivial one is a difficult task. Before stating our definition of knots and links, it worth mentioning that the study of knots started in 19th century with Gauss and Lord Kelvin, as tools involved in physical problems, while the development of the mathematical theory started in the 20th century with Dehn, Reidemeister and Alexander. Since then, significant applications of knot theory outside topology, were discovered, as those in algebraic geometry, mechanical statistic and biochemistry.

## 2. Basic notions

2.1. **The objects: knots and links.** We will consider only knots and links in $S^3$. Since $S^3$ can be viewed as $\mathbb{R}^3 \cup \infty$ and each 1-dimensional submanifold in $S^3$ may avoid a point, up to equivalence, (see Subsection 2.2 and Exercise 2.3.1) we can consider knots and links also in $\mathbb{R}^3$.

**Definition 2.1.1.** *A link with $k$-components is the image of an embedding of a disjoint union of $k$ copies of $S^1$ in $S^3$. A link with a single component is called a knot. Equivalently a link is a subset of $S^3$ homeomorphic to a disjoint union of circles.*

Some care is needed with the regularity of the embedding. If we require just a topological embedding, wild knots and links may appear as the one depicted in Figure 1. In order to avoid these kind of topological objects, we require that the embedding is $C^\infty$ or PL. Sometimes such links are called tame links.

**Remark 2.1.2.** *In dimension at most 3 the categories $C^\infty$ and PL are equivalent (see [11]).*

In Figure 2 there are some examples of knots and links.

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[1] In the last decade a similar theory for compact curves with boundary, called knotoids, was developed, see [18]

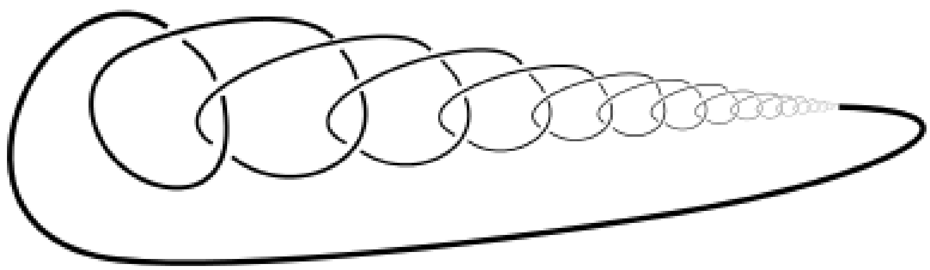


FIGURE 1. A wild knot

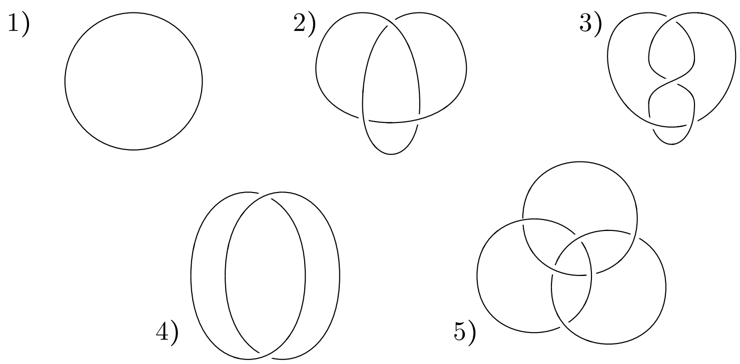



FIGURE 2. Some examples of links: 1) trivial knot, 2) trefoil knot, 3) figure-eight knot, 4) Hopf link, 5) Borromean rings.

2.2. **The equivalence relation between knots and links.** Knots and links are considered up to the natural equivalence relation given by deformations: roughly speaking, two links are equivalent it they can be deformed one into each other without breaking nor crossing strands.

**Exercise 2.2.1.** *Convince yourself, and at least one friend, that the two knots depicted in Figure 3 are equivalent. (On the right-side knot consider the two strands connected at infinity).*

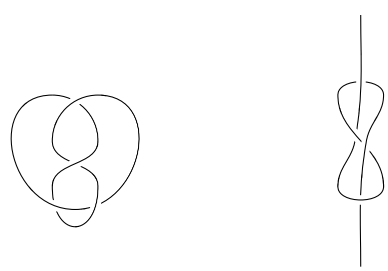


FIGURE 3. Two representations of the figure-eight knot

**Exercise 2.2.2.** *Try to convince at least yourself that the trefoil and the figure-eight knots are not equivalent.*

**Exercise 2.2.3.** *Are all links and knots depicted in Figure 2 not equivalent to each other?*

Again, some care is needed when ones goes to mathematically defines such equivalence.

A classical caveat here is that continuous deformations are not enough: every knot can be deformed to the unknot by means of a continuous deformation which is a en embedding at all times! This can be done by pulling tight the strands of the knot. At the limit the knotted region disappear into a point. Figure 4 depicts such procedure.

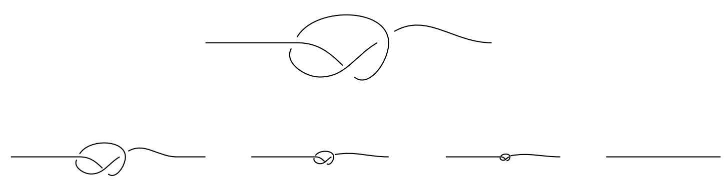


FIGURE 4. How to continuously deform every knot to the unknot

The first possibility to formalize the equivalence relation between links is via ambient homeomorphisms:

**Definition 2.2.4** (Ambient Homeomorphism equivalence)**.** *Two links $L_1, L_2$ in $S^3$ are equivalent if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ so that $f(L_1) = L_2$.*

This viewpoint is clean and global, but in some sense it hides the concept of deformation, which can be formalized via the notion of isotopy.

**Definition 2.2.5.** *A (continuous, smooth, PL)* isotopy *between two maps $f, g : A \to A$ is a (continuous, smooth, PL) map $F : A \times [0,1] \to A$ such that, if we set $F_t(a) = F(a,t)$, then*

$$F_0 = f, \qquad F_1 = g$$

*and any $F_t$ is a (continuous, smooth, PL) isomorphism.*

So if $F$ is a continuous isotopy, then each $F_t$ is a homeomorphism; if it is smooth, then any $F_t$ is a diffeomorphism, and so on.

**Definition 2.2.6.** *Given $K_1, K_2 \subset A$, an* ambient isotopy *between $K_1$ and $K_2$ is an isotopy $F : A \times [0,1] \to A$, between the identity of $A$ and a map $f : A \to A$, such that $F_1(K_1) = K_2$.*

**Definition 2.2.7** (Ambient Isotopy Equivalence)**.** *Two links $L_1, L_2$ in $S^3$ are equivalent if there is an ambient isotopy between $L_1$ and $L_2$.*

The notion of continuous deformation has a $PL$-counter part, that can be described as a finite sequence of some standard combinatorial moves.

A $PL$-link is just a (not necessarily connected) simple (i.e. without self-crossings) closed (i.e. without free endpoints) polygonal line in $\mathbb{R}^3 = S^3 \setminus \{1 \text{ point}\}$.

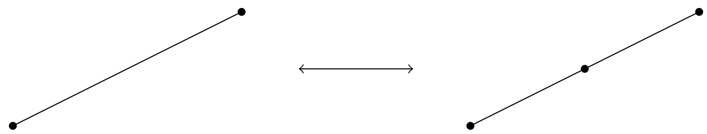


FIGURE 5. Simplicial subdivision

The first move we are allowed to use is the **subdivision**: this simply means that we are allowed to add vertices to interior of segments, or remove the common vertex of two consecutive, aligned, segments.

The second move we use is the **$\Delta$-move**. Any pair of consecutive segments in $\mathbb{R}^3$ determines a Euclidean triangle. If such triangle does not intersects our link, we are allowed to replace the pair of segment with the third segment in the boundary of the triangle. And vice versa.

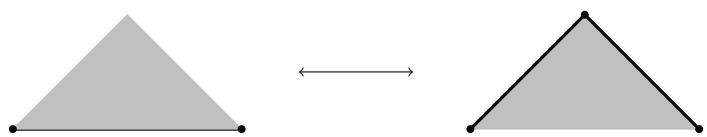


FIGURE 6. $\Delta$-move

**Definition 2.2.8** (Delta Equivalence)**.** *Two PL-links $L_1$ and $L_2$ are equivalent if they can obtained one from the other by a finite sequence of subdivisions and $\Delta$-moves and their inverses.*

The above three notions of equivalence are in fact equivalent to each other:

**Theorem 2.2.9.** *Two links $L_1, L_2$ in $S^3$ are equivalent if one (hence all) of the following equivalent conditions is fulfilled:*

- *There is an ambient isotopy between $L_1$ and $L_2$;*
- *there is an orientation preserving homeomorphism $f : S^3 \to S^3$ such that $f(L_1) = L_2$;*
- *they are equivalent up to subdivisions and $\Delta$-moves.*

*Proof.* The proof of this theorem can be found in [10, Appendix A]. □

Note that the homeomorphism is required to be orientation preserving. This is crucial in some aspects of chemistry. Given a knot (or link) in $\mathbb{R}^3 \subset S^3$, one can consider its reflected image along a plane which does not intersect it (such reflection is an orientation reversing homeomorphism of $S^3$). This is called the **mirror image** of the knot.

**Definition 2.2.10.** *A knot is called chiral if it is* **not** *equivalent to its mirror image.*

An example of chirality in chemistry is the glucose (or dextrose, which in Italian is also called *destrosio*, which sounds like "rightose"). Its mirror image is the fructose (or levulose, which in Italian is also called *levulosio* or *zucchero invertito*). They are basically the same molecule, but they interacts differently. For instance, glucose crystallises while fructose doesn't (that's why in fruit we find mainly fructose and not glucose, and that's why in patisserie it is used fructose for certain cakes).

An example of chiral knot (but for the proof of its chirality we have to develop some more theory) is the trefoil knot: it exists in its left and right versions, like glucose (See Figure 7).

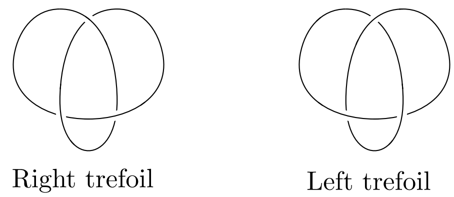



FIGURE 7. The right and left trefoil knots.

A knot which is not chiral is called **achiral** (or **amphicheiral**). In the spirit of Theorem 2.2.9, one can see that a knot $K$ is achiral if and only if there is an *orientation-reversing* homeomorphism $f : S^3 \to S^3$ such that $f(K) = K$.

**Exercise 2.2.11.** *Each orientation preserving homeomorphism of the sphere is isotopic to the identity (see* [2, Section 2.2]*): use this fact to prove the previous statement.*

An example of such is the figure-eight knot. The proof of achirality of figure-eight knot is easy: see Figure 8.

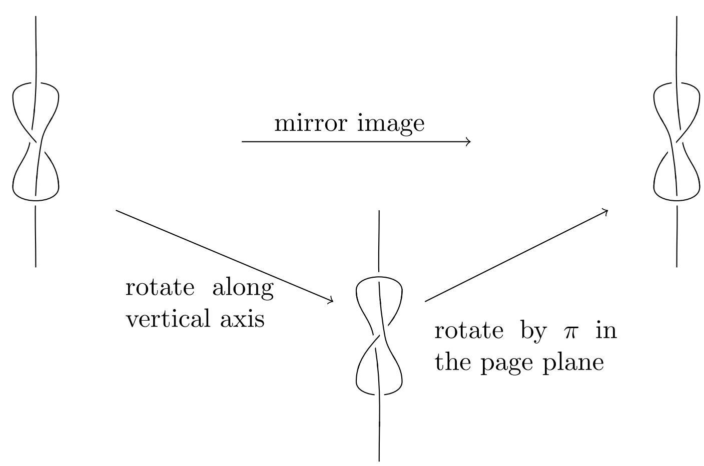



FIGURE 8. The figure-eight knot is achiral

Chirality or achirality are quality of knots that are clearly well defined w.r.t. the equivalence relation: by definition if two knots are equivalent, then either both are chiral or both are achiral. This is the first example of **invariant** of knots, that is to say, a quality that remains *unvaried* under the deformations defining the equivalence relation.

Such quality allow us to distinguish knots: if we know that the trefoil is chiral, then it follows that the trefoil and the figure-eight knots are not equivalent to each other: that's simply because one is chiral and the other is achiral.

**Further reading.** With the same ideas, one can define other invariant qualities of knots. For instance, one can consider oriented knots, and ask whether or not a knot can be deformed to itself, but with the opposite orientation. This notion is called **inversibility**. For example, the trefoil knot is chiral (not equivalent to its mirror image) but inversible (equivalent to itself with orientation reversed): can you prove this? Can you give more example of invariant qualities of this type? How many?

Two oriented knots can be connected in a very intuitive way, forming a third knot. This procedure is called **sum** (sometimes *connected sum*) of knots and it is depicted in Figure 9.

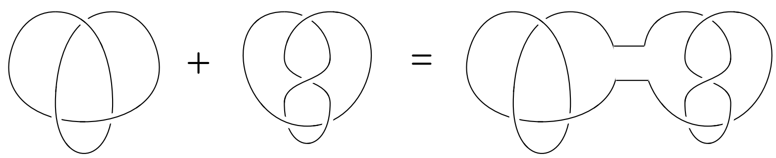


FIGURE 9. Sum of knots

More precisely, the sum of two oriented knots $K_1, K_2$ can be defined as follows. First put knots in $S^3$ so that there is a sphere $S$ having one knot on one side and one on the other. For each $i = 1, 2$ chose a point $x_i$ in $K_i$. Chose an embedded arc $\gamma$ connecting $x_1$ and $x_2$ outside the knots, intersecting $S$ only in one point and having a neighborhood $U$ which intersects $K_1 \cup K_2$ only in two small arcs near $x_1, x_2$ respectively. Then, for $i = 1, 2$, remove from $K_i$ a small interval $I_i \subset U \cap K_i$ around $x_i$, creating in this way two end-points $x_i^+, x_i^-$, labelled according with a chosen orientation of $K_i$. Finally connect $x_1^+$ to $x_2^-$ and $x_1^-$ to $x_2^+$ using paths parallel to $\gamma$ and contained in $U$.

**Exercise 2.2.12.** *Show that the sum of two knots is well-defined, that is to say, its equivalence class does not depend on: the representatives of $K_1$ and $K_2$ in their respective classes, the choices of the orientation, $x_i, \gamma, I_i$ and the paths joining $x_i^\pm$.*

The above exercise is very important especially for beginners. That's because one "sees" the solution, but then one has to translate what one "sees" in a formal proof. The most common example of this situation is (and very often used) is the following, known as disk theorem (from which you may want to take inspiration if you got stucked in the exercise).

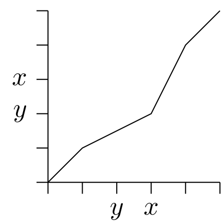
**Theorem 2.2.13** (Disk Theorem). *Let $D^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ be the closed disk of dimension $n$ and let $x, y \in int(D)$. Then there is an isotopy of $D$ that moves $x$ to $y$ and fixes the boundary. More precisely, there is a homeo $f : D \to D$ so that $f(x) = y$, which is isotopic to the identity via an isotopy which is the identity in a neighborhood of $\partial D^n$. Moreover all maps can be chosen to be smooth or PL.*

*Proof.* Let's start with the 1-dimensional case. Let $x, y \in (0, 1)$. It is not restrictive to suppose $x > y$. One can easily construct a piece-wise affine map $f : [0, 1] \to [0, 1]$

which is the identity near $0,1$ and sends $x$ to $y$. For instance consider the map

$$f(t)=\begin{cases} t & \text{for } t\in[0,\frac{y}{2}] \\ \dfrac{y}{2x-y}(t-x)+y & \text{for } t\in[\frac{y}{2},x] \\ \dfrac{1-x-2y}{1-3x}(t-x)+y & \text{for } t\in[x,\frac{1-x}{2}] \\ t & \text{for } t\in[\frac{1-x}{2},1] \end{cases}$$

whose graphic looks like:




If one wants a smooth map it is easy to produce such a map by smoothing corners via functions like $e^{-1/x^2}$. Now that one has $f$, the isotopy is just

$$F(s,t)=st+(1-s)f.$$

Note that for any $s$ the derivative in $t$ of $F(s,t)$ is always positive, so $t\mapsto F(s,t)$ is a homeo for any $s$.

Now, we pass to the high dimensional case. First we prove the theorem for the space $D^{n-1}\times[0,1]$ with both $x,y$ on the segment $\{0\}\times[0,1]$. If $\rho(p)$ denotes the radius of a point in $p\in D^{n-1}$, and $F$ is as above, then the map

$$g(p,t)=(p,F(\rho(p),t))$$

is an homeo that maps $(0,x)$ to $(0,y)$ and it is the identity at the boundary. The isotopy with the Identidy map is $G(r,(p,t))=rId+(1-r)g(p,t)$ (check that for any $r$ the map $(p,t)\mapsto G(r,(p,t))$ is a homeo). One can deform such map in order to be smooth (or PL), and so that it is the identity on a neighborhood of the boundary.

Finally, given two points in $D^n$, one can always find an homeo $\phi$ from $D^{n-1}\times[0,1]$ to a neighborhood $V$ of the two points, mapping $(x,0)$ to the first and $(y,0)$ to the second. Then $\phi\circ G\circ\phi^{-1}$, which extends to the identity outside $V$, is the requested isotopy that moves the first point to the second. □

**Exercise 2.2.14.** *Show that the sum of knots is abelian and associative. Show that the unknot is an (the) neutral element for the sum.*

The above exercise says that the set of equivalence classes of knots is an abelian monoid under sum. It can be shown that there is no inverse (other than for the unknot), but this is not an immediate thing.

The converse of the sum, is the so-called **Prime decomposition**. Given a knot $K$ which is the sum of two knots $K_1$ and $K_2$, it is clear that there exists an embedded 2-sphere in $S^3$ which cut $K$ in exactly two points so that $K_1$ is on one side of the sphere and $K_2$ is on the other.

Therefore, given a knot $K$ in $S^3$, if there is a PL 2-sphere $S$ which intersects $K$ transversally in exactly two points, we can cut $K$ with $S$, and then close up the result using segments coming from (different sides of) $S$. We obtain two knots $K_1$ and $K_2$ so that $K=K_1+K_2$. It may happen that one of the $K_i$'s is the trivial knot (hence the other is $K$), and in this case we call such decomposition a **trivial decomposition**.

**Definition 2.2.15.** *A knot is called* **prime** *if it has no non-trivial decomposition as sum of two knots.*

**Exercise 2.2.16.** *Show that trefoil knots and figure-eight knot are prime. (Hint: you may (ore may not) want to wait a couple of sections before diving in this exercise).*

Analogously to what happens for integers numbers with respect to the product each knot could be uniquely decomposed as sum of prime knots.

**Further reading.** Sum of knots is strictly related to connected sum of manifolds: to sum two manifold one just remove one small ball from each manifold and then glue together the result along the boundary sphere just created. Can you see relations between sum of knots and connected sum of (1-dimensional and) 3-dimensional manifolds?

**Curiosity:** When one binds shoes, if he does the connected sum of two trefoils of the same type (also called Granny knot, or "false knot", in Italian "nodo dell'asino") the bow does not hold; if, on the other hand, we sum two opposite trefoils (known as square knot) then the bow holds.

2.3. **From 3D to 2D: diagrams.** Till now we talked about knot and links in $S^3$ and we depicted knots and links on a plane. Such attitude can be formalized in a rigorous way.

First of all we can consider knots in $\mathbb{R}^3$. This is because a knot $K$ is homeomorphic to $S^1$ which is not homeomorphic to $S^3$. In particular $K$ is not $S^3$ and there is a point of $S^3$ which does not belong to $K$. By removing that point we obtain a knot in $\mathbb{R}^3$. (Same for links).

The class of $K$ as knot in $\mathbb{R}^3$ does not depend on the removed point. This is because (by using the Disk Theorem one can see that) given $x, y \in S^3 \setminus K$, there is an isotopy of $S^3$ which fixes $K$ and sends $x$ to $y$.

**Exercise 2.3.1.** *Prove the above sentence.*

Now, given $K$, chose a plane which does not intersect $K$. Since $K$ is compact, it is in particular contained in a ball; hence there are plenty of such planes.

Next, consider the orthogonal projection of $K$ to such plane. The result is a curve in the plane, which may self-cross in several points (and in fact will self intersect unless we started with the unknot). Note that if $K$ is a polygonal simple curve in $\mathbb{R}^3$, then its projection is a polygonal curve in $\mathbb{R}^2$. (The same for links)

Not all projections are good for our purposes. We will consider only **regular** projections, that is to say projections such that

- they are locally injective in $K$ (any point of $K$ has a neighborhood such that the restriction of the projection to such neighborhood is injective).
- There are only simple intersections: no multiple nor tangential crossings are allowed (see Figure 10).

Usually, in the $PL$-category, we require that crossings do not contain vertices. Using $PL$ category it is easy to see that regular projections exists, and moreover

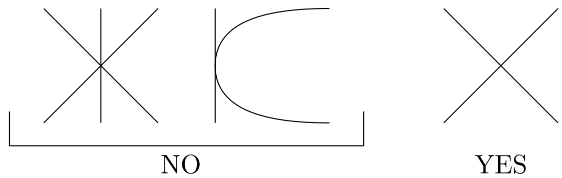



FIGURE 10. Regular projections

that almost all of them are regular (they form an open, dense set whose complement has measure zero).

Given a regular projection, in order to recover the knot from its image on the plane it suffices to specify, at each crossing, which strand passes below and which one above. There are only two possibility, that we realises graphically as usual (Figure 11, but if one wants to write computer programs has to specify a binary tag at each crossing)

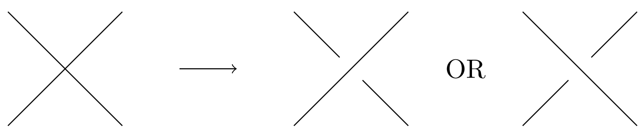



FIGURE 11. Crossing resolutions

The result of the image of a regular projection plus the specification of crossings is what is called a **knot diagram** (the same for links). Roughly speaking is like we had a real knot made of rope, we put it on the table and look from above. All knots and links depicted so far were knots and links diagrams. :)

More precisely, in order to define a projection we need a orthogonal decomposition of $\mathbb{R}^3$ as $\mathbb{R}^2 \times \mathbb{R}$, where $\mathbb{R}^2$ is our projecting plane. Thus any point in a knot has a "plane" coordinate and a "vertical" coordinate. The projection is given by just taking the plane coordinate. To recover the knot/link one only need to know the height function at any point. Under/over crossings, just says which strand has higher height function. Then one can easily show that any two height functions satisfying the crossing conditions produce isotopic knots/links in $\mathbb{R}^3$.

**Curiosity:** the previous way to label crossings of a diagram in order to reconstruct the link is not the only one. In the paper [1] of 1928, one of the first about knot theory, the author use the following method: at each crossing, put a dot in two of the four corners in such a manner that '*an insect crawling in the positive sense along the "lower" branch trough a crossing would always have the two dotted corners on its left.*' Can you depict the link diagrams of Figure 2 using this convention?

2.4. **Reidemeister moves and link-equivalence.** If we have a knot $K$ and we move it with an isotopy, the corresponding diagrams chances. Fortunately they changes in a controlled way. More concretely there is a handful of simple moves on diagrams, called Reidemeister moves, so that any isotopy translates in a finite sequence of such moves. Such moves are describe by words as follows and depicted in Figure 12

(1) Twist/untwist a strand on itself;
(2) locally slide a strand under a neighbor;
(3) move a strand completely over or under a crossing.

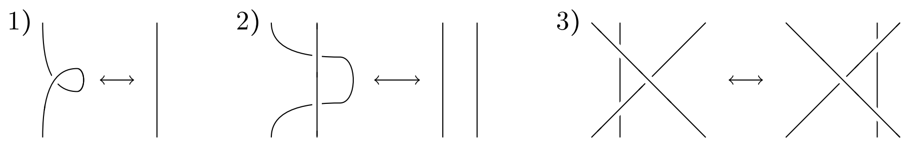



FIGURE 12. Reidemeister moves

**Theorem 2.4.1.** *Two links $K_1, K_2$ are equivalent if and only if their diagrams are connected by a finite sequence of Reidemeister moves and isotopies (as subsets of $\mathbb{R}^2$).*

*Proof.* We work in the $PL$ category. By Theorem 2.2.9 it suffices to prove that $\Delta$-moves translate to finite sequence of Reidemeister moves and isotopies, and vice-versa.

Consider a $\Delta$-move associated to a triangle $T$. We can subdivide $T$ in smaller triangle, as small as needed and apply the following argument to any single triangle. Moreover, up to isotopy we may assume that the projection is regular before and after the $\Delta$-move.

We may thus assume that $T$ is small enough so that its projection, apart from the side on which the $\Delta$-move takes place, intersects the diagram of $K$ in a connected set, contains in its interior at most one vertex/crossing of the diagram of $K$, and if this is the case, all strands emanating from such vertex/crossing exit from $T$ through edges. Therefore, and taking in mind that our projection is regular before and after the $\Delta$-move, one can easily see that the possible case are:

(1) Empty intersection;
(2) $T$ intersects a single segment of $K$, not crossing vertices of $T$;
(3) $T$ intersects a single segment of $K$ which crosses also a vertex of $T$;
(4) $T$ contains a vertex of $K$ in its interior;
(5) the interior of $T$ contains a crossing whose strands touches all edges of $T$;
(6) the interior of $T$ contains a crossing whose strands touches only two edges of $T$;

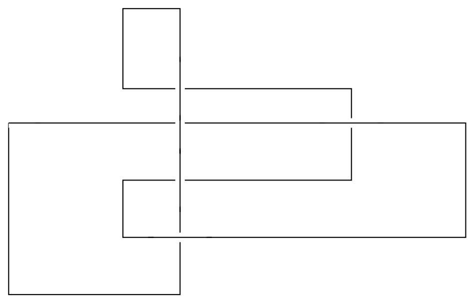
One has to note that since in a $\Delta$-move the interior of the triangle does not intersect the link, all crossing appearing at the boundary of $T$ with the remaining of the link are all of the same type: all under or all over crossings. Then, a case by case check shows that:

(1) In this case the $\Delta$-move corresponds to an isotopy;
(2) here we have either isotopy or Reidemeister move 2;
(3) that's Reidemeister 1;
(4) either isotopy of Reidemeister 2;
(5) this is either Reidemeister 3 or a combination of 2 plus 3;
(6) this is either Reidemeister 3 or a combination 2, 2, 3.

The converse is even easier: isotopies of diagrams lift to isotopies of links, up to isotopy Reidemeister moves can be made $PL$ and it is readily checked that all of them can be obtained via $\Delta$-moves.

□

**Exercise 2.4.2.** *Show by means of Reidemeister moves that the following knot is equivalent to the right trefoil knot.*



**Exercise 2.4.3.** *Prove that, given a link diagram of $K$; the mirror image of $K$ is represented by the same diagram but with all crossing flipped: overcrossings becomes undercrossings.*

Given a link diagram, there are crossings — like for instance that appearing on Reidemeister move 1 — that are easily removed by thinking to twist a portion of the link in $\mathbb{R}^3$ around that crossing. More precisely, a crossing of a link diagram is called **removable** if there is a circle in the plane, intersecting the crossing transversally and not intersecting the link elsewhere. (See Figure 13).

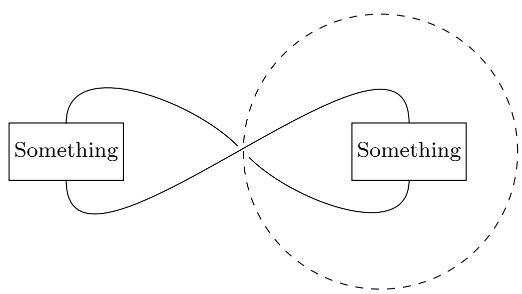



FIGURE 13. A removable crossing

**Definition 2.4.4.** *A link diagram is called* **reduced** *if it has no removable crossing.*

## 3. Combinatorial Invariants

3.1. **Invariants.** An invariant of knots or links is a way to associate a mathematical object to each knot/link so that it depends only on the equivalence class of the knot/link. In other words an invariant is something associated to a link, which is invariant under the transformations defining equivalence (isotopies, $\Delta$-moves, Reidemeister moves. . . ).

Invariants can be numerical, but can be of any (mathematical) nature. We've already seen invariant qualities such chirality and inversibility. Another easy example is the **number of connected component** of a link. This is clearly invariant under mentioned transformations, and knots are precisely those link whose "connected component"-invariant is one. WE denote with $n(L)$ the number of connected component of a link $L$.

So, two knots that are equivalent must have the same invariants. Therefore, invariants are often used to detect knots.

**Spoiler:** the two major problems in detecting knots, that is to say:

- given a knot diagram recognize algoritmically whether or not it is the unknot; and more generally:
- give two link diagrams decide algoritmically whether or not they represent the same knot;

are solvable in theory. So we have a theoretic complete classification of knots. But the theoretic algorithms solving such problems are practically not feasible, and in practice invariants are very useful.

An example of such unfeasability is the following theorem, which is considered a very good result:

**Theorem 3.1.1** (Lackenby [12])**.** *Any diagram of the unknot with n crossings may be reduced to the trivial diagram using at most* $(236n)^{11}$ *Reidemeister moves.*

So, in principle, given a diagram, you can compute all sequence of diagrams obtained with $(236n)^{11}$ moves and see if you find the unknot. But the number of all such diagram is highly exponential in $n$. (Did you know that $10^9$ seconds are more or less 30 years?)

The best result so far for detecting the unknot is an algorithm (announced by the same Lackenby) solving the problem in $n^{c\log(n)}$ steps, for some constant $c$.

**Further reading.** There are many approaches to decision problems in knot theory, and many algorithm – more or less efficient – to solve them. They use tools from many mathematical areas: Combinatorics, 3-manifolds theory, logic, and also some deep learning stuff. Which is your favourite algorithm?

**Exercise 3.1.2.** *Play the following game with a friend: Take simple figure-eight and a trefoil knot diagrams. Each player manipulate one of them via Reidemeister moves, without telling the other which knot was. Then compare the two diagrams. Wins the first player which is able to say if the two diagrams correspond to the same knot or not.*

There are several ways to produce invariants: define objects that are invariant on the noose (like the above "connected component"-invariant); define object associated to knot-diagrams and show that are invariant under Reidemeister moves (this is the way polynomial invariants are defined, see Section 6); or associate a quantity/quality to each link diagram and then among all the diagrams representing the same link, chose the simplest one. The latter is the way used to define combinatorial invariants.

3.2. **The crossing number.** Any link diagram has a well defined number of crossings, which is clearly NOT invariant under Reidemeister moves. No problem: take the minimum possible.

**Definition 3.2.1.** *The* **crossing number** *of a link in $S^3$ is the minimum number of crossing of diagrams representing it.*

For example, the crossing number of the unknot is 0.

**Exercise 3.2.2.** *Show that there are no nontrivial links with crossing number one.*

**Exercise 3.2.3.** *Show that the crossing number of the Hopf link is* 2*. (Hint: it is clearly at most* 2*; if it were less, then the Hopf link would be trivial; in this case there would be a en embedded disk $D_1$ bounding the first component and disjoint from the second; but from the diagram we can see that there exists a disc $D_2$ bounding the first component and intersecting the second only once; gluing $D_1$ and $D_2$ along the common boundary we get an immersed PL sphere which intersects the second component only once; arguing by induction on connected components on $D_1 \cap D_2$ one can see that we can reduce to the case where the sphere is embedded, hence bounding a three ball: the second component then would enter that ball and never exit, providing a contradiction. All these reasoning can be rigoroussly formalized, even in a shorter way if one can use homology.)*

**Exercise 3.2.4.** *Show that the crossing number of the trefoil knot is* 3*. (After having solved this exercise: Are you sure your proof is correct?).*

**Exercise 3.2.5.** *Show that there are no nontrivial knots with crossing number* 2*.*

**Exercise 3.2.6.** *List all possible knots with crossing number at most* 3*.*

You got it. One can enumerate knots by crossing number. And in fact, there exists tables of knots and links listed by crossing numbers. You may find it in any book of knot theory. (Or in internet for instance here:
`http://katlas.org/wiki/The_Rolfsen_Knot_Table`)

**Caveat:** Tables are usually up to mirror images, so for instance you'll find only one trefoil knot, not both.

**Further reading** One of the most famous conjecture still open about crossing number is the additivity under sum of knots

**Conjecture 3.2.7.** *If $K = K_1 + K_2$ then $Cr(K) = Cr(K_1) + Cr(K_2)$.*

It is easy to see that $Cr(K) \leq Cr(K_1) + Cr(K_2)$ just by summing two minimal diagrams. On the other direction, the best known result so far is $152Cr(K) \geq Cr(K_1) + Cr(K_2)$.

**Exercise 3.2.8.** *Show that above inequalities imply that a knot with crossing number* 238 *(or* 922*) has no inverse under sum.*

On the other hand, there are classes of knots where additivity is known: for instance that of alternating knots.

3.3. **Alternating links.** Given a link diagram, one can chose and follow a strand: it will pass sometimes under other strands, sometime over. We refer to these situations as under-crossing or over-crossing

**Definition 3.3.1.** *A link diagram is* **alternating** *if any strand cyclically meets under- and over-crossings alternately. A link is called alternating if it admits an alternating diagram.*

Proving that a link is alternating is easy: it suffices to show an alternating diagram. For example, trefoil and figure-eight knot are alternating knots, Hopf link is an alternating link. Much more difficult is to prove that a link is not alternating. But non-alternating knots do exist. (The smallest non-alternating knot has 8 crossings. Want to know who is it? Check some knots tables).

Alternating links and knots have particularly good properties (whose proves however, require sophisticated techniques and usually a strong use of polynomial invariants). Among others we have:

- The number of crossing of a reduced alternating diagram realises the minimum, i.e. the crossing number of the link they represent. (This was known as one of the Tait conjectures). We will deal with it in Section 6.1
- The (sum of alternating knots is alternating, and the) crossing number is additive in the family of alternating knots.
- An alternating knot $K$ is prime if and only if "it looks prime". That is, if their alternating reduced diagrams are prime (any circle $C \subseteq \mathbb{R}^2$ meeting the diagram transversally in two non-crossing points, split $\mathbb{R}^2$ in two regions, one of them contains only a simple arc of the diagram. Proven in [14].)
- If a knot is prime and alternating, then any diagram realising the crossing number is alternating. (This is false in the composite case, a counterexample being the square knot, that is the sum of a right and left trefoil.)

More geometric properties of alternating links will be described later.

**Further reading.** Search for complete proves of above results.

**Curiosity:** Such knowledge of alternating knots can be very useful in real life. For instance, in climbing one uses often the figure-eight knot for securing itself. It may happen in a situation of stress that one has doubts about its knot: well, if it has four crossings and it is alternating (and no removable crossing), then it is the figure-eight, otherwise not.

Note that unless we use some strong result that we stated without a proof, we are still not able to prove that the figure-eight knot is non-trivial.

3.4. **Colorability.** In a link diagram with under/over crossing depicted as usual, strands are divided in connected components. Namely any strand-component starts and ends when it meets undercrossings.

A link diagram is **tricolorable** if its strand-components can be colored by using three colors in such a way that:

- At least two colors are effectively used;
- At each crossing, the strand-components we see have either all the same color, or all different colors.

Note that the diagram of the unknot is not tricolorable, just because it has only one strand-component, so only one color is effectively used.

At this point one would like to declare a knot tricolorable if it has a diagram which is tricolorable. But we can do better: tricolorability is a quality of diagrams which is invariant under Reidemeister moves! This is easily seen from Figure 12. Hence, if a knot has a tricolorable diagram, then all its diagrams are tricolorable.

**Definition 3.4.1.** *A knot (or link) is* **tricolorable** *if its diagrams are tricolorable.*

**Theorem 3.4.2.** *The trefoil knot is tricolorable. The figure-eight knot is not tricolorable.*

*Proof.* Just try. □

**Corollary 3.4.3.** *The trefoil knot is not the unknot, in particular it has crossing number* 3*. (Remember Exercise 3.2.4?). The trefoil knot is not equivalent to the figure-eight knot.*

One can generalizes the tricolorability to $p$-colorability, with $p$ prime. Instead of color use tags in $0, 1, ..., p-1$ taken modulo $p$. Then ask that at any crossings, if $x$ is the color of the over-crosser strands, and $y, z$ the other two, require

$$y + z \equiv 2x \ (mod\ p).$$

**Exercise 3.4.4.** *Show that in case* $p = 3$ *this is exactly the tricolorability.*

A diagram is $p$-colorable if admits a $p$-coloring with at least two colors effectively used. Checking $p$-colorability of a diagram boils down to solve a linear system with coefficients in the finite field $\mathbb{Z}/p\mathbb{Z}$. More precisely, if a knot diagram has $n$ crossings, then one has $n$ strand-components, and therefore the system is a system of $n$ equations in $n$ variables $x_1, \dots, x_n$. Since equations are invariant under translations $x_i \mapsto x_i + k$, one always has the solution $(k, \dots, k)$. So the system always has rank at most $n-1$. Solutions multiple of $(1, \dots, 1)$ correspond to trivial colorings with only one color, we refer to them to monochromatic solutions. The requirement that the system has a non-monochromatic solution then translates in asking that the rank of the system is at most $n-2$.

**Exercise 3.4.5.** *Show that p-colorability of diagrams is invariant under Reidemeister moves, hence it is am invariant of knots. (Hint: write the two linear systems after and before a Reidemester move and show that a system has a non-monochromatic solution if and only if the other system has one.)*

So, now $p$-colorability is a link invariant.

**Exercise 3.4.6.** *Show that the figure-eight knot is* 5*-colorable (using four colors). In particular it is not trivial, and has crossing number* 4*.*

3.5. **Bridge number.** If we draw a link diagram just by taking a knotted rope, throwing it on the table and look from above, we can imagine that the rope is a street of a city. In this case the over-crossings are brigdes. Clearly if a strand meets two consecutive over-crossings, one can consider this a single bridge.

Formally, a bridge decomposition of a knot diagram is a subdivision of the link it represents in two families of arcs:

$$b_1, \dots, b_k \quad a_1, \dots, a_k$$

such that $b$-arcs (the bridges) meet only over-crossings in its interior, and $a$-arcs under-crossings. If we have two consecutive $b$-arcs we can glue them together, and similarly with $a$-arcs. So we may assume that neighbors of $b$-arcs are $a$-arcs and vice versa. This proves that we have the same number of $a$- and $b$- arcs (up to unknotted components). This number is called **bridge number** of the decomposition. It is easily see that any diagram has a unique combinatorial type of decomposition with above properties, just by looking at maximal strand-components meeting only over-crossings.

**Definition 3.5.1.** *The bridge number of a link is the minimum bridge number of its diagrams.*

**Example 3.5.2.** *The knot of Figure 14 has bridge-number two.*

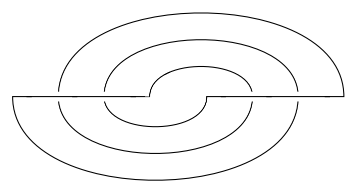


FIGURE 14. A knot diagram with two bridge (litlle secret: it is a trefoil)

Particularly interesting are the so-called **two-bridge links**, that is, those having bridge-number two.

**Exercise 3.5.3.** *Show that the trefoil knot is a two-bridge knot (hint: see Figure 15, compare with Figure 14.)*

**Exercise 3.5.4.** *Show that the figure-eight knot is a two-bridge knot. (Hint: see Figure 16).*

**Exercise 3.5.5.** *Show that there are no non-trivial knots with bridge-number one.*

One nice characteristic of two-bridge links is that they all can be arranged in a symmetric position similar to that in Figure 14, and that the first under-crossing that we meet from a bridge parameterises them.

More precisely, given a diagram with bridges $b_1, b_2$, up to isotopy the arcs $a_1, a_2$ can be arranged in such a way they under-cross alternatively $b_1$ and $b_2$. Then, up to isotopy, we can do also the converse (i.e. $b$-arcs meets alternatively $a_1$ and $a_2$).

**Remark 3.5.6.** *The above condition can be achieved as follows: if there is a situation in which, say the arc $a_1$ crosses two consecutive times $b_1$ in points $p, q$, then the sub-arcs $pq$ in $a_1$ and $b_1$ form an embedded circle, which by Jordan theorem, bounds two disk in $\mathbb{R}^2 \cup \{\infty\} = S^2$. One of them do not contains $b_2$. Looking at the intersection pattern of $a$-arcs with that disk, and arguing by innermost components one reduces the crossing number of the two-bridge diagram. By induction (or, just by considering a two-bridge diagram that minimizes the number of crossings) we reduce to the case where we do not have consecutive crossings of an $a$-arc with a $b$-arc. The same reasoning works to prove that there are no non-trivial one-bridge knots.*

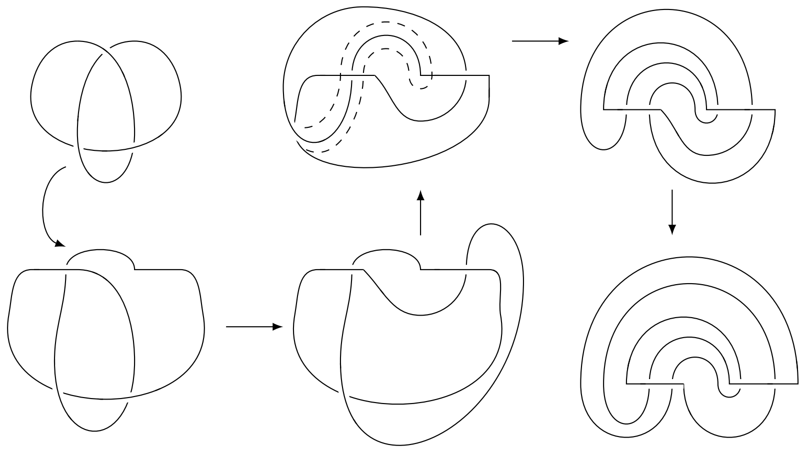


FIGURE 15. How to deform the standard trefoil diagram in a two-bridge diagram. Third and fifth steps are just taking the lowest strand and pass it, in $\mathbb{R}^3$, on the upper part of the diagram (or if you prefer moving it through the $\infty$ of $\mathbb{R}^2$). Fourth step is sliding the undercrossing along dashed lines. By rotating clockwise the left horizontal segment by $\pi$, one gets the diagram of Figure 14

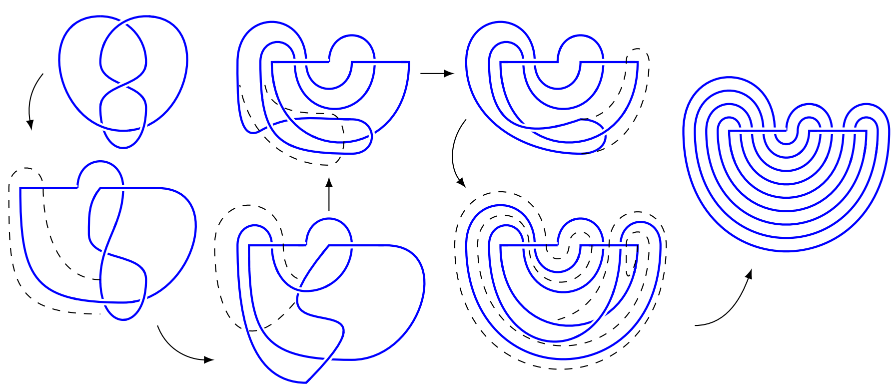


FIGURE 16. How to produce a two bridge diagram of the figure-eight knot. At any step, dashed lines indicate the following sliding of strands.

So we can arrange bridges horizontally and number the under-crossings, together with $b$-end-points, as follows ($0, k$-points are the $b$-end-points, others are under-crossings):

Then we have only to show where the left-zero-point jump on the right and the remaining of the diagram is uniquely determined.

**Exercise 3.5.7.** *Prove the last sentence. (Hint: when the jump of* 0 *is determined, by some instance of Jordan curve theorem, jump of* 1 *is also determined, and so on).*

**Exercise 3.5.8.** *Prove that in the above construction, if $k$ is odd then the result is a knot, otherwise it is a link with two components.*

**Exercise 3.5.9.** *Prove that two-bridge links are alternating.*

$n$-bridge links have also nice "braid"-diagrams. More specifically, given an $n$-bridge diagram of $L$ in $\mathbb{R}^2$, one can recover $L$ in $\mathbb{R}^3$ by defining height functions on any $a$- and $b$-arcs. On bridges, we choose a function with only one maximum, say at level 1 and no other critical point. At $a$-arcs we choose functions with only one minimum, say at level $-1$ and no other critical point. Now project the link on a vertical plane, and move it a little if necessary to obtain a regular projection. One obtained a link-diagram with $n$ maximum at level 1, $n$ minimum at $-1$ and no other critical points. That is to say, one takes a braid with $2n$ strands (see Section 7) and cap the top layer and the bottom one with arcs. In Figure 17 is depicted such kind of diagram for the figure-eight knot.

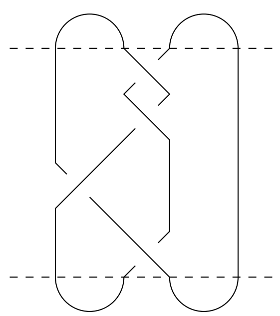


FIGURE 17. A two-cap-braid diagram of the figure-eight knot

**Further reading.** It can be shown that the bridge number well-behaves under connected sum:

$$Br(K_1 + K_2) = Br(K_1) + Br(K_2) - 1.$$

This in particular implies that no non-trivial knot has a sum-inverse; or that non-prime knots have bridge number at least three (so for instance two-bridge knot are prime – see also [6, Prop. 7.8] for an independent proof of that).

There are many different way of defining the bridge number, you may search and find your favourite.

3.6. **More and more invariants...** One can define many other combinatorial invariants. Another famous one is the unknotting number: it is the minimum swap of crossing necessary to unknot a knot. You may search and find your favourite invariant.

## 4. Topological invariants: looking outside the knot

4.1. **The complement.** A mind twisting game you can do with colleagues is the following: ask for two submanifold of $\mathbb{R}^3$, of the same dimension, that are NOT homeomorphic but such that their complements are homeomorphic. You can stand hours looking at them trying to figure out such an example… But: if you twist the question the other way, that is: ask for two submanifolds of $\mathbb{R}^3$ that ARE homeomorphic but whose complements are not homeomorphic, then you will receive immediately the answer: two non-equivalent knots!

It follows directly from Definition 2.2.4, that if two knots are equivalent then their complements in $S^3$ are homeomorphic. In other words, the homeomorphism type of the knot-complement is a knot-invariant. As we already mention, invariants need not necessarily to be numbers, or quantities. In this case the invariant is pretty topological.

As in the above game, even if passing to complements may seems naive, it is a very powerful tool.

In fact, it is a very deep result that the knot complement is a complete invariant, that is to say:

**Theorem 4.1.1** ([7])**.** *Two knots in $S^3$ are equivalent if and only if their complements are homeomorphic.*

(Note that since 3-manifolds are in principle classified, this theorem provides another theoretic possible classification of knots.)

**Further reading.** Link complement is a link-invariant, but it is not complete. If one looks at knots inside a 3-manifold, so not necessarily in $S^3$, then also in this case the above theorem fails (also for knots).

Now is time to make things more precise. What do we mean exactly by knot-complement? Well, one could just take the complement as subsets in $S^3$. This produces an open 3-manifold. It is sometime better to work with compact manifolds, so usually, when one speaks about knot complement what he really means is the complement, in $S^3$, of an open tubular neighborhood of the knot. The result now is a compact manifold with boundary (which is homeomorphic to the 2-torus) whose interior is homeomorphic to the topological complement of the knot. If you aim to avoid confusion a good choice is to call "exterior" the complement of the tubular neighborhood.

**Example 4.1.2.** *The exterior of the unknot is a solid tours.*

4.2. **Knot groups.** Since the knot complement is a knot invariant, every topological invariant of the knot complement turns out to be an invariant of the knot. The most classical example is the fundamental group of the knot complement, which is also referred to as the knot group:

$$\pi_1(S^3 \setminus K).$$

There is a nice and effective way to compute the knot group, called **Wirtinger presentation**: Given a knot $K$, chose a knot diagram, say in the horizontal plane, and place the base-point above the knot. For any connected component of the knot diagram (remember our convention that under-crossings disconnect the strand passing under) chose a loop going from the base-point around the strand.

**Exercise 4.2.1.** *Show that such loops generates $\pi_1(S^3 \smallsetminus K)$.*

Then one can easily see that if $x, y, z$ are the classes of loops corresponding to the three strands occurring at a crossing, say with $x$ over-crossings, then we can chose orientations of $x, y, z$ so that

$$xyx^{-1} = z$$

as element of the knot group. Therefore $\pi(S^3 \smallsetminus K)$ is generated by above loops and relations.

In order to write explicitly a presentation, choose an orientation of the diagram-plane and one of the knot, and cyclically names $\gamma_i$ the connected components (with indices taken modulo $n$ where $n$ is the number of components).

We call a crossing **positive** if the under-crossing direction is obtained by rotating counterclockwise the over-crossing direction by $\pi/2$ (see the right picture of Figure 23).

To each $\gamma_i$ associate the (class of) loop(s) $x_i$ that starts from the base point, goes once around $\gamma_i$, and crosses positively $\gamma_i$. Then if $\gamma_i, \gamma_{i+1}$ under-crosses $\gamma_k$ we have either

$$x_k^{-1} x_i x_k = x_{i+1} \qquad \text{or} \qquad x_k^{-1} x_{i+1} x_k = x_i$$

depending whether the crossing is respectively positive or negative.

Check pictures of this construction in any text book of knot theory. The fact that the fundamental group of the complement is exactly what we described can be formalised via Van Kampen Theorem (see for instance [17]).

**Exercise 4.2.2.** *Compute groups of left and right trefoils. (They should be isomorphic, right? Why? Exhibit an isomorphism.)*

**Exercise 4.2.3.** *If one has $n$ crossings then one gets $n$ relations. Show that one is consequence of the others. (If your are lazy you are allowed to check Proposition 4.2.8. But: are you sure to be so lazy?)*

**Exercise 4.2.4.** *Compute the group of the figure-eight knot. (You will simplify so to have two generators and one relation).*

*Let's do this exercise.* Relations are computed in Figure 18. Now,

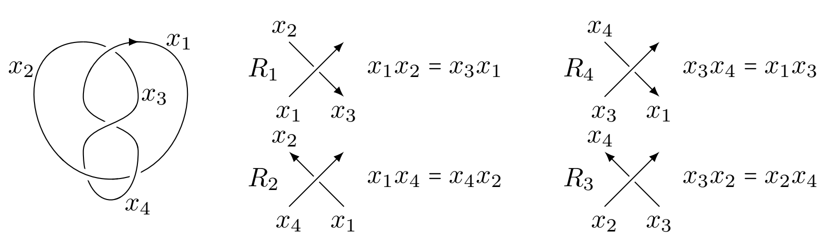



FIGURE 18. Wirtinger relations for the figure-eight knot.

$$R_4: \quad x_3x_4 = x_1x_3 \quad \text{is equivalent to} \quad x_3x_4x_2 = x_1x_3x_2 \qquad \text{but}$$

$$x_3x_4x_2 \overset{R_2}{=} x_3x_1x_4 \overset{R_1}{=} x_1x_2x_4 \overset{R_3}{=} x_1x_3x_2$$

Therefore the knot group of the figure-eight is generated by $x_1, x_2, x_3, x_4$ with relations

$$x_3 = x_1x_2x_1^{-1} \quad x_4 = x_2^{-1}x_3x_2 \quad x_1x_4 = x_4x_2.$$

First two relations combined describe $x_3, x_4$ in terms of $x_1, x_2$. So the group is generated by only $x_1, x_2$ with the only last relation, that becomes

$$x_1 x_2^{-1} x_3 \cancel{x_2} = x_2^{-1} x_3 x_2 \cancel{x_2} \quad \text{that is} \quad x_1 x_2^{-1} x_1 x_2 x_1^{-1} = x_2^{-1} x_1 x_2 x_1^{-1} x_2$$

□

Observe that relations we obtain are similar to those in permutation groups. This basically can be used to build morphisms from knot groups to groups of permutations, showing that in fact knot groups are not abelian.

**Exercise 4.2.5.** *By using the presentation of the figure-eight knot complement found in previous exercise, find a non-trivial morphism to the symmetric group in four elements. Use it to deduces that the figure-eight knot as non-abelian fundamental group, in particular it is not the unknot (if it does not work with your morphism, try a "less commutative" one).*

**Exercise 4.2.6.** *Do the same for trefoils.*

**Exercise 4.2.7.** *Show that two-bridge knot complements have fundamental group generated by two elements and one relation. (Hint: use the Wirtinger presentation with a diagram obtained as discussed at page 17, see also Figure 14. The generators can be chosen as those corresponding to bridges.)*

**Proposition 4.2.8.** *One of the relations in the Wirtinger presentations can be deduced from others.*

*Proof.* Choose a diagram, orient $K$, number strand-components cyclically, and define $\sigma(i)$ and $\epsilon_i = \pm 1$ so that relations are $x_{i+1} = x_{\sigma(i)}^{\epsilon_i} x_i x_{\sigma(i)}^{-\epsilon_i}$. Then, arguing recursively, one sees the last relation is

$$x_1 = x_{\sigma(n)}^{\epsilon_n} x_{\sigma(n-1)}^{\epsilon_{n-1}} \cdots x_{\sigma(1)}^{\epsilon_1} x_1 x_{\sigma(1)}^{-\epsilon_1} \cdots x_{\sigma(n-1)}^{-\epsilon_{n-1}} x_{\sigma(i)}^{-\epsilon_i}.$$

Now, observe that the word $g = x_{\sigma(n)}^{\epsilon_n} x_{\sigma(n-1)}^{\epsilon_{n-1}} \cdots x_{\sigma(1)}^{\epsilon_1}$ describe just a curve parallel to the knot. In particular both $x_1$ and $g$ are elements of $\pi_1(\partial N(K))$ where $N(K)$ is the tubular neighborhood of $K$ removed to build the knot exterior. But $\partial N(K) = T^2$ has abelian fundamental group, hence all its elements commute. So the last realtion is always satisfyed once previous ones are. Clearly this argument is independent from the chosen numbering of strand components, so one can deduce any relation from others. □

So, now, if knot groups are not abelian, what is their abelianisations? That's pretty easy: it is always $\mathbb{Z}$. This can be seen directly from the Wirtinger presentation: when you add commutativity, the Wirtinger relations just say that the generators are all the same! So the abelianised is the free group in one generator, that is to say, $\mathbb{Z}$. (This is often stated by saying that the first homology group of a knot exterior is $\mathbb{Z}$. The homology of a knot exterior can easily be computed also via usual exact sequences, like Mayer Vietoris or that of the pair).

**Exercise 4.2.9.** *Compute the homology of a Knot exterior.*

**Exercise 4.2.10.** *Extend the above discussion to links.*

4.3. **Seifert surfaces and genus.** Now, we still look outside the knot, but we focus not on the complete exterior. Heuristically, the unknot is the only knot that bounds a disc embedded in $S^3$ (can you prove this?). So one may ask if other knots bound some surface and what kind of information one can retrieve from such surfaces.

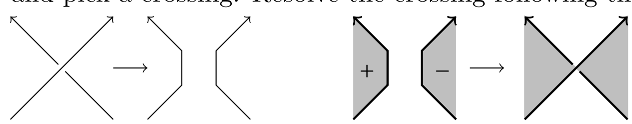
Since the 2-homology of $S^3$ is trivial, any link bounds a 2-cycle. It is a classical (but not easy) theorem of topology that one dimensional and codimension-one cocyles can be desingularised (and also 2-dim and codim, provided we work with oriented manifolds). That is to say, any cycle is homologous to an embedded surface. In particular any knot bound an embedded surface (wait a moment. . . embedded? It sounds strange, right? But yes: embedded.).

**Definition 4.3.1.** *A* **Seifert** *surface of a link $K$ is a compact connected oriented surface embedded in $S^3$ such that*

$$\partial S = K.$$

Any link has plenty of Seifert surfaces. We describe now a standard algorithm to easily get one.

**Algorithm for Seifert surfaces**. Very concisely, one can work by induction on the number of crossing of a diagram. No crossing, is some Jordan curve theorem. Now orient $K$ and pick a crossing. Resolve the crossing following the orientation.




By induction there is an oriented connected surface bounding the link represented by the new diagram. Note that such surface does not disconnect $S^3$. So we can connect the two strands where we had the crossing with an embedded arc not crossing the surface. Since the orientation of the surface induces that on the boundary, we can add a (twisted) band to the surface in such a way to preserve orientation and regenerate the original knot. The new oriented surface now bounds $K$.

You may want to do it more algorithmically and at once, without induction. Is the same: resolve each crossing following the orientations. Now you have a diagram without crossings, that is a handful of oriented Jordan curves in the planes, each one bounding a disc, whose orientation is decided by the orientation of the boundary. You may have some disc inside others: no problem we are in $\mathbb{R}^3$ you can move upwards a little the inner discs. Now attach twisted bands as before. Everything is coherent with a global orientation of the surface (up to isotopy, one always can put the discs so that near crossings they look as in above picture).

**Remark 4.3.2.** *Note that this algorithm also shows that the complement of a link diagram can be chessboard-colored: starts from a region, declare it black and follow the Seifert surface just constructed. Moreover, the black regions have signs $+,-$ that one meet alternately along the surface (this is often referred to by saying that the dual graph of the black regions is bipartite. This graph is known as Seifert graph).*

**Exercise 4.3.3.** *Compute Seifert surfaces of trefoil and figure-eight knot.*

Clearly, topological invariant of Seifert surfaces turns out to be knot-invariants. A numerical invariant that can be cooked in this way is the genus of a knot:

**Definition 4.3.4.** *The genus of $K$ is the least possible genus of a Seifert surface of $K$.*

**Remark 4.3.5.** *The unknot is the only knot with genus* 0*. This is because having genus* 0 *means bounding a disk.*

**Theorem 4.3.6.** *The genus is additive under connected sum.*

*Proof.* Let $K = K_1 + K_2$ and let $F_1, F_2$ be minimal Seifert surfaces for $K_1, K_2$ respectively. Since genus is additive by connected sum we get $g(K) \leq g(K_1) + g(K_2)$. Conversely, if $F$ is a minimal Seifert surface for $K$, we can use a sphere $S$ decomposing $K$ as $K_1 + K_2$, and put $F$ in generic position with respect to $S$. Thus $F \cap S$ is an arc joining the two points of $K \cap S$ plus some curves in $S$. Take an innermost curve $\gamma$. It bounds a disc $D$ in $S$ not intersecting $K$. Now we cut $F$ along $\gamma$ and we glue back two parallel copies of $D$ obtaining a new (possibly disconnected) surface $F'$ bounding $K$. Is $\gamma$ do not disconnect $F$, then $F'$ is a Seifert surface for $K$ with smaller genus, which is impossible since $F$ is minimal. So $\gamma$ must disconnect. It follows that $F'$ has two connected components. We keep only the one containing $K$ (by the way, the other has to be a sphere because of the minimality assumption on $F$, that is, $\gamma$ bounds a disc also in $F$), and this is a Seifert surface for $K$ intersecting $S$ in less components. By induction we can eliminate all closed curves form $F \cap S$. This provide a splitting of $F$ as the connected sum of two Seifert surfaces $F_1, F_2$ of $K_1$ and $K_2$, which may a priori be not mimimal, but in any case provide the needed inequality $g(K) = g(F) = g(F_1) + g(F_2) \geq g(K_1) + g(K_2)$ (hence a posteriori the $F_i$'s are minimal). □

Note that since non-trivial knot exists, we then have knots of arbitrarily high genus constructed via connected sum.

**Corollary 4.3.7.** *No knot except the unknot has an inverse under connected sum. Moreover, genus* 1 *knots are prime.*

*Proof.* Let $K_1, K_2$ be non trivial knots. Then both $g(K_1)$ and $g(k_2)$ are at least one (Remark 4.3.5). Then $g(K_1 + K_2) = g(K_1) + g(K_2) \geq 2$. So one hand we have that the sum cannot be trivial. On the other hand we have that the genus of any non-prime knot is at least 2. □

The genus of the Seifert surface constructed from a diagram as described above, can be easily computed via a Euler characteristic count: collapse discs to points and twisted bands to segments. You obtain a graph, that is the Seifert graph, to which the Seifert surface retract (a so-called spine) and from its characteristic you recover the genus of the surface. So diagrams provide upper bounds on the genus. Explicitly, if $d$ is the number of discs and $n$ is the number of crossings we have

$$d - n + 1 = 2 - 2g \qquad g = \frac{n + 1 - d}{2}.$$

**Exercise 4.3.8.** *Show that both trefoil and figure-eight knots have genus at most* 1*.*

**Corollary 4.3.9.** *Both trefoil and figure-eight have genus exactly* 1*. In particular, they are prime.*

*Proof.* We know by colorability (or because their groups are not abelian) that both are nontrivial. So they have genus at least one (Remark 4.3.5). By above exercise, their genus is exactly one. By Corollary 4.3.7 they are prime. □

From the Seifert surface, one can construct other invariants. A famous instance of that is the Seifert matrix, a matrix that can be associated to a diagram. Reidemeister invariance translates in matrix-operations. We will go into that when dealing with the Alexander polynomial.

4.4. **Peripheral systems.** Given a knot $K$ in $S^3$, consider a small tubular neighborhood $U$ of $K$. It is a solid torus, knotted in $S^3$. The boundary torus $T = \partial U$ of $U$, has fundamental group $\mathbb{Z}\times\mathbb{Z}$. In general, there is no standard way to choose generators of $\mathbb{Z}\times\mathbb{Z}$; but in the case of knots this is possible. First, we need a couple of purely topological facts.

Given a compact oriented $n$-manifold $M$, possibly with boundary, there is a natural intersection form

$$H_{n-1}(M,\partial M;\mathbb{Z})\times H_1(M;\mathbb{Z})\to\mathbb{Z} \qquad (\sigma,c)\mapsto\sigma\cdot c$$

defined as follows.

We are mainly interested in cases $n = 2,3$. We describe the argument for $n = 3$, for $n = 2$ being even easier. Given $\sigma\in H_2(M,\partial M)$ and $c\in H_1(M)$, by classical desingularisation theorems, we may assume that the class $\alpha$ is represented by an oriented surface $\Sigma$, non necessarily connected, but embedded in $M$; and that, similarly, $c$ is represented by an embedded oriented curve $\gamma$. Now, if necessary, modify via isotopy $\Sigma$ and $\gamma$ so that they meet transversely. At any intersection point assign $+1$ if the orientation given by $\Sigma$ and $\gamma$ agree with that of $M$, and $-1$ otherwise. Then make the algebraic sum of all those $\pm1$. The result is the intersection $\sigma\cdot c$. For the argument $n = 2$ replace $\Sigma$ with a curve. Note that both $M$ and $\Sigma$ may have empty boundary. (See any book of algebraic topology for details).

**Lemma 4.4.1.** *Let $M$ be a compact, oriented 3-manifold so that $\partial M$ is a torus. Let $\gamma_1,\gamma_2$ be simple closed curves in $\partial M$ which are the boundary of surfaces $\Sigma_1,\Sigma_2\subset M$. Then $\gamma_1,\gamma_2$ can be made disjoint via isotopy; in particular, if both are homotopically non trivial, then they are parallel up to isotopy.*

*Proof.* $\Sigma_1$ represent a class in $H_2(M,\partial M)$. Therefore the intersection $\Sigma_1\cdot\gamma_2$ is well defined. But since the intersection form is defined in homology and and $\gamma_2=\partial\Sigma_2$, hence is null in homology, then $\Sigma_1\cdot\gamma_2 = 0$. Since $\gamma_2\subset\partial M$, then $\Sigma_1\cap\gamma_2=\gamma_1\cap\gamma_2$. So

$$\gamma_1\cdot\gamma_2 = 0$$

in $\partial M$. Standard arguments on curves on the torus now conlcude. □

**Exercise 4.4.2.** *Write precisely the "standard arguents" needed in the above proof. (Hint: first do the case where one of the $\gamma_i$'s disconnects; then use an infinite cyclic covering of the torus where $\gamma_1$ lifts to a simple closed curve. The cover can be constructed by simply cut $T^2$ along $\gamma_1$ and glue $\mathbb{Z}$ copies of that along boundary components in such a way the orientation is preserved. Show that such cover is a cylinder. What's goes wrong in higer genus surfaces?)*

**Corollary 4.4.3.** *Let $K$ be a knot in $S^3$ and let $F_1,F_2$ be to Seifert surfaces for $K$. Let $V$ a tubular neighborhood of $K$, small enough so that both $F_i$ intersect $V$ in an annulus. Then the curves $F_1\cap\partial V$ and $F_2\cap\partial V$ are parallel up to isotopy.*

*Proof.* Just apply above lemma to $M = S^3\setminus V$. The curves $F_i\cap\partial V$ are homotopically nontrivial because $\partial F_i = K$. □

We are now in position to complete our task. A **meridian** of $K$ is every simple closed curve $m$ in $T$ which bounds a disc in $U$. They are all isotopic to each other be Lemma 4.4.1. A meridian is therefore well defined up to isotopy and orientation. **A longitude** of $K$ is any other simple closed curve $l$ in $T$ which intersects $m$ only once. Heuristically a longitude runs along the knot. Curves $m, l$ generates $\pi_1(T)$. There are many longitudes, in fact, given one, say $l$, they are all of the form $l' = \pm l + km$ for some $k \in \mathbb{Z}$.

However, we can chose **the** longitude of $K$ in a unambiguous way. Namely: chose the trace of any Seifert surface of $K$ (they are all isotopic by Corollary 4.4.3). Again, the longitude is well defined up to isotopy and orientation.

**Remark 4.4.4.** *The longitude is the only non-trivial curve which is trivial in the homology of the knot exterior.*

So, given a knot $K$, one can chose $m, l$ in a unique way by (orienting $K$) and choosing $m, l$ as above so that they form a positive basis of the tangent plane of $T$ at the unique point they meet (where $T$ is oriented as $\partial U$, and $U$ inherits the orientation from that chosen for $S^3$. Changing the orientation of $K$ simultaneously change those of both $m, l$.). While $m, l$ are well defined isotopy classes of curves in $T$, the element they represents in the fundamental group of the exterior of $K$ is well defined only up to common conjugation.

The triple $(\pi_1(S^3 \setminus K), [m], [l])$ is called **peripheral system** of $K$ (it is well-defined up to isomorphism. Note that $x \mapsto x^{-1}$ is an isomorphism, so the orientation of $K$ don't really matter). The following, is a celebrated theorem of Waldhausen:

**Theorem 4.4.5.** *Two knots $K_1$ and $K_2$ with peripheral systems $(G_1, m_1, l_1)$ and $(G_2, m_2, l_2)$ respectively are equivalent if and only if there is a group isomorphism $f : G_1 \to G_2$ such that $f(m_1) = m_2$ and $f(l_1) = l_2$.*

**Exercise 4.4.6.** *Write explicitly peripheral systems for right and left trefoils.*

There is easy algorithm to writhe a peripheral system in the Wirtinger presentation. All generators are meridian: choose one $x_0$. Now start following the knot and put a $x_i^{\pm 1}$ at any **under**-crossing you meet, where the $x_i$ corresponds to the strand that over-crosses and the sign depends on the sign of the crossing. The results represent a curve parallel to the knot in a tubular neighborhood of the knot. So it intersects once a meridian and it is therefore a putative candidate longitude. Now, remember that the longitude is the unique which is null in homology: add $x_0^k$ where $k$ is minus the sum of exponents of the $x_i$'s you wrote so far. That's the longitude.

**Exercise 4.4.7.** *Compute the peripheral system of the trefoil with the described algorithm and check that it coincides (up to common conjugation) to that you found in previous exercise.*

We remark that meridian and longitude are never trivial in the fundamental group of the torus exterior, unless $K$ is the trivial knot. This is an instance of the following more general fact.

**Theorem 4.4.8.** *Let $K$ be a non-trivial knot in $S^3$ and let $V$ be a tubular neighborhood of $K$. Then $\pi_1(\partial V)$ injects in $\pi_1(S^3 \setminus K)$.*

*Proof.* If it would not be the case, then by the so-called Loop Theorem (which is a difficult result), there would exist a non-trivial simple closed curve in $\partial V$ bounding an embedded disc in $S^3 \setminus V$. By Lemma 4.4.1 such a curve must be the longitude. Thus $K$ would bound a disk, hence it would be trivial. □

**Corollary 4.4.9.** *No non-trivial knot has abelian knot group.*

*Proof.* We know that the abelianisation is $\mathbb{Z}$, therefore if the group would be abelian, then it would be $\mathbb{Z}$. But there are no injective morphisms from $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. □

4.5. **Linking number.** This is an invariant of links. The Hopf link is a link formed by two circles "linked once one around each other". The three circles of Borromean link are mutually unlinked (if the third circle was not there, you could separate them.)

This can be formalised in (at least) a couple of nice equivalent way. We just defined Seifert surfaces, so let's use them.

First, orient $S^3$. Given two oriented knots (i.e. an oriented link with two components) $K_1$ and $K_2$ let $F$ be a Seifert surface of $K_1$. We may assume that $K_2$ meets $F$ transversally. At any intersection point assign +1 if the orientation given by $F$ and $K_2$ agrees with that of $S^3$ and $-1$ otherwise. Then makes the algebraic sum of all those $\pm1$. The result is a number, which is the **linking number** of $K_2$ with respect $K_1$, and usually denoted by

$$lk(K_1, K_2).$$

The careful reader surely noticed that $lk(K_1, K_2)$ is just the intersection $F \cdot K_2$. The notation suggests that it does not depend on the chosen Seifert surface. Indeed, if we have another $F'$, we can reverse the orientation of $F'$ and join them together to obtain o closed surface $\Sigma = F \cup F'$. Since the second homology group of $S^3$ is trivial, then algebraic intersection between $K_2$ and $\Sigma$ is zero (roughly, $K_2$ must enter $\Sigma$ the same amount of times that it exits). So the intersection with $F$ cancel that with $-F'$, hence the algebraic intersection of $K_2$ and $F$ is the same as that with $F'$.

A simple way to compute the linking number is the following. Draw a diagram of the link ($\mathbb{R}^2$ is oriented as usual). We already know how to assign $\pm1$ to crossings: well the linking number is just the half of algebraic sums of signs of crossing between the two link components.

**Exercise 4.5.1.** *Show that the linking number defined via crossings is invariant under Reidemeister moves (hint: since we are looking at crossings between different components, type* (1) *moves do not hurt).*

**Exercise 4.5.2.** *Show that the two ways of defining linking numbers agree. (Take a Seifert surface constructed with the algorithm, consider $K_2$ as a loop, decompose it in a Wirtinger-way and play with diagrams).*

This in particular show that $lk(K_1, K_2) = lk(K_2, K_1)$ because the second construction is symmetric.

If one has a link with many components, one can compute the linking number of all pairs of components: this is a link invariant.

**Caveat:** Even if two components of a link $L$ have zero linking number, they still may be "linked" in $L$. A classical example is the Whitehead link.

**Further reading.** Linking number can be defined from "framed" knots, that is knots where an orientation and a normal vector are chosen at any point. One can then compute the so called self-linking number (using the vector normal to the plane diagram, often called black-board frame).

Linking numbers also appears in may aspects of Physic, Chemistry and Biology (think to the DNA).

## 5. Torus knots

5.1. **Defintion and first properties.** Let's step back one dimension for two seconds: what can we say about knots and links on surfaces? Well, in $\mathbb{R}^2$ and $S^2$ there are only unknots: this is Jordan curve theorem. But in a two-dimensional torus $T^2$ we have plenty of nontrivial simple closed curves. If we place $T^2$ in $S^3$ in a unknotted way (that is, the boundary of the tubular neighborhood of the unknot), such curves define knots in $S^3$, which are called torus knots/links.

Given an unknotted torus $T^2$ in $R^3$, its meridian and longitude are well defined (up to orientation, see page 25). Be careful: in $S^3$ an unknotted torus bounds two solid tori: so given an unknotted torus in $S^3$, the pair meridian/longitude is well defined, but one can interchange meridian and longitude: both of them bound a disk: one on one side of the torus, one on the other side.

So, once chosen, the pair meridian/longitude defines a basis of $\pi_1(T^2) \simeq H_1(T^2) \simeq \mathbb{Z} \times \mathbb{Z}$. Any closed curve $\gamma$ has coordinates $(p,q)$ in such basis. (This means that $\gamma$ goes $p$ times around the meridian and $q$ times around the longitude.)

For any $(p,q)$ there is a unique simple closed (multi) curve in $T^2$ up to isotopy with coordinates $(p,q)$. Such (multi) curve has $\gcd(p,q)$ parallel components. Thus if $p,q$ are coprime this construction produces a knot, otherwise a link with $\gcd(p,q)$ strands. We refer to knots and links obtained in this way as **torus knot/link** of type $(p,q)$.

**Remark 5.1.1.** *Torus knots of type $(\pm 1, q)$ or $(p, \pm 1)$ are trivial, so we always assume both $p,q$ different from $\pm 1$.*

Note that a knot of type $(p,q)$ is also a knot of type $(q,p)$: this is just because we can interchange the role of meridian and parallel.

**Exercise 5.1.2.** *Choose an unknotted solid torus in $\mathbb{R}^3$ and show that a $(p,q)$ knot on its boundary is equivalent to a $(q,p)$ knot by exhibiting an isotopy. (Hint: displace the curve to "the other solid torus" and then rotate that torus an put it in the place of the original one)*

**Exercise 5.1.3.** *Show that trefoil knots are the $(2,3)$ and $(2,-3)$ torus knot.*

Note that $(p,-q)$ always produces the mirror image of the $(p,q)$ link: this can be seen by using a reflection on a plane.

Moreover, if $K$ is an oriented $(p,q)$ knot, then it is clear that an $(-p,-q)$ knot represents $K$ with the inverse orientation. But now a rotation of $\pi$ along a symmetry axis of the torus shows that the knot of type $(p,q)$ and that of type $(-p,-q)$ are the same! We then have proven that:

**Proposition 5.1.4.** *Torus knots are invertible.*

Summarizing, so far we have:

**Lemma 5.1.5.** *Given $p, q$ coprime, the torus knots of type*

$$(p, q), (q, p), (-p, -q), (-q, -p)$$

*are all equivalent.*

Torus knot appears naturally at singularities of algebraic curve. Namely, consider in $\mathbb{C}^2$ the equation

$$x^p + y^q = 0.$$

This defines a singular surface $\Sigma$ (a complex curve) in $\mathbb{R}^4 \simeq \mathbb{C}^2$. The intersection of that curve with the unit sphere $S^3$ in $\mathbb{C}^2$ is a torus knot of type $(p, q)$. This can be seen directly by hands, by setting $(x, y) \in \mathbb{C}^2$ in polar coordinates: $x = re^{it}, y = \rho e^{i\theta}$. From the equation we get

$$r^p e^{ipt} = -\rho^q e^{iq\theta} \Rightarrow r^p = \rho^q$$

and since we are on the sphere we have

$$r^2 + \rho^2 = 1.$$

This determines $r$ (we want $r$ to be real and positive), hence $\rho$. So $\Sigma \cap S^3$ lies in the product of two $S^1$ (namely $|x| = r$ and $|y| = \rho$), hence is a torus link. Once settled moduli, the equation on arguments says $pt = \pi + q\theta$, which determines a curve of type $(p, q)$ in $S^1 \times S^1$.

Torus knots exteriors have particularly easy fundamental group, always with two generators and one relations:

$$< a, b \mid a^p = b^q > .$$

**Exercise 5.1.6.** *Prove it. (Hint: Given a torus knot, divide $S^3$ in two solid tori with common boundary $T^2$, now remove $K$ and use a direct instance of Van Kampen.)*

**Exercise 5.1.7.** *Compute meridian and longitude of torus knots in the presentation $< a, b | a^p = b^q >$.*

5.2. **Classification of torus knots.** Next theorem shows that Lemma 5.1.5 was sharp.

**Theorem 5.2.1.** *Two nontrivial torus knots of type $(p, q)$ and $(p', q')$ respectively are equivalent if and only if $(p', q')$ is one of*

$$(p, q), (q, p), (-q, -p), (-p, -q).$$

*A torus knot of type $(p, q)$ is trivial if and only if one of $p, q$ is $\pm 1$.*

*Proof.* For the first claim, since we have Lemma 5.1.5, it remains to show the converse, that is, that if a knot of type $(p, q)$ is equivalent to one of type $(p', q')$, then $(p', q')$ is one of the above pairs. We will exploit deeply the structure of the fundamental group of knot exteriors and peripheral systems. Although Wirtinger presentation is nice, it is not the best for working with torus knots. Let's start with some notation (see Figure 19): our torus $T^2$ splits $S^3$ in two solid tori $T_i, T_e$ ($T_i$ is the inner torus in the picture, $T_e$ is the external one). We name $x$ the generator of $\pi_1(T_e)$ and $y$ that of $\pi_1(T_i)$. The knot $K$ of type $(p, q)$ (depicted in blue) then turns $p$ times along $x$ and $q$ times along $y$. We chose a curve $C$ (depicted in red) which is parallel to $K$, for instance it can be chosen to be one of the two component of

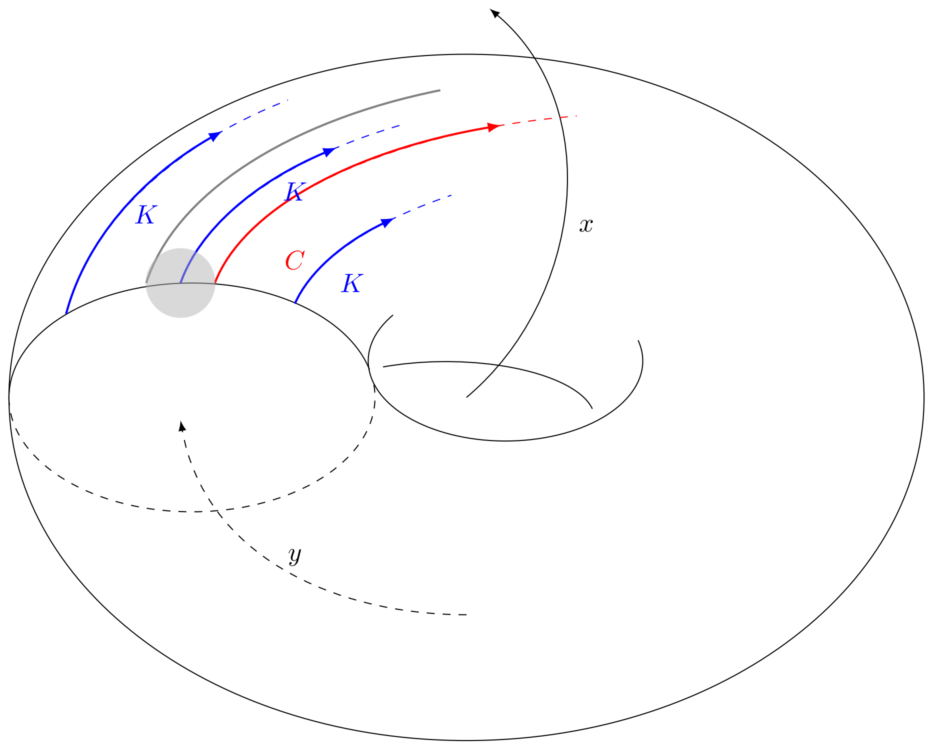



FIGURE 19. Curves on torus knot complement

the intersection of $\partial N(K)$ with $T^2$, where $N(K)$ is the tubular neighborhood of $K$ removed from $S^3$ to get the exterior of $K$. The curve $C$ generates the fundamental group $(T_i \cap T_e) \smallsetminus K$, and it is $x^p$ in $\pi_1(T_e)$ and $y^q$ in $\pi_1(T_i)$. This in particular shows, via a direct application of Van Kampen, that

$$\pi_1(S^3 \smallsetminus K) = \langle x, y \mid x^p = y^q \rangle$$

We name $t$ the element

$$t = x^p = y^q \in \pi_1(S^3 \smallsetminus K)$$

**Lemma 5.2.2.** *If both $p, q$ are different from $\pm 1$, then the center of $\pi_1(S^3 \smallsetminus K)$ is the subgroup generated by $t$. The quotient $\pi_1(S^3 \smallsetminus K)/\langle t \rangle$ is isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$. (Here we used the notation $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$).*

*Proof.* It is clear that $t$ commutes with any word in $x, y$, so it is central. The quotient $\pi_1(S^3 \smallsetminus K)$ is clearly isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$, where the isomorphism is given by mapping a word $w = x^{a_1} y^{b_1} x^{a_2} y^{b_2}$ to its reduction where exponents $a_i$'s are taken modulo $p$ and $b_i$'s modulo $q$. (Which is well defined because any two words representing the same element are related by a finite set of replacements $x^p \leftrightarrow y^q$, and two words differing by only one such replacement have the same image.)

Now, if both $p, q$ are different from $\pm 1$, the group $\mathbb{Z}_p * \mathbb{Z}_q$ has trivial center (because a nontrivial word $g$ commuting with $x$ must start with $x$, and if it commutes with $y$, then starts with $y$), which implies that $\langle t \rangle$ is in fact the center of $\pi_1(S^3 \setminus K)$. □

**Corollary 5.2.3.** *A torus knot of type $(p, q)$ is trivial if and only if at least one of $p, q$ is $\pm 1$. In particular the second claim of Theorem 5.2.1 is proved.*

*Proof.* Clearly if one of $p, q$ is $\pm 1$ then the knot is trivial (Remark 5.1.1). Vice versa, Lemma 5.2.2 says that if both $p, q \neq \pm 1$ then the knot group is not abelian. But the trivial knot group is $\mathbb{Z}$, which is abelian. □

Now we need to describe a peripheral system (that is, meridian and longitude) in terms of $x, y$. Let $L$ be a curve on the torus $T^2$ which which intersects $K$ positively and once (that is to say $K \cdot L = 1$). Such curve has coordinates $(c, d)$ in $\pi_1(T^2)$ with $pd - qc = 1$. One would like to say that $L = x^c y^d$ in $\pi_1(S^3 \setminus K)$ but $L$ actually intersects $K$. We can move $L$ a little so that it does not intersect $K$ anymore: we can do it in two directions and the difference of the two resulting curves is the meridian of $K$. More precisely, we look what happens locally at the intersection point (see Figure 20).

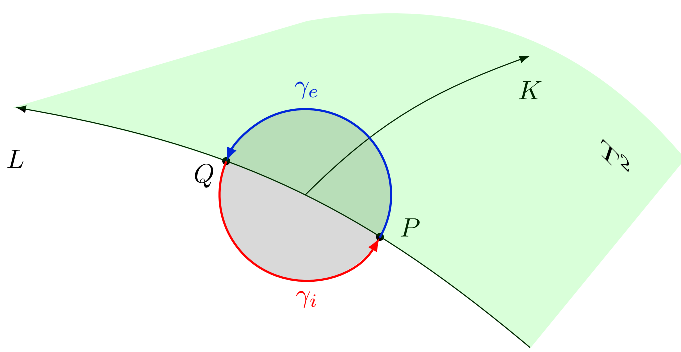



FIGURE 20. The meridian of $K$

Let $m$ be the meridian of $K$ given as the boundary of a disk centered at $K \cap L$. Let $P, Q$ be the intersection points of $m$ with the torus $T^2$ where $K$ lives, and we may assume that they both belong to $L$. Points $P, Q$ split $L$ in two segments $L_0 = PQ$ (that crosses $K$) and $L_1 = QP$ (the other). Similarly, $m$ is split in $\gamma_e$ (belonging to the exterior torus $T_e$) and $\gamma_i$ (belonging to the interior torus $T_i$). We have

$$m = \gamma_e \gamma_i = \gamma_e L_1 L_1^{-1} \gamma_i$$

Moreover, $\gamma_e L_1$ is homotopic to $x^c$ in $\pi_1(T_e, P)$ and $L_1^{-1}\gamma_i$ to $y^{-d}$ in $\pi_1(T_i, P)$. It follows that the meridian of $K$ in $\pi_1(S^3 \setminus K, P)$ is

$$m = x^c y^{-d}.$$

We now compute the longitude. The curve $C$ is a candidate longitude because it is parallel to $K$, therefore the longitude is of the form $Cm^\alpha$ for some $\alpha \in \mathbb{Z}$. Since

$C$ is $t = x^p = y^q$ in $\pi_1(S^3 \smallsetminus K)$, it follows that all putative longitudes $l$ are of the form

$$l = t(x^c y^{-d})^\alpha \qquad \text{which in homology becomes} \qquad l = t x^{c\alpha} y^{-d\alpha}.$$

Since $l$ is characterized by the fact that it is null in $H_1(S^3 \smallsetminus K)$, we see that we must have

$$\alpha = pq.$$

We can resume what we just proved in the following statement.

**Lemma 5.2.4.** *The peripheral system meridian/longitude of a $(p,q)$-knot is*

$$m = x^c y^{-d} \qquad l = x^p (x^c y^{-d})^{pq} \qquad \text{with} \;\; pd - cq = 1.$$

We are now in position to complete the proof of Theorem 5.2.1. It two torus knots of type $(p,q)$ and $(p',q')$ are equivalent, then they have isomorphic groups. Looking at quotients modulo centers, we see that (if both $p,q \neq \pm 1$) the pair $\{|p|,|q|\}$ must equal $\{|p'|,|q'|\}$. It follows $(p',q')$ is one of the following:

$$(p,q),(q,p),(-p,-q),(-q,-p),(-p,q),(p,-q),(q,-p),(-q,p).$$

Therefore we need to prove that last four possibility cannot occur. Note that by Lemma 5.1.5 all of them define equivalent knots. So we are left to prove that:

**Lemma 5.2.5.** *The coprime pairs $(p,q)$ and $(p,-q)$ define either inequivalent or trivial knots, the latter occurring only if one between $p,q$ is $\pm 1$.*

*Proof.* By Corollary 5.2.3 we may assume bot $p,q \neq \pm 1$. By Lemma 5.1.5 we may assume $p > 0$. Let $K$ be a torus knot of type $(p,q)$ and $K'$ be one of type $(p,-q)$. Set

$$G = \pi_1(K) = \langle x, y \mid x^p = y^q \rangle \qquad G' = \pi_1(K') = \langle X, Y \mid X^p = Y^q \rangle$$

(yes, I know, you would have expected $Y^{-q}$, but up to change $Y$ with $Y^{-1}$ it is the same). The peripheral systems are:

$$m = x^c y^{-d} \qquad l = x^p m^{pq} \qquad pd - cq = 1$$

$$m' = X^r Y^s \qquad l' = X^p (m')^{-pq} \qquad ps + qr = 1$$

(again, we have $Y^s$ instead of the expected $Y^{-s}$ by coherence with above choice on generator $Y$). Note that these equations imply that $c,r$ are either zero or coprime with $p$, and similarly $s,d$ with $q$.

We need to show tha $K$ and $K'$ are not equivalent. We argue by the way of contradiction and suppose that $K$ and $K'$ are equivalent. Then there is a group isomorphism $\phi : G \to G'$ such that

$$\phi(m) = m' \qquad \phi(l) = l'.$$

The element $\phi(x)$ generates a cyclic group of order $p$ when projected in $\mathbb{Z}_p * \mathbb{Z}_q = G/Z(G)$ ($Z(G)$ denotes the center of $G$). Similarly for $\phi(y)$. It follows that there exist $g_1, g_2 \in G'$, $h_1, h_2 \in Z(G')$ and $\mu, \eta \in \mathbb{Z}$ coprime with $p$ and $q$ respectively, so that

$$\phi(x) = g_1 X^\mu g_1^{-1} h_1 \qquad \phi(y) = g_2 Y^\eta g_2^{-1} h_2.$$

Condition $\phi(m) = m'$ becomes

$$g_1 X^{c\mu} g_1^{-1} g_2 Y^{-d\eta} g_2^{-1} h_1^c h_2^{-d} = X^r Y^s.$$

We now project such equality to the abelianised of $G'/Z(G')$, which is $\mathbb{Z}_p \oplus \mathbb{Z}_q$, and where conjugations and the $h_i$'s disappear, obtaining

$$(c\mu, -d\eta) = (r, s) \in \mathbb{Z}_p \oplus \mathbb{Z}_q$$

that is to say

$$c\mu \equiv r \pmod p \qquad - d\eta \equiv s \pmod q.$$

Finally, once we know $\phi(m) = m'$, equation $\phi(l) = l'$ becomes

$$g_1(X^p)^\mu g_1^{-1} h_1^p (m')^{pq} = X^p (m')^{-pq}$$

which, projected on $G'/Z(G')$ (which is $\mathbb{Z}_p * \mathbb{Z}_q$) becomes

$$[m']^{pq} = [m']^{-pq}$$

so we have

$$\text{either} \quad r \equiv 0 \pmod p \qquad \text{or} \quad s \equiv 0 \pmod q)$$

which, combined with above equations leads to

$$\text{either} \quad c = r = 0, \text{ which implies } d = s = p = 1$$

$$\text{or} \qquad s = d = 0, \text{ which implies } -c = r = q = \pm 1.$$

But this contradicts our hypothesis that both $p, q \neq \pm 1$. □

The proof of Theorem 5.2.1 is now complete. □

**Corollary 5.2.6.** *Right ans Left trefoils are not equivalent. In particular trefoils are chiral.*

By the very same reason (that is, that $(p, -q)$ gives the mirror of $(p, q)$) we have:

**Corollary 5.2.7.** *All torus knot are chiral.*

In particular the figure-eight knot is not a torus knot. So there are (prime) knots that are not torus knot (one never knows...)

### 5.3. **Primality, genus, bridge and crossing numbers of torus knots.**

**Theorem 5.3.1.** *Nontrivial torus knots are prime.*

*Proof.* Let $K$ be a torus knot of type $(p, q)$ which live in an unknotted torus $T$ in $S^3$. Let $K = K_1 + K_2$ be a decomposition of $K$, realised by a 2-sphere $S$, embedded in $S^3$ and intersecting $K$ only in two points. Put $T$ and $S$ so that they intersect transversally. So $T \cap S$ is a handful of simple closed curves.

Let $\gamma \subseteq T \cap S$ be a curve not intersecting $K$ and which is homotopically trivial in $T$ (if any), whence it bounds a disk $D$ in $T$. Since $K$ is non-trivial, it lives outside $D$. Up possibly to pass to an innermost curve, we may assume that $D \cap S = \gamma$. We cut $S$ along $\gamma$ and glue back two parallel copies of $D$. We obtain two spheres $S', S''$. Since $H_2(S^3) = 0$ and since $S \cap K$ consists in two points, only one between $S', S''$, say $S'$, intersects $K$. Therefore, we may replace $S$ with $S'$ and getting a new sphere that realises the splitting $K = K_1 + K_2$, with less connected components in $T \cap S$. Therefore, by induction we may reduce to the case where all curves in $T \cap S$ which do not intersect $K$, are nontrivial in $T$.

Let $\gamma \subseteq T \cap S$; it bounds a disk $B$ in $S$. Up possibly to pass to an innermost curve, we may assume that $B \cap T = \varnothing$. Therefore $\gamma$ is a meridian of $T$ with respect to one of the two solid tori bounded by $T$. Now we have three cases;

(1) If $\gamma \cap K = \emptyset$, then $\gamma$ – which is non trivial at this point of the proof – is parallel to $K$. In this case $K$ is a meridian, which imply that $K$ is the unknot.
(2) If $\gamma \cap K$ is a single point, then $K$ intersects a meridian of $T$ in one point, so it is of type $(1,q)$, whence it is trivial again.
(3) If $\gamma \cap K$ is two points, then either there are other components in $T \cap S$ – and this case reduces to case (1) — or $T \cap S = \gamma$. In this case $\gamma$ bounds two disks in $S$, one inside each of the two solid tori bounded by $T$. This is possible only if $\gamma$ is trivial in $T$, so it bounds a disk $E$ also in $T$. In this case the curve $E \cap K$ represent one of $K_1, K_2$ in the splitting, which is therefore the trivial splitting $K+$ the unknot.

Resuming, we proved that either $K$ is trivial or the splitting is trivial, q.e.d. □

A torus knot of type $(p,q)$ can be obtained a follows: take $S^1$ with $p$ marked point at angular distance $2\pi/p$ one from the following. Then consider the cylinder $S^1 \times [0,1]$ and glue $S^1 \times \{0\}$ and $S^1 \times \{1\}$ by a rotation of $q2\pi/p$. The result is a torus $T$ and segments $\{k2\pi/p\} \times [0,1]$ give rise to a curve of type $(p,q)$ in $T$.

In order to produce a knot diagram from this construction, we just have to embed $T$ in $R^3$ "as you imagine" and project to the horizontal plane.

Algorithmically, one takes $p$ parallel strands in the plane, then do $q$ times the following operation: take the right strand and pass it to the left under all other strands. Then connect final endpoints with initial ones with $p$ parallel arcs.

In Figure 21 we depicted the case $p > q$) and in Figure 22 the case $p < q$. (They are two diagrams of the same knot).

Such diagrams are quite optimal:

**Exercise 5.3.2.** *Show that the Seifert surface obtained from that diagrams has genus*

$$g = \frac{(p-1)(q-1)}{2}.$$

*Observe that this construction in particular implies that a torus knot of type $(p,q)$ has genus at most $(p-1)(q-1)/2$.*

**Remark 5.3.3.** *It can be shown that indeed the above estimate for torus knots is sharp: a torus knot of type $(p,q)$ has genus exactly $(p-1)(q-1)/2$. (See* [15, Chapter 7]*.)*

**Exercise 5.3.4.** *Show that the bridge number of diagrams of $(p,q)$ knots, constructed as above is $q$. (So it is 3 in Figure 21, and 7 in Figure 22).*

**Remark 5.3.5.** *It can be shown that the bridge number of a $(p,q)$ knot is actually* $\min(p,q)$. *(See* [15, Chapter 7]*.)*

**Exercise 5.3.6.** *Show that the crossing number of diagrams of $(p,q)$ knots, constructed as above, is $q(p-1)$. (So it is 18 in Figure 21 and 14 in Figure 22).*

**Remark 5.3.7.** *It can be shown that the crossing number of a $(p,q)$ knot is actually* $\min(p(q-1), q(p-1))$. *(See* [15, Chapter 7]*.)*

**Further reading.** An important family of knots are those whose complement admits a complete hyperbolic structure of finite volume. In this case the knot is called hyperbolic. Torus knots do not admit such structures because the knot complement has (at least) one incompressible annulus: $T \setminus K$. Also, hyperbolic

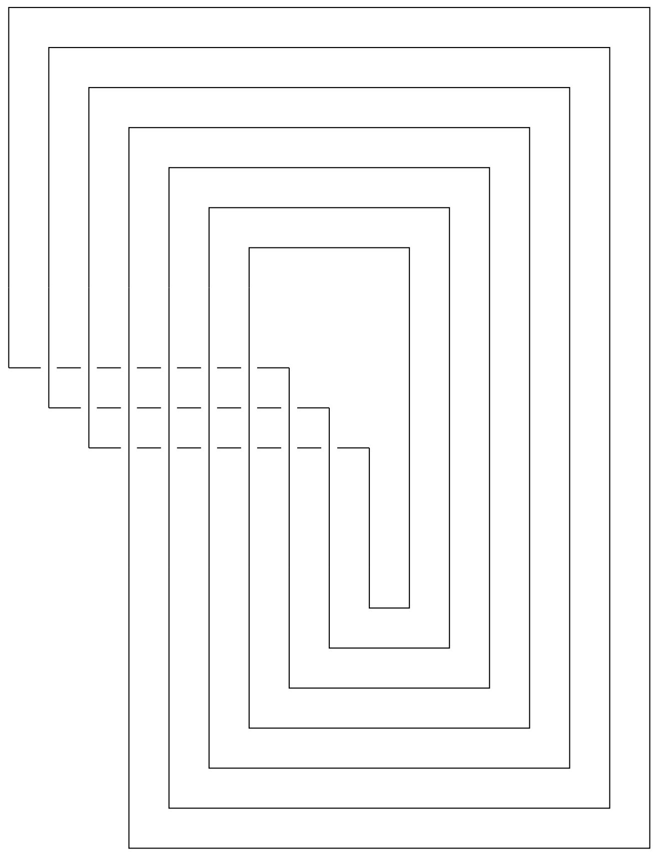


FIGURE 21. How to obtain a diagram of a torus knot of type $(p, q)$. Here we depicted the $(7, 3)$ knot.

knots must be prime (again non prime knots exhibit an incompressible annulus: the separating sphere minus $K$). These are the only obstructions to hyperbolicity for alternating knots: alternating prime knots are either torus knots or hyperbolic. (Be careful: not all torus knot are alternating: $(5, 3), (4, 3), (5, 4)$ are examples of non-alternating torus knots.)

If you are interested in colorability of torus knots, you may check [5].

## 6. Polynomial invariants

So far we have met numerical invariants (crossing number, genus, bridge number...) which are not very powerful with respect to the problem of classification of knots/links, but are very easy to compare. On the other side, there are topological invariant (the complement, the knot group, the peripheral system,..) that are very powerful with respect to classification, but very hard to compare. Now, we will introduce something that stand in the middle: invariants of (Laurent) polynomial type.

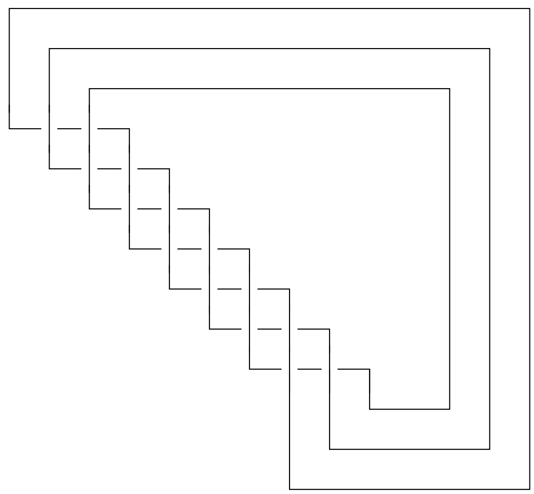
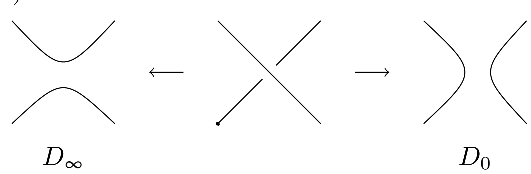


FIGURE 22. A diagram of the $(7,3)$ knot (which, of course, is equivalent to the $(7,3)$ one).

6.1. **The bracket and the Jones polynomial.** Let's start with an apparently naïve question: is there a way to construct a link invariant by resolving the singularities of one of its diagrams? The answer is (unexpectedly) yes, but of course we have to do it carefully.

First of all notice that we have two ways to resolve the singularity at a crossing point (why in the algorithm for Seifert surfaces there was just one possibility? Give yourself an answer!)




Moreover, observe that after a resolution a trivial component may appear. This couple of remarks motivates the idea of associating a polynomial $\langle D \rangle \in \mathbb{Z}[A, B, d]$ to a diagram $D$ with the following properties

1) $\langle \bigcirc \rangle = 1$
2) $\langle D \cup \bigcirc \rangle = d\langle D \rangle$
3) $\langle \times \rangle = A \langle \asymp \rangle + B \langle )( \rangle$.

The meaning of the third one is that the three diagrams involved differs only in a portion as indicated in the small pictures. In the future this is the way you should intend such equations involving portions of diagrams.

**Remark 6.1.1.** *Notice that from the third rule we have* $\langle \times \rangle = A \langle )( \rangle + B \langle \asymp \rangle$ *(just rotate the sheet by* $\pi/2$*). An easy way to remember the rule is to label with* $A$ *the two region that are swept out by turning the overcrossing line counterclockwise until it coincides with the undercrossing one:* $A$ *multiplies the resolution that connects the* $A$*-regions. Of course you can replace* $A$ *with* $B$ *and counterclockwise with clockwise.*

By applying recursively the third property we can reduce the computation of the polynomial to those of trivial links and from 1 and 2 we got $\langle \bigcirc_k \rangle = d^{k-1}$ with $\bigcirc_k$ the trivial link with $k$ components. So, the three property allows to compute $\langle D \rangle$.

**Exercise 6.1.2.** *Convince yourself that the computation is independent from the order chosen for the crossing resolutions and so the polynomial is uniquely defined by the three properties.*

You may want to have an explicit formula for $\langle D \rangle$, the idea is: make a choice of resolution at each crossing, solve all the crossing at the same time, recording as a monomial in the unknowns $A$, $B$ and $d$ the choices you've made and the number of trivial components that you got, and then sum up over all possible choices. More precisely, call a **state** of a given diagram $D$ a choice at each crossing of one of the two resolution; associate to each state $s$ the number $a(s)$ of resolutions connecting $A$-regions and the number $k(s)$ of connected component of the diagram obtained by performing all the resolutions. With this notation we get the following formula

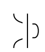
$$\langle D \rangle = \sum_{s \in S(D)} d^{k(s)-1} A^{a(s)} B^{c(D)-a(s)}$$

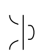
where $c(D)$ denotes the crossing number of the diagram and $S(D)$ denotes the set of all states of $D$ (how many are they?).

**Exercise 6.1.3.** *Prove the above formula.*

As you may suspect, $\langle D \rangle$ is not a link invariant. Anyway, we can check how it behaves under Reidemeister moves and see if it is possible to adjust $A$, $B$ and $d$ to obtain a link invariant.

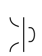
**Lemma 6.1.4.** *The following formula holds*

$\langle$  $\rangle = AB \langle\rangle\langle\rangle + (ABd + A^2 + B^2) \langle \asymp \rangle .$

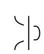
*Hence* $\langle$  $\rangle = \langle\rangle\langle\rangle$ *for all diagrams if*

$$B = A^{-1} \quad \text{and} \quad d = -A^2 - A^{-2}.$$

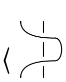
*Proof.* We have

$\langle$  $\rangle = A \langle \asymp \rangle + B \langle$  $\rangle = A \left( A \langle \asymp \rangle + B \langle\rangle\langle\rangle \right) + B \left( A \langle$  $\rangle + B \langle \asymp \rangle \right)$

$$= (A^2 + ABd + B^2) \langle \asymp \rangle + AB \langle\rangle\langle\rangle .$$

So if we want to have we have $\langle$  $\rangle = \langle\rangle\langle\rangle$ we have to impose $AB = 1$ and $A^2 + ABd + B^2 = 0.$ $\square$

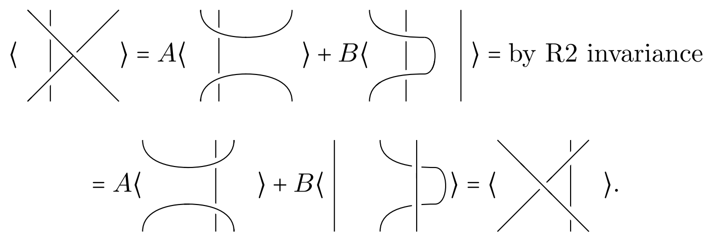
So for the above choices of $B$ and $d$ we get the invariance under Reidemeister 2.

**Exercise 6.1.5.** *Check that the same holds with* $\langle$  $\rangle = \langle\rangle\langle\rangle$. *Is it necessary?*

Miraculously, the same choices guarantee the invariance also under Reidemeister 3. Indeed, any choice that works for R2 works also for R3.

**Lemma 6.1.6.** *R2 invariance for $\langle\ \ \rangle$ implies type R3 invariance.*

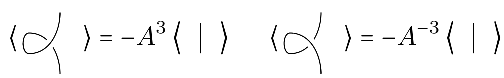
*Proof.*




□

The invariance of a link diagram under planar isotopy, R2 and R3 corresponds to 3-dimensional equivalence of framed links (see Further reading at page 27). From now on the we assume that $B$ and $d$ are chosen as above. The corresponding polynomial, that is an element of $\mathbb{Z}[A, A^{-1}]$, is known as the **bracket polynomial** and is an invariant of framed links.

In order to have a link invariant we have to check what happens with R1. A direct computation proves that




**Exercise 6.1.7.** *Do the required computations.*

So, at first glance, it seems that the only possibility for having a link invariant would be choosing $A = 1$ that, however, trivializes $\langle\ \ \rangle$.

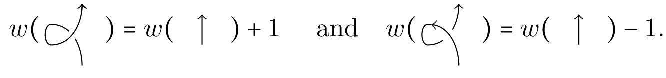
**Exercise 6.1.8.** *If we were satisfied with having a numerical invariant (instead of a polynomial one) are there different possibilities from fixing $A = 1$? Is there also possibilities that do not produce a trivial invariant?*

Thinking more carefully, another possibility is to multiply the bracket polynomial by another quantity able to balance the change of $\langle\ \ \rangle$ under R1. The quantity we need is a generalization of the linking number that sum over all the crossings (and not only over those involving different components).

**Definition 6.1.9.** *Let $D$ be an oriented link diagram. The **writhe** $w(D)$ of $D$ is the sum of signs of all crossings.*

**Exercise 6.1.10.** *Show that the writhe is invariant under Reidemeister 2 and 3.*

As the bracket polynomial, the writhe is an invariant for oriented framed links, but not for oriented links. Indeed, independently of the chosen orientation we have




**Curiosity:** It is thought that nineteen-century knot tabulators believed that the writhe of a diagram was a knot invariant, at least for reduced diagrams. That lead to a famous error of the inclusion, in the early knot tables, of both a knot and its reflection, listed as $10_{161}$ and $10_{162}$ with diagrams having writhe $-8$ and $10$. The error was detected by Kenneth Perko in the 1970's (search for Perko pair).

Using the writhe we can balance the change in $\langle\ \rangle$ due to R1. More precisely, a straightforward computation show that the polynomial

$$f_L(A) = (-A)^{-3w(D)} \langle D \rangle \in \mathbb{Z}[A, A^{-1}]$$

is invariant under all the Reidemeister moves and so it is an invariant of oriented link.

**Exercise 6.1.11.** *Check the previous statement.*

This polynomial is known as **Jones polynomial**. To be precise, what generally is called Jones polynomial is

$$V_L(t) = \left((-A)^{-3w(D)} \langle D \rangle\right)_{|_{A=t^{-\frac{1}{4}}}} \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}].$$

Before explaining the reason of this change of unknown, and why $V_L$ is an element of $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$, let's explore some properties of $f_L(A)$.

**Proposition 6.1.12.** *It holds*

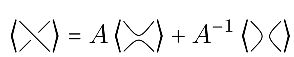
1) $f_L^*(A) = f_L(A^{-1})$, *with* $L^*$ *the mirror image of* $L$
2) $f_{-L}(A) = f_L(A)$, *with* $-L$ *the link obtained by reversing the orientation of each component of* $L$.
3) $f_{K_1+K_2} = f_{K_1} f_{K_2}$, *with* $K_1, K_2$ *knots.*

*Proof.* We have $w(D) = w(-D) = -w(D^*)$. Since $\langle D \rangle$ does not depend on the orientation we got the second statement. For the first, notice that at each crossing the resolution multiplied by $A$ is the one that in the mirrored crossing is multiplied by $A^{-1}$ (see Remark 6.1.1). Let $D_1 + D_2$ be the diagram of $K_1 + K_2$ obtained by "summing" the diagrams $D_1$ and $D_2$ of $K_1$ and $K_2$: clearly $w(D_1 + D_2) = w(D_1) + w(D_2)$ and $c(D_1 + D_2) = c(D_1) + c(D_2)$; moreover, there is a bijection $S(D_1) \times S(D_2) \to S(D_1 + D_2)$ that maps the couple $(s_1, s_2)$ to the state $s_1 \cup s_2$ having the same choices at crossings; the statement follows by observing that $a(s_1 \cup s_2) = a(s_1) + a(s_2)$ and $k(s_1 \cup s_2) = k(s_1) + k(s_2) - 1$. $\square$

So if $K$ is a knot $f_K$ does not depends on the orientation chosen to compute it.

**Exercise 6.1.13.** *For* $K$ *the trefoil knot (left or right the one you like most) compute* $f_K$ *and use the result to prove (again) that the trefoil knot is chiral.*

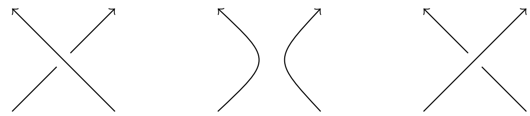
Now we need to take a look again to the property



Does it hold something similar for $f_L$? The answer is yes.

**Theorem 6.1.14.** *If* $L_+$, $L_-$ *and* $L_0$ *are three links having diagrams* $D_+$, $D_-$ *and* $D_0$ *that differ as indicated in Figure 23 in a specific crossing we have*

$$A^4 f_{L_+} - A^{-4} f_{L_-} = (A^{-2} - A^2) f_{L_0}. \tag{1}$$



$D_-$ $D_0$ $D_+$

FIGURE 23

*Proof.*

$$\langle \times \rangle = A^{-1} \langle \asymp \rangle + A \langle )( \rangle$$

$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle )( \rangle$$

Hence

$$A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle )( \rangle .$$

Using the fact that $w(D_+) - 1 = w(D_0) = w(D_-) + 1$ we get

$$-A^4 f_{L_+} + A^{-4} f_{L_-} = (A^2 - A^{-2}) f_{L_0}$$

□

If we do the change of variable $A = t^{-\frac{1}{4}}$ equation (1) becomes

(2) $$t^{-1} V_{L_+} - t V_{L_-} = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{L_0}$$

and is called skein relation of the Jones polynomial.

Vaughan Jones defined the Jones polynomial in 1984 using representation theory of braid groups and proved that the polynomial is characterized by (2) and $V_{\bigcirc} = 1$. A couple of year after Luis Kauffman described the relation of the Jones polynomial with the bracket polynomial.

**Further reading.** Jones described a connection between the Jones polynomial and the Pott model in statistical mechanics, a model describing interacting spins in a crystalline lattice. The relation could be described using Kauffman approach and involves the dichromatic polynomial (of the Seifert graph), an important invariant in graph theory.

**Exercise 6.1.15.** *Using* (2) *and* $V_{\bigcirc} = 1$ *prove that* $V_{\bigcirc^k} = (-1)^{k-1} \left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)^{k-1}$. *Hint: start with a diagram of the trivial link with one crossing and use induction on the number of components.*

Equation (2) relates three links differing from each by a cross swap and/or a crossing resolution. Since each link can be changed to the unlink by cross swaps (can you find an algorithm to do this?), it is possible to compute $V_L$ by recursively apply (2) together with the formula for $V_{\bigcirc^k}$. This way to compute $V_L$ is more efficient then our initial definition (can you see why?).

**Exercise 6.1.16.** *Compute the Jones polynomial of the trefoil knot using* (2) *and compare it with the computation done in Exercise 6.1.13.*

**Exercise 6.1.17.** *Prove that if* $L$ *is a link with* $k$ *components then* $V_L(1) = (-2)^{k-1}$. *Hint: what information you get from* (2) *when* $t = 1$*?*

**Curiosity:** If you want to have a look to the Jones polynomials of prime knots with at most 8 crossings (up to mirror images) you may look at Appendix II of [15]. An example of two knots having the same Jones polynomial is given in [15, p. 227-228]. Whether there exists a non-trivial knot with trivial Jones polynomial is still an open conjecture.

**Further reading.** A way to produce knots with the same Jones polynomial is using the so called mutation. A famous couple of mutant knots is the Kinoshita-Terasaka knot and the Conway knot. Look for the definition of mutant pair and try to understand why such a couple of knots have the same Jones polynomial.

In Section 3.2 we saw that a very natural invariant for knots and links is the crossing number: the Jones polynomial gives a way to find a lower bound for it and a way to compute it for alternating links: more precisely it gives a way to prove one of the Tait conjecture (see pag.14).

**Proposition 6.1.18.** *A reduced alternating diagram $D$ of a knot $K$ is minimal (i.e., $c(D) = c(K)$).*

The above results follows at once from the following statement

**Theorem 6.1.19.** *Let $K$ be a knot and $D$ a diagram of $K$. Then $B(V_K) \leq c(D)$, where $B(V_K)$ denotes the breadth of $V_K$. Moreover if $D$ is alternating and reduced then the equality holds.*

We recall that the **breadth** of a polynomial in one unknown is the difference of degree between the monomial having maximum degree and that of minimum degree.

*Proof.* First of all notice that $B(V_K) = \frac{1}{4}B(f_K) = \frac{1}{4}B(\langle D\rangle)$. Denote with $M$ (resp. $m$) the degree of the monomial of $\langle D\rangle$ having the maximum (resp. minimum) degree. Recall that in $\langle D\rangle$ each state $s$ contributes a term of the form $A^{a(s)}A^{-b(s)}(-A^2 - A^{-2})^{k(s)-1}$, with $b(s) = c(D) - a(s)$; the exponent of the highest power of $A$ in this expression is $a(s) - b(s) + 2k(s) - 2$. Let us consider the state $s_A$ obtained by choosing at each crossing the resolution that connects $A$-regions. The contribution of $s_A$ is $A^{c(s)}(-A^2 - A^{-2})^{k(s)-1}$ and the highest power of $A$ occurring in this expression is $A^{c(D)+2k(s_A)-2}$. Now any state $s$ of $S(D)$ can be achieved by starting with $s_A$ and changing, one a time, the resolution at the necessary crossings. In other words there exists a sequence of states $s_A = s_0, s_1, s_2, \ldots, s_n = s$ such that each state differs from the previous one by the choice done in just one crossing in with a resolution of $A$-type is replaced by the other one. Clearly $a(s_{i+1}) = a(s_i) - 1$, $b(s_{i+1}) = b(s_i) + 1$ and $k(s_{i+1}) = k(s_i) \pm 1$. So the exponent of the highest power of the term corresponding to $s_{i+1}$ is $a(s_i) - b(s_i) - 2 + 2(k(s_i) \pm 1) - 2$ that is less or equal to the highest contribution of $s_i$. It follows that $M \leq c(D) + 2k(s_A) - 2$. Applying the same reasoning to the mirrored diagram $D^*$, we get that $m \geq -c(D) - 2k(s_B) + 2$, where $s_B$ is the state obtained by choosing at each crossing the resolution that does not connect $A$-regions. So $B(\langle D\rangle) \leq 2(c(D) + k(s_A) + k(s_B) - 2)$. To obtain the first statement it is enough to prove that $k(s_A) + k(s_B) \leq c(D) + 2$. This could be proved by induction on $c(D)$: for $c(D) = 0$ is clearly true; given a diagram $D$ with $n$ crossing, select one of its crossings: for at least one of the two resolutions the resulting diagram $D'$ is connected, let's suppose that is the $A$-resolution; we have

$k(s_A(D)) = k(s_A(D')$ while $k(s_B(D)) = k(s_B(D') \pm 1$ so

$$k(s_A(D)) + k(s_B(D)) = k(s_A(D')) + k(s_B(D')) \pm 1 \leq (n-1) + 2 \pm 1 \leq n + 2.$$

If $D$ is alternating then when we place $A$'s and $B$'s around a crossing, we see that all the vertices in any region are either all labelled $A$'s or all labelled $B$'s. So if we consider a black and white chessboard-colouring of the diagram complement $k(s_A)$ and $k(s_B)$ are the number of white and black regions (up to exchanging the colours). Moreover, if in the transition from $s_0 = s_A$ to $s_1$ the number of regions does not decrease, then some black region would touch both sides of a crossing so the diagram wouldn't be reduced. This means that the degree of the terms corresponding to states different from $s_A$ are less then $c(D) + 2k(s_A) - 2$ and so $M = c(D) + 2k(s_A) - 2$. Analogously we get $m = -c(D) - 2k(s_B) + 2$ and so $B(\langle D \rangle) = 2(c(D) + k(s_A) + k(s_B) - 2)$. To end the proof we can observe that, for alternating knots, $k(s_A) + k(s_B) = c(D) + 2$ since this sum is equal to the total number of region of the complement of the diagram. □

**Exercise 6.1.20.** *Explain the last sentence of the above proof. (Hint: you can use the fact that the Euler characteristic of the disc is one).*

**Exercise 6.1.21.** *Is it possible to generalize the above proof to links?*

**Further reading.** In the process of discovering new polynomial invariants for links, Jones established a connection between knot theory and statistical mechanics. Such relation is explored in [9] using the Kauffman bracket approach and passing through a graph theory invariant, the dichromatic polynomial.

6.2. **The Alexander polynomial and (Alexander)-Conway polunomial.** We will go into the definition of another polynomial invariant, that, as the previous one, is a Laurent polynomial in one variable with integer coefficients. Its name is Alexander polynomial and it can be used to obtain a lower bound for the genus: Moreover, it will give us the opportunity to explore the topology of the link complement, since it is associated to a particular covering of the link complement, let's see how.

Recall from covering theory, that, to each subgroup of the fundamental group of (a sufficiently nice) topological space $X$ there corresponds a covering space (if you need a refresh on covering spaces you may look at [8, Section 1.3] or [8, Chapter 7]). The space we want to focus on is the exterior $E = E(L)$ of a link $L \subseteq S^3$. For a knot, the Hurewicz map gives a surjective homomorphism $\pi_1(E) \to H_1(E) \cong \mathbb{Z}$, where the isomorphism is induced by the canonical choice for the meridian (see Exercise 4.2.9). If we a have a link with more then one component, we can "sum up" the contribution of each component. More precisely, if $L = K_1 \cup \cdots \cup K_n$ is a link with $n$components we define $\psi : \pi_1(E) \to \mathbb{Z}$ by

$$[\gamma] \mapsto \sum_{i=1}^{n} \psi_i([\gamma])$$

where $\psi_i : \pi_1(E) \to H_1(E) \to H_1(S^3 \setminus K_i) \cong \mathbb{Z}$.

**Exercise 6.2.1.** *Prove that if $\gamma$ is a knot then $\psi([\gamma]) = \sum_{i=1}^{n} lk(\gamma, K_i)$.*

Let $p : \widetilde{E}_\infty \to E$ the covering associated to $\ker \psi$; since $\ker \psi$ is a normal subgroup the covering is regular and so

$$\mathrm{Aut}(\widetilde{E}_\infty) = \frac{\pi_1(E)}{\ker\psi} \cong \mathbb{Z}$$

and $\mathrm{Aut}(\widetilde{E}_\infty)$ acts transitively on the fiber over each point. When $L$ is a knot this covering is called universal cyclic covering; also in the general case general let's call this covering (by a little abuse of notation) **the cyclic covering** of $L$.

The ring $\mathbb{Z}[t,t^{-1}]$ acts over $H_1(\widetilde{E}_\infty)$ as follows

$$\sum_{i=-h}^{k} a_i t^i \cdot [c] = \sum_{i=-h}^{k} a_i (\tau_\sharp)^i([c])$$

where $\tau$ is the generator of $\mathrm{Aut}(\widetilde{E}_\infty)$ corresponding to $1 \in \mathbb{Z}$ and $\tau_\sharp$ is the automorphism induced in homology by $\tau$.

**Definition 6.2.2.** *The* ***Alexander module*** *of $L$ is $H_1(\widetilde{E}_\infty)$ endowed with its $\mathbb{Z}[t,t^{-1}]$-module structure.*

**Curiosity:** It is possible to define the Alexander module of a KNOT in a purely algebraic way (without any reference to coverings). Given a group $G$ denote with $G'$ the commutator subgroup of $G$ and let $G'' = (G')'$. Convince yourself that the (underlying set of the) Alexander module of a knot $K$ is $G'/G''$ with $G = \pi_1(S^3 \setminus K)$. How is defined the action?

Clearly the Alexander module is an invariant of the link. We can use a classical theory developed for finitely presented module over a commutative and unitary ring to define "computable" and "comparable" invariants for the link: one of those will be the Alexander polynomial. Before that, we want to give an explicit construction of $\widetilde{E}_\infty$.

6.2.1. *A construction for $\widetilde{E}_\infty$.* Let $S$ be a fixed oriented Seifert surface for $L$. If you cut $S^3$ along $S$ the resulting open manifold has "two copies" of $S$: the general idea for constructing $\widetilde{E}_\infty$ is to take (countably many) copies of this manifold and glue them together by identifying the positive copy of $S$ in one manifold with the negative copy of $S$ in the successive one. More precisely, denote with $N = N(\mathrm{int}(S)) \cong \mathrm{int}(S) \times (-1,1)$ an open tubular neighborhood of the interior of $S$ with $\mathrm{int}(S)$ identified with $\mathrm{int}(S) \times \{0\}$ and the notation chosen so that the meridian of every component of $L$ enters the neighborhood at $\mathrm{int}(S) \times \{-1\}$ and leaves it at $\mathrm{int}(S) \times \{1\}$. Let $N^-$ and $N^+$ be the subset of $N$ corresponding to $\mathrm{int}(S) \times (-1,0)$ and $\mathrm{int}(S) \times (0,1)$. Let $Y_i$ and $N_i$ copies of $Y$ and $N$ for $i \in \mathbb{Z}$. The manifold $\widetilde{E}_\infty$ is obtained by identifying $N_i^- \subseteq N_i \subseteq Y_i$ with $N_{i-1}^- \subseteq Y_{i+1}$ and $N_i^+ \subseteq N_i \subseteq Y_i$ with $N_{i+1}^+ \subseteq Y_{i+1}$ and the covering map is the projection $p : \widetilde{E}_\infty \to E$ that sends $Y_i$ to $Y$ and $N_i$ to $N$.

**Exercise 6.2.3.** *Prove that the space we constructed above is indeed the cyclic covering we were looking for: use the definition to check that the map defined above is indeed a covering; check that the covering automorphism is infinite cyclic generated by the map that shift all the indices by one and that $\ker\psi \subset p_*(\widetilde{E}_\infty)$ (use Exercise 6.2.1 and the interpretation of the linking number as intersection with a Seifert surface). Conclude using the classification theorem of covering spaces.*

**Exercise 6.2.4.** *If $K$ is the trivial knot how is $\widetilde{E}_\infty$?*

6.2.2. *Elementary ideals of finitely presented modules.* Let $R$ be a commutative ring with unity. A module $F$ over $R$ is **free** with generators $f_1, \dots, f_k$ if every $x \in F$ is of the form $\sum_{i=1}^{k} a_i f_i$ for a unique choice of $a_i \in R$. In this case $F \cong R^k$. A module $M$ over $R$ is **finitely presented** if it admits a **finite presentation**, i.e., an exact sequence

$$F \to E \to M \to 0$$

with $E$ and $F$ two finitely generated free modules over $R$. Essentially the generators of $E$ correspond, once projected to $M$, to generators of $M$, while the images in $E$ of the generators of $F$ correspond to relation in $M$. If we fix basis for $F$ and $E$, the map $F \to E$ is represented by a matrix $A \in M_{m\times n}(R)$ with $m$ and $n$ the cardinality of the basis of $E$ and $F$, respectively. The matrix $A$ is called a **presentation matrix** for $M$. Clearly a module has many presentation matrices, the following theorem, whose proof can be found in [13, Therem 6.1], tells us how they are related.

**Theorem 6.2.5.** *Two matrices with entries in $R$ present the same module $M$ if and only if one is obtained from the other by a (finite) sequence of the following moves and their inverses*

1) *permutation of rows (or columns)*
2) *addition of an $R$-multiple of a row (or column) to another row (or column)*
3) $(A) \leftrightarrow (A|\boldsymbol{0})$
4) $(A) \leftrightarrow \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline & \\ \mathbf{0} & 1 \end{array}\right).$

The first two moves are just elementary moves involved in Gauss algorithm for echelon form, the third corresponds to adding a trivial relation, while the last consists in adding a new generator and a relation that kills it.

**Definition 6.2.6.** *Let $M$ be a finitely presented $R$-module and $A$ a presentation matrix for $M$ with $m$ rows. The* ***r-th elementary ideal*** *$E_r(M)$ of $M$ is the ideal in $R$ generated by the $(m-r+1)\times(m-r+1)$ minors of $A$.*

Thanks to the previous theorem and to the property of the determinant, the definition is well posed, that is, the elementary ideal are independent of the matrix used to compute them. By Laplace theorem $(k+1)\times(k+1)$ minors are linear combinations of $k\times k$ ones, so $E_r(M) \subseteq E_{r+1}(M)$. By convention we set $E_r(M) = 0$ if $r \le 0$ and $E_r(M) = R$ if $r > m$.

Now we can go back to our topological setting of an oriented link $L$ in $S^3$ and focus on $R = \mathbb{Z}[t, t^{-1}]$ and $M = H_1(\widetilde{E}_\infty)$, the Alexander module of $L$. The **r-th Alexander ideals** of an oriented link $L$ are the elementary ideal of the Alexander module of $L$. The **r-th Alexander polynomial** of $L$ is a generator of the smallest principal ideal that contains the $r$-th Alexander ideal of $L$. It is defined up to units of $\mathbb{Z}[t, t^{-1}]$, that is elements of the form $\pm t^{\pm k}$. The first Alexander polynomial of $L$ is called **the Alexander polynomial** of $L$ and is denoted with $\Delta_L$. If $f, g$ are polynomial in $\mathbb{Z}[t, t^{-1}]$ we set $f \doteq g$ if $f = \pm g t^{\pm k}$

**Exercise 6.2.7.** *Prove that the ring $\mathbb{Z}[t, t^{-1}]$ is a U.F.D. (unique factorization domain) but is not a P.I.D. (principal ideal domain). So the $r$-th Alexander polynomial is the G.C.D. of the $(m-r+1)\times(m-r+1)$ minors of any presentation matrix for the Alexander module.*

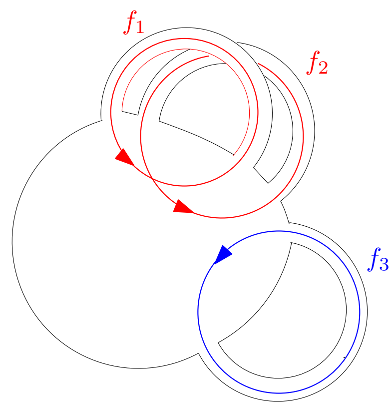



FIGURE 24. A basis for the homology of a surface of genus 1 and with 2 boundary components.

From Exercise 6.2.4 (did you solve it?), if $K$ is the trivial knot then $\widetilde{E}_\infty = \mathbb{R}^3$. So $H_1(\widetilde{E}_\infty) = 0$ having presentation matrix (1). As a consequence $E_1 = \mathbb{Z}[t, t^{-1}]$ and $\Delta_K(t) = 1$. However, for a non trivial link the definition of the Alexander polynomial is quite implicit, so we need to develop a strategy to compute it. To do so, we will introduce a $\mathbb{Z}$-valued bilinear form associated to a Seifert surface.

6.2.3. *Seifert form and Seifert matrix.* Consider an oriented Seifert surface $S$ for an oriented link $L \subseteq S^3$. Denote with $n$ the number of components of $L$ and with $g$ the genus of $S$. By the theory of surfaces $H_1(S) \cong \mathbb{Z}^{2g+n-1}$ and a set of generators of the homology are (the images underground the homeomorphism between a standard surface and $S$ of) classes of loops represented in Figure 24 (for the case $g = 1$ and $n = 2$).

**Proposition 6.2.8.** *It holds $H_1(S^3 \smallsetminus S) \cong H_1(S) \cong \mathbb{Z}^{2g+n-1}$. Moreover, there exists a bilinear form*

$$\beta : H_1(S^3 \smallsetminus S) \times H_1(S) \to \mathbb{Z}$$

*such that*

$$\beta([c], [d]) = lk(c, d)$$

*whenever $c$ and $d$ are simple oriented closed curves (in $S^3 \smallsetminus S$ and $S$ respectively).*

*Proof.* Let $V = \overline{N(S)}$ be a closed tubular neighborhood of $S$ and set $V' = \overline{S^3 \smallsetminus N(S)}$. By representing $S$ as a disk with (knotted bands), we can understand that $V$ is a (knotted) handlebody of genus $2g+n-1$ with each (knotted) 1-handle coming from each (knotted) band of $S$. So $V \cap V' = \partial V = \partial V' \cong \Sigma_{2g+n-1}$ a closed, connected, orientable surface of genus $2g+n-1$. As a basis for $H_1(\partial V)$, we can take classes of loops $f'_1, \dots f'_{2g+n-1}, e_1, \dots, e_{2g+n-1}$ so that $f_i$ is homologous to $f'_i$ in $H_1(V)$ and $e_j$ bounds a disk in $V$ "dual" to $f_j$, i.e. intersecting $f_j$ transversally in one point and disjoint from $f_k$ when $k \neq j$. Since $V \cup V' = S^3$, From the Mayer-Vietoris sequence, we have

$$0 = H_2(S^3) \to H_1(V \cap V') \xrightarrow{\phi} H_1(V) \oplus H_1(V') \to H_1(S^3) = 0.$$

Our choice of bases gives $\phi([f_i']) = ([f_i], x_i)$ and $\phi([e_j]) = (0, y_j)$, for some elements $x_i, y_j$ in $H_1(V')$. Since $\phi$ is an isomorphism clearly the $[e_j]$'s are a basis of $H_1(V') \cong H_1(S^3 - S)$ and so $H_1(S) \cong H_1(S^3 \setminus S)$. Fix an orientation for $e_j$ and $f_i$ so that $lk(e_j, f_i) = \delta_{ij}$. Now define

$$\beta\left(\sum_{i=1}^{t} a_j[e_j], \sum_{i=1}^{t} b_i[f_i]\right) = \sum_{i=1}^{t} a_i b_i,$$

with $t = 2g + n - 1$. Clearly $\beta$ is a $\mathbb{Z}$-valued bilinear form, we have just to check its behaviour on simple closed oriented curve $c$ and $d$. Let $[c] = \sum_{k=1}^{t} a_k[e_k]$ and $[d] = \sum_{h=1}^{t} b_h[f_h]$.

From Exercise 6.2.1 in $H_1(S^3 \setminus f_i)$ we have

$$lk(c, [f_i]) = [c] = \left[\sum_{j=1}^{t} a_j[e_j]\right] = \sum_{k=1}^{t} a_k\, lk(e_k, f_i) = a_i.$$

Thus in $H_1(S^3 \setminus c)$ it holds

$$lk(c, d) = [d] = \left[\sum_{h=1}^{t} b_h[f_h]\right] = \sum_{h=1}^{t} b_h\, lk(f_h, c) = \sum_{h=1}^{t} b_h a_h = \beta\left([c], [d]\right).$$

□

We want to use $\beta$ to define a bilinear form, called Seifert form, on $H_1(S)$. Remember, that, since $S$ is oriented $V \cong S \times [-1, 1]$ hence we have two embeddings $i^{\pm} : S \to S^3$ given by $i^{\pm}(x) = (x, \pm 1)$. Denote by $x^{\pm}$ the image of $x$ via the map induced in homology by $i_{\pm}$. The **Seifert form** is a $\mathbb{Z}$-valued bilinear form $\alpha : H_1(S) \times H_1(S) \to \mathbb{Z}$ defined by

$$\alpha(x, y) = \beta(x^-, y).$$

Translating by 1 in the positive direction we have $\beta(x^-, y) = \beta(x, y^+)$. Moreover if $c, d$ are simple closed curves in $S$, then $\alpha([c], [d]) = lk(c^-, d) = lk(c, d^+)$. The Seifert form could be used to find a presentation matrix for the Alexander module.

**Theorem 6.2.9.** *Let $L$ be an oriented link and $S$ a Seifert surface for $L$. A presentation matrix for the Alexander module of $L$ is $tA - A^T$ where $A$ is the matrix associated to the Seifert form $\alpha$ on $H_1(S)$ with respect to a fixed basis of $H_1(S)$.*

The matrix $A$, which is not unique, is called **Seifert matrix** for $L$: it is possible to define an equivalence relation on matrices connecting any two Seifert matrices (see [15]). A proof of the above result can be find in [13].

**Remark 6.2.10.** *Since $tA - A^T$ is a square matrix (of order $2g(S) + n(L) - 1$) the first Alexander ideal of $H_1(\widetilde{E}_\infty)$ is principal and so $\Delta_L = \det(tA - A^T)$.*

**Example 6.2.11.** *For $n \in \mathbb{Z}$, consider the oriented knot $K_n$ of Figure 25 having $|2n-1|$ crossings in the lower part of the diagram being positive (as those depicted) if $2n-1 > 0$ and negative (mirrored with respect to the figure) otherwise. The grey one is a Seifert surface $S_n$ for $K_n$ having genus 1: to check it either use the formula of page 23 or observe that $S$ retracts to a theta graph. A basis for $H_1(S)$ is $([f_1], [f_2])$ and the associated Seifert matrix is*

$$\begin{pmatrix} 1 & 0 \\ -1 & n \end{pmatrix}.$$

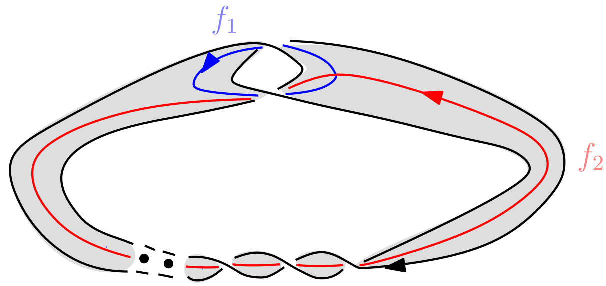



FIGURE 25. The diagram of $K_n$

*So,*

$$\Delta_{K_n} = \det\begin{pmatrix} t-1 & 1 \\ -t & n(t-1) \end{pmatrix} = n(t^2 - 2t + 1) + t.$$

*Note that $K_0$ is the unknot and $\Delta_{K_0} = t$: this result is in accord with the computation of page 6.2.2. The knot $K_1$ has an alternating diagram with 3 crossing so it is the treifol knot (anyway a good exercise is to deform this diagram to a more standard one for the trefoil). Its Alexander polynomial is $\Delta_{K_1} = t^2 - t + 1$.*

Let's list some properties of the Alexander polynomial.

**Proposition 6.2.12.** *Let $L$ be a link. Then*

(1) $\Delta_L(t) \doteq \Delta_L(t^{-1})$
(2) *if $K$ is a knot then $\Delta_K(1) = \pm 1$.*
(3) *if $L$ is a link with at least two components then $\Delta_L(1) = 0$.*
(4) $\Delta_L \doteq \Delta_{L^*}$
(5) $\Delta_L \doteq \Delta_{-L}$
(6) *if $K_1, K_2$ are oriented knots then $\Delta_{K_1+K_2} \doteq \Delta_{K_1}\Delta_{K_2}$*
(7) $B(\Delta_L) \le 2g(L) + n(L) - 1$.

*Proof.* (1) Using the properties of the determinant and the transpose we have $\Delta_L(t^{-1}) = \det(t^{-1}A - A^T) = (-t^{-1})^k \det(-A + tA^T) = (-t^{-1})^k \det((-A + tA^T)^T) = (-t^{-1})^k \det(tA - A^T) \doteq \Delta_L(t)$, with $k$ the order of $A$.
(2) If $K$ is a knot then any Seifert surface has just one boundary component and, by considering the core of the bands, we can take a basis $([f_1], \cdots, [f_{2g}])$ of $H_1(S)$ such that the $f_i$'s are simple closed curves, $f_{2j-1}$ and $f_{2j}$ intersect once, and there are no intersections between other couples of curves. If $A$ is the associated Seifert matrix $(A - A^T)_{ij} = lk(f_i^-, f_j) - lk(f_i^+, f_j)$ and this is the algebraic intersection of $f_i$ and $f_j$ on $S$, so

$$A - A^T = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_g \end{pmatrix},$$

where $B_i = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As a consequence $\Delta_K(1) = \det(A - A^T) = \pm 1$.
(3) If $n(L) \ge 2$, by taking, as before, the core of the bands, we have a basis $([f_1], \cdots [f_{2g}], [f_{2g+1}], \cdots [f_{2g+n-1}])$ so that the columns of $A - A^T$ corresponding to $[f_{2g+1}], \cdots [f_{2g+n-1}]$ are zero, since, if $i > 2g$, then $f_i$ has no intersection with the

other curves.
(4) Let $r : S^3 \to S^3$ be a reflection so that $r(L) = L^*$. If $S$ is a Seifert surface for $L$ then $S^+ = r(S)$ is a Seifert surface for $L^*$. Moreover, if $A$ is the Seifert matrix associated to a basis $\mathcal{B}$, then $-A$ is a Seifert matrix associated to the basis $r(\mathcal{B})$. So $\Delta_{L*} = \pm\Delta_L$.
(5) If $S$ in an oriented Seifert surface for $L$ then $-S$ (i.e., $S$ with opposite orientation) is an oriented Seifert surface for $-L$. So if $A$ is a Seifert matrix for $L$, a Seifert matrix for $-L$ is $A^T$, since $i^-$ exchanges with $i^+$.
(6) If $A_1$ and $A_2$ are Seifert matrices for $K_1$ and $K_2$, then

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

is a Seifert matrix for $K_1 + K_2$ (with respect to a Seifert surface obtained gluing Seifert surface for $K_1, K_2$ and to a basis of the homology that is the union of the bases used for $A_1$ and $A_2$).
(7) For each Seifert surface $S$ for $L$, the matrix $tA - A^T$ has order $2g(S) + n(L) - 1$, so the degree in $t$ of the Alexander polynomial is less or equal to $2g(S) + n(L) - 1$. □

The Alexander polynomial is defined up to units of the Laurent ring: is there a way to normalize it? Observe that for knots, the previous proposition ensures that it is always possible to choose a representative (that we still denote with $\Delta_K$) such that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$; this two properties define uniquely the representative and such a normalization of the Alexander polynomial is called the **Alexander-Conway** polynomial (or the Conway normalization of the Alexander polynomial).

**Exercise 6.2.13.** *If $A$ is a Seifert matrix for a knot, prove that* $\det(t^{\frac{1}{2}}A - t^{-\frac{1}{2}})$ *is its Alexander-Conway polynomial.*

Despite its definition, for knots $\det(t^{\frac{1}{2}}A - t^{-\frac{1}{2}}A^T) \in \mathbb{Z}[t, t^{-1}]$. For link with at least two components the reasoning is different since $\Delta_L(1) = 0$. Anyway it is possible to prove that $\det(t^{\frac{1}{2}}A - t^{-\frac{1}{2}}A^T) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ does not depend on the Seifert matrix used to compute it and, as before, it is called **Alexander-Conway** polynomial. For such normalization a skein relation holds.

**Theorem 6.2.14.** *The Alexander-Conway polynomial $\Delta_L$ of a link $L$ satisfies*

1) $\Delta_{\bigcirc} = 1$
2) $\Delta_{L_+} - \Delta_{L_-} = \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)\Delta_{L_0}$

*where $L_+$, $L_-$ and $L_0$ are three links having diagrams $D_+$, $D_-$ and $D_0$ that differ as indicated in Figure 23 in a specific crossing. Moreover, from 1) and 2) it follows that it holds $\Delta_{\bigcirc^k} = 0$ for $k \geq 2$.*

*Proof.* The first statement follows from Example 6.2.11. In order to prove the second statement let $S_0$ be a Seifert surface for $L_0$ constructed via the Seifert algorithm. We can construct Seifert surfaces for $L_+$ and $L_-$ by adding bands at the specific crossing. If $([f_2], \ldots, [f_n])$ is a basis for $H_1(S_0)$, we obtained bases for $H_1(S_\pm)$, by adding the class of a loop $[f_1]$ as in Figure 26

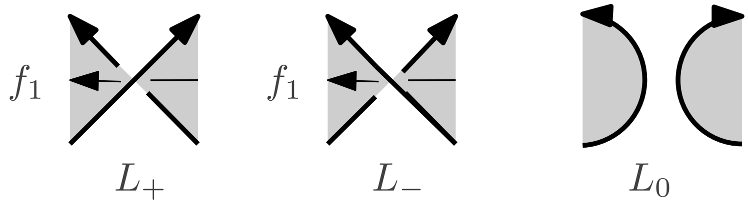



FIGURE 26. The curve $f_1$.

By computing the Seifert matrices $A_0$ and $A_\pm$ associated to $S_0$ and $S_\pm$ with respect to bases $([f_2],\dots,[f_n])$ and $([f_1],[f_2],\dots,[f_n])$ we have

$$A_+ = \begin{pmatrix} n & \mathbf{a} \\ \mathbf{b} & A_0 \end{pmatrix} \qquad A_- = \begin{pmatrix} n-1 & \mathbf{a} \\ \mathbf{b} & A_0 \end{pmatrix},$$

for $n \in \mathbb{Z}$. Now

$$\Delta_{L_+} = \det\left(t^{\frac{1}{2}}A_+ - t^{-\frac{1}{2}}A_+^T\right) = \det\begin{pmatrix} n\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) & t^{\frac{1}{2}}\mathbf{a} - t^{-\frac{1}{2}}\mathbf{b}^T \\ t^{\frac{1}{2}}\mathbf{b} - t^{-\frac{1}{2}}\mathbf{a}^T & t^{\frac{1}{2}}A_0 - t^{-\frac{1}{2}}A_0^T \end{pmatrix},$$

$$\Delta_{L_-} = \det\left(t^{\frac{1}{2}}A_- - t^{-\frac{1}{2}}A_-^T\right) = \det\begin{pmatrix} (n-1)\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) & t^{\frac{1}{2}}\mathbf{a} - t^{-\frac{1}{2}}\mathbf{b}^T \\ t^{\frac{1}{2}}\mathbf{b} - t^{-\frac{1}{2}}\mathbf{a}^T & t^{\frac{1}{2}}A_0 - t^{-\frac{1}{2}}A_0^T \end{pmatrix}.$$

So using the multilinearity of the the determinant (on the first column)

$$\Delta_{L_+} - \Delta_{L_-} = \det\begin{pmatrix} t^{\frac{1}{2}} - t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}}A_0 - t^{-\frac{1}{2}}A_0^T \end{pmatrix} = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\det\left(t^{\frac{1}{2}}A_0 - t^{-\frac{1}{2}}A_0^T\right) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\Delta_{L_0}.$$

The last statement is a straightforward computation analogous to that of Exercise 6.1.15. □

**Exercise 6.2.15.** *Use the above characterization to compute (again) that the Alexander-Conway polynomial of the trefoil knot is* $t^{-1} - 1 + t$.

6.3. **A two variable generalization: the HOMFLY-PT polynomial.** So far, we have seen two one-variable polynomial invariants. They have some common features (a good behaviour under the sum of knots, a skein-type relation,...), but they are essentially different: namely there are knots having the same Alexander polynomial and different Jones one and knots having the same Jones polynomial and different Alexander one. Do these two polynomials have a common generalization? The answer come from the following theorem whose proof can be found in [13, Theorem 15.2]

**Theorem 6.3.1.** *There exists a well-defined unique map*

$$P : \{\text{oriented links in } S^3\} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$$

*such that*

1) $P_{\bigcirc} = 1$
2) $lP_{L_+} + l^{-1}P_{L_-} + mPL_0 = 0$

*where* $L_+$, $L_-$ *and* $L_0$ *are three links having diagrams* $D_+$, $D_-$ *and* $D_0$ *that differ as indicated in Figure 23 in a specific crossing. Moreover, conditions 1) and 2) determine* $P$ *on all links.*

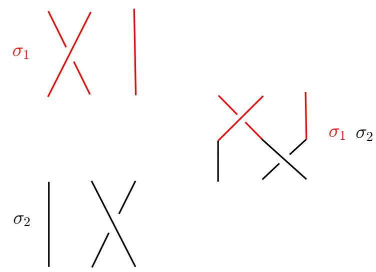



FIGURE 27. Product of braids.

The above polynomial is called HOMFLY-PT polynomial and specializes to the Alexander-Conway's one for $l = i$ and $m = i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, and to the Jones' one for $l = it^{-1}$ and $m = i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$.

**Curiosity:** the letters composing the name of the polynomial are the initials of the surnames of the mathematicians that constructed the polynomial: Hoste, Ocneanu, Millet, Freyd, Lickorish, Yetter, Przytycki, Traczyk.

**Further readings.** There exists another 2-variable generalization of the Jones polynomial: the Kauffman polynomial, constructed in the bracket spirit. However, it doesn't contain the Alexander polynomial as a specialization.

## 7. Braid groups: an algebraic approach to knot theory

So far we represented links using diagrams, in this section we do a quick tour into another possible representation of links that uses braids. Let us start with a definition.

Let $D^2 \times [0,1]$ be a solid cylinder and let $P_1, \dots, P_n$ be fixed points internal to $D^2$. A **braid** on $n$ strands is a compact properly embedded 1-submanifold monotonic with respect to a height function and with $n$ components. More precisely $T \subseteq D^2 \times [0,1]$ is a braid if

1) $T \cong \underbrace{[0,1] \sqcup \cdots \sqcup [0,1]}_{n}$
2) $\partial T \subseteq D^2 \times \{0,1\}$
3) $\pi_{|_\alpha} : D^2 \times [0,1] \to [0,1]$ is monotonic for each $\alpha$ component of $T$.

As for links, we consider braids up to isotopies of $D^2 \times I$ that keep fixed pointwise the boundary and satisfies the monotonic condition at each time. The set of braids (up to isotopy) on $n$ strands is denoted with $B_n$. Also for braid we can introduce the notion of regular diagrams and isotopy of braids becomes isotopy of diagrams (with ends fixed) and Reidemeister moves of type 2 and 3 (why not 1?).

Since braids are essentially sequences of paths they could be composed: for $\sigma_1, \sigma_2 \in B_n$ the braid $\sigma_1\sigma_2$ is obtained by attaching the lower face $D^2 \times \{1\}$ of the first solid cylinder with the upper face $D^2 \times \{0\}$ of the second one and scaling $[0,2]$ to $[0,1]$ (see Figure 27).

With respect to this operation, called product, the braid set is a group called the braid group.

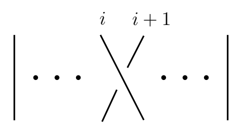



FIGURE 28. The generator $\sigma_i$.

**Exercise 7.0.1.** *Check explicitly the above statement. Hint: the inverse of a given braid $\sigma$ is obtained by taking its reflection through $D^2 \times \{0\}$.*

There are several different equivalent definitions of $B_n$, we list just some of them (for the equivalence of these definitions see [2, Chapter 9]:

- $B_n = \mathrm{MCG}_n(D^2)$, the mapping class group of a $n$-punctured disk: the isotopy classes of orientation preserving homeomorphisms of $D^2$ that keep the boundary fixed pointwise and $\{P_1, \dots, P_n\}$ setwise with the operation given by composition
- $B_n = \pi_1(C_n(D^2))$, the fundamental group of the configuration space of $n$-points in $D^2$: $C_n(D^2)$ is the quotient space of $\{(P_1, \dots, P_n) \mid P_i \in D^2, P_i \neq P_j \text{ if } i \neq j\}$ under the action of the symmetric group $\mathfrak{S}_n$ on $n$-letters.

The group $B_n$ admits the following presentation ([2, Chapter 9])

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, \dots, n-2\rangle,$$

where $\sigma_i$ is the braid depicted in Figure 28

**Exercise 7.0.2.** *Check that $\sigma_i$'s are generators and that the relations hold true.*

Clearly $B_1$ is the trivial group and $B_2$ is the infinite cyclic group.

**Exercise 7.0.3.** *Prove that $B_3$ isomorphic to the fundamental group of the trefoil knot. (Hint: compute the Wirtinger presentation and play with relations)*

If $n \leq m$ we have a natural inclusion $B_n \to B_m$ obtained adding $m - n$ trivial strands on the right. Moreover there is a surjection $B_n \to \mathfrak{S}_n$ that sends $\sigma_i$ to the transposition $(i\ i+1)$. The kernel $P_n$ of this map is a subgroup of $B_n$ called **pure braid group**.

**Exercise 7.0.4.** *Characterize $P_n$ as a subgroup of $\mathrm{MCG}_n(D^2)$ and $\pi_1(C_n(D^2))$. Use the map $B_n \to \mathfrak{S}_n$ to prove that $B_n$ is not commutative for $n \geq 3$.*

**Exercise 7.0.5.** *Prove that $\sigma_i$ has infinite order (hint: use the map $\mathfrak{S}_n \to (\mathbb{Z}, +)$) that sends all $\sigma_i$'s into 1. Depict the braid $(\sigma_i\sigma_2\cdots\sigma_{n-1})^n$ and prove that is central.*

7.1. **From braid to links.** When dealing with bridge number we mentioned that each $n$-bridge link admits a diagram with $n$ maximum at level 1, $n$ minimum at $-1$ and no other critical points (see Figure 17): in other words an $n$-bridge link is obtained by "capping" with arcs the top and bottom layers of a braid with $2n$ strands (see Figure 29).

Such a closure of a braid is called **plat closure** and is defined only for braids with an even number of strands. The bridge number of a link $L$ is the minimum $n$ so that $L$ is the plat closure of a braid in $B_{2n}$ (here you are another definition of the bridge number!).

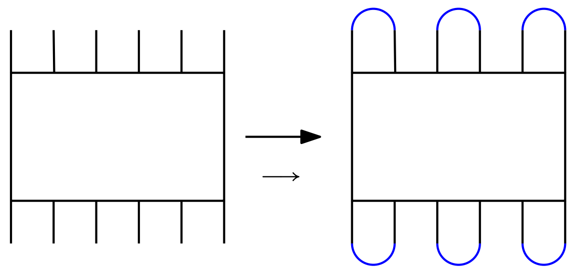


FIGURE 29. The plat closure of a braid with an even number of strands.

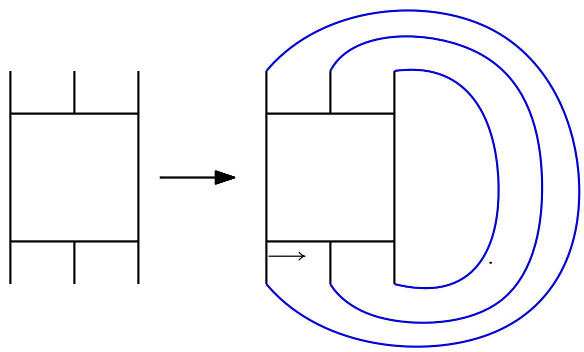


FIGURE 30. The ordinary closure of a braid.

Another way to close up a braid with an arbitrary number of strands is the one depicted in Figure 30: it is defined for braids having an arbitrary number of strands and it is called **ordinary closure** or just **closure** of the braid.

**Exercise 7.1.1.** *Describe the links that you get as the closure of elements in $B_2$.*

As in the case of plat closure, every link is the the closure of a braid. This theorem is called Alexander Theorem and a you can find a proof in [15, Theorem 10.3.1] or [16, Section 6.5]. In the following, following [16, Section 6.5], we sketch an algorithm that, given a diagram of a link $L$, allows to find a braid having $L$ as closure.

Start by observing that if we orient each strand of a braid from top to bottom, then the closure inherit a coherent orientation and the corresponding diagram has the following property: there exists a point such that every connected component of the diagram, seen from the point, runs from right to left (see Figure 31). We say that the diagram winds around the point.

So, if an oriented diagram of a link $L$ winds around a point $Q$, to get a braid whose closure is $L$ it is enough to cut open the diagram along a ray exiting from $Q$ and intersecting the diagram only in regular points. The general strategy is to modify the diagram so as to obtain one that admits this point; to do so we can use $\Delta$ moves: indeed, as Figure 32 shows, with respect to a point $Q$ that is internal to the triangle an edge running from left to right is replaced by a couple of edges running from right to left.

**Exercise 7.1.2.** *Apply the above algorithm to the diagram of the figure eight knot depicted on the left side of Figure 3.*

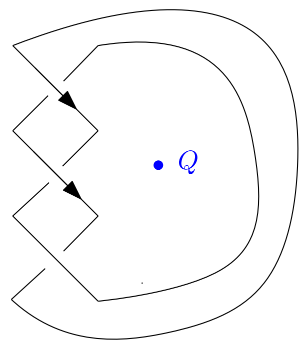



FIGURE 31. The diagram of a closed braid winds around $Q$.

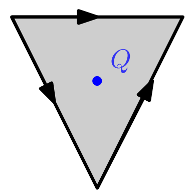



FIGURE 32. Changing how a segment runs with respect to $Q$ $f_1$.

The minimum $n$ such that a given link is the closure of a braid in $B_n$ is called **braid index** of the link. Clearly only the trivial knot has braid index 1.

**Exercise 7.1.3.** *Prove that the links having braid index two are $(2,k)$ torus links, with $k \neq 0, \pm 1$.*

Generally different braids may have the same closure. The following theorem characterize how they are related (see [3, Theorem 2.3] for a proof).

**Theorem 7.1.4** (Markov 1935)**.** *Two braids $\beta \in B_n$ and $\beta' \in B_{n'}$ have isotopic closures if and only if they are connected by a finite sequence of the following moves (and their inverses)*

*conjugation:* $\gamma \to \omega\gamma\omega^{-1}$, *for* $\gamma, \omega \in \ B_k$
*stabilization:* $B_k \ni \gamma \to \gamma\sigma_k \in B_{k+1}$.

**Exercise 7.1.5.** *Prove that two braids that differ for a conjugation or a stabilization have equivalent closures.*

**Exercise 7.1.6.** *Prove that a link with braid index $n$ is the plat closure of a braid with $2n$ strands. Conclude that $Br(L) \leq b(L)$.*

**Further readings.** Also for plat closure there exists an equivalence theorem due to Joan Birman [4]. The moves connecting two braids having the same plat closure are a stabilization moves and moves that correspond to generators of a (non normal) subgroup of the braid group called Hilden subgroup.

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