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Hyperbolicity equations for cusped 3*-manifolds and volume-rigidity of representations*

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Introduction

Ideal triangulations and hyperbolic manifolds

One of the most useful tools for studying hyperbolic structures on 3-manifolds is the technique of ideal triangulations, introduced by Thurston in [26] to study the hyperbolic structure of the complement of the figure-eight knot. An ideal triangulation of an open 3-manifold M is a description of M as a disjoint union of copies of the standard tetrahedron with vertices removed (ideal tetrahedron), glued together by a given set of pairing maps between the 2-dimensional faces. If M is equipped with an ideal triangulation τ, the idea is to construct a hyperbolic structure on M by defining it on each tetrahedron and then by requiring that such structures are compatible with a global one on M. See [26], [20], [7], [1], for more details.

This process is similar to the definition of a hyperbolic structure on M via an $(\mathbb{H}^3, \mathrm{Isom}(\mathbb{H}^3))$-atlas. In fact, there exists a parallelism between the classical theory of (X, G)-structures defined using open coverings and the theory of (X, G)-atlantes on triangulated objects (see Section 1.2 and Chapter 2). Namely, once one has a triangulated object, the simplices of maximal dimension play the role of local charts and the pairings between the faces of the triangulation play the role of changes of chart.

A complete, finite volume hyperbolic structure with geodesic faces on an oriented tetrahedron is described by a complex number with positive imaginary part, called *modulus*. Similarly, a finite area, complete similarity structure with straight edges on an oriented triangle is described by a complex number with positive imaginary part, also called *modulus*. These two situations are strictly related. In fact, horospherical sections near the vertices of a hyperbolic ideal tetrahedron of modulus z give Euclidean triangles (up to scaling) with modulus z. In particular the hyperbolic structure of an ideal tetrahedron is completely determined by the similarity structure of any of its horospherical triangles. The notion of modulus coherently extends to numbers in $\mathbb{C} \setminus \{0, 1\}$, with the meaning that a real modulus describes the structure of a flat tetrahedron (contained in a plane but with four distinct vertices), and a modulus with negative imaginary part describes a negatively oriented tetrahedron. See Section 2.3 for details.

If M is a finite volume complete hyperbolic orientable 3-manifold, then it is well-known that it is diffeomorphic to the interior of a compact 3-manifold $\overline{M}$ whose boundary consists of tori. The starting point of this work is to consider the following data:

- A 3-manifold M satisfying the above topological conditions.
- An ideal triangulation τ of M. By chopping-off a regular neighborhood of the vertices (keep in mind the correspondence: hyperbolic ideal tetrahedra $\leftrightarrow$ Euclidean horospherical triangles), τ induces a triangulation of $\overline{M}$ via truncated tetrahedra, and so a classical triangulation of the tori of $\partial\overline{M}$.
- A choice $\mathbf{z} = (z_1, \ldots, z_n)$ of a complex modulus z_i for each tetrahedron Δ_i of τ. Such a choice of moduli induces a choice of moduli, also called $\mathbf{z}$, for the triangulation induced on $\partial\overline{M}$.

The first natural questions are: *Does* $\mathbf{z}$ *define a hyperbolic structure on* M*? Does* $\mathbf{z}$ *define a similarity structure on* $\partial\overline{M}$*?* As usual, the first natural questions are the hardest to answer. First of all, one has to check the local compatibility of the choice of moduli. When one has a classical (X, G)-atlas defined on an open cover $\{U_i\}$ of M, the changes of chart satisfy the so-called co-cycle conditions:

$$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \mathrm{Id}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$.

In the setting of triangulated objects and hyperbolic/Euclidean world, this translates to algebraic equations on the moduli, called *compatibility equations*. Moreover, when all the moduli of $\mathbf{z}$ have positive imaginary part, the (X, G)-atlas induced by $\mathbf{z}$ actually is a classical (X, G)-atlas, so $\mathbf{z}$ defines an (X, G)-structure on M (more precisely, hyperbolic on M and similarity on $\partial\overline{M}$, see [26]). When the moduli are in $\mathbb{C} \setminus \{0, 1\}$, so that negative and flat tetrahedra appear, the situation becomes more involved (see below). If $\mathbf{z}$ defines a structure on M, then one can ask about completeness of such a structure, and this translates into other algebraic equations on the moduli, called *completeness equations*. More generally, by requiring that the metric completion of the structure induced by $\mathbf{z}$ is a chosen Dehn filling of M, one gets a system of equations, called *hyperbolic Dehn filling equations*, that coincide with the completeness equations in the case of the empty filling. Finally, if the moduli lie in

$\{z \in \mathbb{C} \setminus \{0, 1\} : \Im(z) \geq 0\}$, then one can continuously define the angle of a modulus as its argument and write down the so-called *equations on the angles* by requiring that the sum of the angles around each edge is exactly 2π (see Sections 2.5 and 2.6 for details on the equations).

A choice of moduli $\mathbf{z}$ such that each modulus lies in $\{z \in \mathbb{C} \setminus \{0, 1\} : \Im(z) \geq 0\}$ and at least one modulus has strictly positive imaginary part is called *positive, partially flat*. The following are the principal known results about the solutions of the above systems:

- If $\mathbf{z}$ is a solution of the compatibility equations and all the moduli have positive imaginary part, then $\mathbf{z}$ defines a hyperbolic structure on M and a similarity structure on $\partial \overline{M}$. Such structures are complete if and only if $\mathbf{z}$ satisfies also the system of completeness equations. More generally, the completion of the structure of M is a prefixed Dehn filling of M if and only if $\mathbf{z}$ satisfies the corresponding hyperbolic Dehn filling equations. See [26].
- If $\mathbf{z}$ is a positive, partially flat solution of the compatibility, completeness equations, and those on the angles, then $\mathbf{z}$ defines a hyperbolic structure on M. See [22].
- Each hyperbolic cusped manifold admits an ideal triangulation with a positive partially flat solution of the compatibility and completeness equation that induces the hyperbolic structure of M. In other words, each hyperbolic cusped manifold can be decomposed in a finite set of positive, partially flat, ideal straight tetrahedra. Such a decomposition is obtained by subdividing the so-called *Epstein-Penner* decomposition. See [7].
- Each solution $\mathbf{z}$ (possibly containing negative moduli) of the compatibility equations sufficiently close to the Epstein-Penner decomposition induces a (incomplete) hyperbolic structure on M. The completion of such a structure is a prefixed Dehn filling of M if and only if $\mathbf{z}$ satisfies the corresponding hyperbolic Dehn filling equations. Moreover, almost all the Dehn fillings of M are obtained in such a way, so they are hyperbolic. This fact is know as Thurston's hyperbolic Dehn filling theorem ([26]). See [21] and [2] for a complete proof.

The meaning of the sentence "$\mathbf{z}$ induces a structure on M" is clear when all the moduli have positive imaginary part, but in general it is no clear how to interpret it. In Chapter 2, I introduce the notion of *Geometric solution* of the above systems of equations, choosing an interpretation in terms of holonomy and developing maps. Roughly speaking, a choice of moduli $\mathbf{z}$ is a geometric solution, say of the compatibility and completeness equations, if there exists a complete finite-volume hyperbolic structure $\mathfrak{S}$ on M such that, if $M_{\mathfrak{S}}$ is M with the hyperbolic structure, there exists a proper degree-one map $f : M \to M_{\mathfrak{S}}$ that on each tetrahedron induces, via pull-back, the structure prescribed by $\mathbf{z}$.

The behavior of the geometric solutions in the hyperbolic 3-dimensional case and in the Euclidean 2-dimensional case (*i.e.* M and $\partial\overline{M}$) are different. In Chapter 3, I study the 2-dimensional case and in Chapter 4 the 3-dimensional one. The principal results are the following:

- In both dimensions 2 and 3, the set of the geometric solutions of the compatibility equations is an open subset of the set of the the algebraic solutions. See Proposition 3.2.10 and Theorem 4.1.12.
- In dimension 2, any solution of the compatibility and completeness equations is geometric provided that the algebraic sum of the areas of the straight versions of the triangles does not vanish. See Proposition 3.2.11. This in particular implies that the geometric solutions of the compatibility and completeness equations are not unique.
- In dimension 2, the geometric solution of the completeness equations are completely characterized by an algebraic condition. See Theorems 3.2.8 and 3.2.9 for details.
- In dimension 3, there exists at most one geometric solution of the compatibility and completeness equations (more generally of the hyperbolic Dehn filling equations). See Theorem 4.1.19.
- These results allow one to prove the hyperbolic Dehn filling theorem starting from a geometric solution different from the Epstein-Penner decomposition.
- In dimension 3, there exist solutions of the compatibility and completeness equations that are not geometric. For such solutions the algebraic sum of the volumes of the straight versions of the tetrahedra can be positive. Moreover, such non-geometric solutions can be geometric if restricted to $\partial\overline{M}$, that is, they do not induce a hyperbolic structure on M but they induce a similarity structure on $\partial\overline{M}$. See Examples 4.2.1 and 4.2.2.
- In dimension 3, there exists an example of a manifold with a positive partially flat solution of the compatibility and completeness equations (but not those on the angles) which is not geometric. See Example 4.2.3.

Volume of representations and rigidity

After Chapters 3 and 4 the problem remains open in dimension 3 of whether or not a solution of the compatibility and completeness equations is geometric. In dimension 2 the algebraic sum of the areas of the straight versions of the triangles is decisive in order to decide if a solution is geometric. Chapter 5 is devoted to study its correspondent in dimension 3. Let $\mathrm{vol}(\mathbf{z})$ be the algebraic sum of the volumes of the straight versions of the tetrahedra of the triangulation. The view-point of holonomy and developing maps leads to try to forget of the triangulation and to look at $\mathrm{vol}(\mathbf{z})$ as a number related to a pair "holonomy representation" - "developing map". More precisely, if M is a cusped manifold and $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a representation, then a number $\mathrm{vol}(\rho)$ is well-defined in such a way that for any solution $\mathbf{z}$ of the compatibility equations one has $\mathrm{vol}(\mathbf{z}) = \mathrm{vol}(h(\mathbf{z}))$ (where $h(\mathbf{z})$ is the holonomy representation of $\mathbf{z}$). The volume of representations is already well-known in the compact case, and deep results about hyperbolic manifolds have been established using it (see for example [6] and [26]). For compact manifolds, one has:

- The volume of ρ is bounded by a multiple of the Gromov norm of M.
- If M is complete hyperbolic, the holonomy of the hyperbolic structure is the only representation of maximal volume. See [6]. Actually, such a rigidity is proved generalizing Gromov's proof of Mostow's theorem, and easily implies the strong version of Mostow rigidity (Theorem 4.1.1 for compact manifolds).

Let W be a compact manifold and let ρ be a representation of its fundamental group into $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$. The volume of ρ is defined by taking any ρ-equivariant map from the universal cover $\widetilde{W}$ to $\mathbb{H}^3$ and then by integrating the pull-back of the hyperbolic volume form on a fundamental domain. This volume does not depend on the choice of the equivariant map because two equivariant maps are always equivariantly homotopic and the cohomology-class of the pull-back of the volume form is invariant under homotopy.

In [6] this definition is extended to the case of a non compact cusped 3-manifold M (see Definitions 5.2.1 and 5.1.5). When M is not compact, some problems of integrability arise if one tries to use the above definition of the volume of a representation. The idea of Dunfield for overcoming these difficulties is to use a particular (and natural) class of equivariant maps, called *pseudo-developing maps* (see Definition 5.1.5), that have a nice behavior on the cusps of M allowing to control their volume. Concerning the well-definition of the volume, working with non-compact manifolds, two pseudo-developing maps in general are not equivariantly homotopic and in [6] it is not proved that the volume of a representation does not depend on the chosen pseudo-developing map. In Chapter 5, I will show that the volume of a representation is well-defined even in the non-compact case, and I generalize to non-compact manifolds the above results know in the compact case. I will restrict to the orientable case. The main results are:

- The volume of a representation is well-defined also in the non-compact case. See Theorem 5.2.9.
- Such a volume can be computed by straightening any ideal triangulation of M. See Theorem 5.2.10.
- The volume of a representation is bounded from above by a multiple of the relative simplicial volume of $(\overline{M}, \partial\overline{M})$. See Theorem 5.3.1.
- If M is complete hyperbolic, then for any representation ρ one has $\mathrm{vol}(\rho) \leq \mathrm{vol}(M)$ and equality holds if and only if ρ is the holonomy of the complete structure. See Theorem 5.4.1.
- This in particular implies that if M is hyperbolic, then for a solution $\mathbf{z}$ of the compatibility and completeness equations one has $\mathrm{vol}(\mathbf{z}) \leq \mathrm{vol}(M)$, and $\mathbf{z}$ is geometric if and only if $\mathrm{vol}(\mathbf{z}) = \mathrm{vol}(M)$.

In Section 5.5, I give some corollaries of the above theorems. In particular I show how from the rigidity theorem for representations (Theorem 5.4.1) one can get a proof of a strong version of Mostow-Prasad rigidity theorem (Theorem 4.1.1). I was informed by B. Klaff that results similar to those proved in Chapter 5 have also been established in [15].

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Chapter 1
Preliminaries

In this chapter I give a short overview of the hyperbolic world and I discuss some preliminary results I need in the sequel. Most of the facts I am going to describe are standard and are extensively treated in several texts (see for example [1] and [24]). So I often omit precise references.

1.1. The hyperbolic space

In this section I describe the most common models of the hyperbolic n-dimensional space $\mathbb{H}^n$. For simplicity, I stick to the dimension 3, but most of the following statements hold in other dimensions. I refer to Chapter A of [1] for complete proofs of the following facts.

Definition 1.1.1. The hyperbolic space $\mathbb{H}^3$ is the only Riemannian manifold of dimension three that satisfies:

- $\mathbb{H}^3$ is connected, simply connected.
- $\mathbb{H}^3$ is complete.
- $\mathbb{H}^3$ The sectional curvature is constant -1.

The group of Riemannian isometries of $\mathbb{H}^3$ is denoted by $\mathrm{Isom}(\mathbb{H}^3)$. The subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ consisting of all orientation-preserving isometries is denoted by $\mathrm{Isom}^+(\mathbb{H}^3)$, while the set of orientation-reversing ones is denoted by $\mathrm{Isom}^-(\mathbb{H}^3)$.

I describe now four models of $\mathbb{H}^3$.

Hyperboloid model. In $\mathbb{R}^4$ consider the standard Lorentzian metric

$$\langle x, y\rangle_L = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

and let $\mathbb{I}^3$ be the upper fold of the hyperboloid consisting of the points with $\langle x, x\rangle_L = -1$, *i.e.*

$$\mathbb{I}^3 = \{x \in \mathbb{R}^4 \ : \ \langle x, x\rangle_L = -1, \ x_4 > 0\}.$$

The tangent space to $\mathbb{I}^3$ at a point x is exactly the orthogonal to x:

$$T_x\mathbb{I}^3 = \{y \in \mathbb{R}^4 \ : \ \langle x, y\rangle_L = 0\}$$

thus the Lorentzian metric, restricted to any $T_x\mathbb{I}^3$ is positive definite.

The hyperbolic metric of $\mathbb{I}^3$ is the metric induced on $T\mathbb{I}^3$ by $\langle\cdot\,,\cdot\rangle_L$. Let $\mathrm{O}(3, 1)$ be the group of linear isomorphisms of $\mathbb{R}^4$ that preserve $\langle\cdot\,,\cdot\rangle_L$. The group of the isometries of $\mathbb{I}^3$ consists of the restrictions of the elements of $\mathrm{O}(3, 1)$ that keep $\mathbb{I}^3$ invariant:

$$\mathrm{Isom}(\mathbb{I}^3) = \{A \in O(3, 1) \ : \ A(\mathbb{I}^3) = \mathbb{I}^3\}.$$

The geodesics of $\mathbb{I}^3$ are obtained intersecting 2-dimensional subspaces of $\mathbb{R}^4$ with $\mathbb{I}^3$. If $x \in \mathbb{I}^3$ and $y \in T_x\mathbb{I}^3 = x^\perp$ with $||y|| = 1$, then the geodesic starting from x with initial speed y is parametrized by arc length by:

$$\gamma(t) = x \cdot \cosh(t) + y \cdot \sinh(t).$$

Disc model. Consider $\mathbb{R}^3$ as a subspace of $\mathbb{R}^4$ by $\mathbb{R}^3 = \{x \in \mathbb{R}^4 \ : x_4 = 0\}$ and let $p : \mathbb{R}^4 \to \mathbb{R}^3$ be the stereographic projection with pole $(0, 0, 0, -1)$. Call $\mathbb{D}^3$ the p-image of $\mathbb{I}^3$. It is readily checked that $\mathbb{D}^3$ is the unitary disc of $\mathbb{R}^3$. The hyperbolic metric of $\mathbb{D}^3$ is the push-forward of the metric of $\mathbb{I}^3$. In coordinates, for $x \in \mathbb{D}^3$, $v \in \mathbb{R}^3 \cong T_x\mathbb{D}^3$ it is given by

$$ds^2_x(v) = \left(\frac{2}{1 - ||x||^2}\right)^2 \cdot ||v||^2$$

where $||\cdot||$ is the Euclidean norm of $\mathbb{R}^3$.

The isometries of $\mathbb{D}^3$ are exactly the diffeomorphisms of $\mathbb{D}^3$ that are conformal w.r.t. the Euclidean metric. The geodesics of $\mathbb{D}^3$ are all the arc of circles that are orthogonal to $\partial\mathbb{D}^3$, including diameters.

Projective model. Let p be the standard projection of $\mathbb{R}^4 \setminus \{0\}$ onto $\mathbb{RP}^3$. In a suitable affine chart of $\mathbb{RP}^3$, the p-image $\mathbb{B}^3$ of $\mathbb{I}^3$ is the unitary ball of $\mathbb{R}^3$. The hyperbolic metric of $\mathbb{B}^3$ is the push-forward of the one of $\mathbb{I}^3$. The geodesics of $\mathbb{B}^3$ are all the Euclidean straight segment with vertices on the boundary. Note that the projection p is not conformal w.r.t. the Euclidean structures.

Half-space model. Let $p : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ be the inversion with respect to the sphere of center $(0, 0, -1)$ and radius $\sqrt{2}$. The p-image of $\mathbb{D}^3$ is the upper half-space of $\mathbb{R}^3$, *i.e.*

$$\Pi^3 = \{(x, y, t) \in \mathbb{R}^3 \ : \ t > 0\}.$$

Note that p is a conformal mapping with respect to the Euclidean structure of $\mathbb{R}^3$. The hyperbolic metric of Π^3 is the push-forward of that of $\mathbb{D}^3$. In coordinates, for $(x, y, t) \in \Pi^3$ and $v \in \mathbb{R}^3 \cong T_{(x,y,t)}\Pi^3$ one has that

$$ds^2_{(x,y,t)}(v) = \frac{||v||^2}{t^2}$$

where $||\cdot||$ is the Euclidean norm of $\mathbb{R}^3$.

The geodesics of Π^3 are all the half-circles that are normal to $\partial\Pi^3$, including vertical straight lines. The isometries of Π^3 are all the conformal (w.r.t. the Euclidean metric of $\mathbb{R}^3$) diffeomorphisms of $\mathbb{R}^3$ that keep Π^3 invariant. In coordinates such maps are those of the form

$$\xi \mapsto \lambda \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} r(\xi) + \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{1.1}$$

where $A \in \mathrm{O}(2, \mathbb{R})$, $\lambda > 0$, $\begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^3$ and r is either the identity or an inversion with respect to a sphere orthogonal to $\partial\Pi^3$.

The space Π^3 can be identified with a $\mathbb{C} \times \mathbb{R}^+$ by

$$(x, y, t) \mapsto (x + \mathbf{i}y, t).$$

Moreover Π^3 can be endowed with an algebraic structure by considering it as a subset of the field $\mathcal{H}$ of quaternions via the inclusion

$$\Pi^3 = \{x + \mathbf{i}y + \mathbf{j}t + \mathbf{k}s \in \mathbb{H} \ : \ s = 0, t > 0\}.$$

This structure is particularly useful to visualize $\mathrm{Isom}^+(\Pi^3)$.

Proposition 1.1.2. *There exists a natural isomorphism* $\Phi : \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{Isom}^+(\Pi^3)$ *given by*

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \xi \mapsto (a \cdot \xi + b) \cdot (c \cdot \xi + d)^{-1}$$

where the product and the inverse are those of $\mathcal{H}$.

Proof. For the proof, to indicate an element $\xi \in \Pi^3$, I will use both notations

$$\xi = z + \mathbf{j}t \qquad \text{and} \qquad \xi = x + \mathbf{i}y + \mathbf{j}t$$

with $z \in \mathbb{C},\ x, y \in \mathbb{R},\ t \in \mathbb{R}^+$.

First of all I prove that $\Phi(A) \circ \Phi(B) = \Phi(AB)$ for $A, B \in \mathrm{PSL}(2, \mathbb{C})$. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad ab - cd = \alpha\beta - \gamma\delta = 1.$$

Recall that for any $\eta, \theta \in \mathcal{H}$ it holds $\eta^{-1} = \overline{\eta}/|\eta|^2$, $|\eta|^2 = \eta\overline{\eta}$, $\overline{\eta\theta} = \overline{\eta}\overline{\theta}$, where the bar indicates the usual conjugation of $\mathcal{H}$. For any $\xi \in \Pi^3$

$$
\begin{aligned}
&\Phi(A)(\Phi(B)(\xi)) = \\
&= \Big(a \cdot (\alpha \cdot \xi + \beta) \cdot (\gamma \cdot \xi + \delta)^{-1} + b\Big) \\
&\qquad\qquad \cdot \Big(c \cdot (\alpha \cdot \xi + \beta) \cdot (\gamma \cdot \xi + \delta)^{-1} + d\Big)^{-1} \\
&= \left(\frac{a \cdot (\alpha \cdot \xi + \beta) \cdot \overline{(\gamma \cdot \xi + \delta)}}{(\gamma \cdot \xi + \delta) \cdot \overline{(\gamma \cdot \xi + \delta)}} + b\right) \\
&\qquad\qquad \cdot \left(\frac{c \cdot (\alpha \cdot \xi + \beta) \cdot \overline{(\gamma \cdot \xi + \gamma)}}{(\gamma \cdot \xi + \delta) \cdot \overline{(\gamma \cdot \xi + \delta)}} + d\right)^{-1} \\
&= \Big(\big[a \cdot (\alpha \cdot \xi + \beta) + b \cdot (\gamma \cdot \xi + \delta)\big] \cdot \overline{(\gamma \cdot \xi + \delta)}\Big) \\
&\qquad\qquad \cdot \Big(\big[c \cdot (\alpha \cdot \xi + \beta) + d \cdot (\gamma \cdot \xi + \delta)\big] \cdot \overline{(\gamma \cdot \xi + \delta)}\Big)^{-1} \\
&= \big[a \cdot (\alpha \cdot \xi + \beta) + b \cdot (\gamma \cdot \xi + \delta)\big] \cdot \big[c \cdot (\alpha \cdot \xi + \beta) + d \cdot (\gamma \cdot \xi + \delta)\big]^{-1} \\
&= \big[(a\alpha + b\gamma) \cdot \xi + (a\beta + b\delta)\big] \cdot \big[(c\alpha + d\gamma) \cdot \xi + (c\beta + d\delta)\big]^{-1} \\
&= \Phi(AB)(\xi).
\end{aligned}
$$

Now I check that the image of Φ is actually contained in $\mathrm{Isom}^+(\Pi^3)$. Any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{PSL}(2, \mathbb{C})$ with $c \neq 0$ splits as

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a/c & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & 1/c \end{pmatrix}.
$$

Since the isometries of Π^3 are the conformal diffeomorphisms of $\mathbb{R}^3$ that keep Π^3 invariant, it suffices to check that for any element A of $\mathrm{PSL}(2, \mathbb{C})$ of the form $\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$, $\Phi(A)$ is an orientation-preserving conformal diffeomorphism that keeps Π^3 invariant.
I check now the first case. For $\xi \in \Pi^3$ one has

$$
\Phi \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix} (\xi) = (\alpha \cdot \xi - 1) \cdot \xi^{-1} = \alpha - \xi^{-1}
$$

which is easily checked to be conformal and orientation-preserving. Moreover, if $\xi \in \Pi^3$ then $-\xi^{-1} \in \Pi^3$, so also the invariance of Π^3 is checked.

For the second case, one has

$$\begin{aligned}\Phi\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}(\xi) &= (\alpha\cdot\xi+\beta)\cdot\alpha = \alpha\cdot\xi\cdot\alpha+\beta\cdot\alpha \\ &= \alpha\cdot(z+\mathbf{j}t)\cdot\alpha+\beta\cdot\alpha = \alpha^2 z+\mathbf{j}|\alpha|^2 t+\beta \\ &= |\alpha|^2\left(\frac{\alpha^2}{|\alpha|^2}z+\mathbf{j}t\right)+\beta\end{aligned}$$

which is conformal, orientation-preserving, and keeps Π^3 invariant.

I check now that Φ is one-to-one. Let $\begin{pmatrix}a & b \\ c & d\end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ such that $\Phi\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \mathrm{Id}$. Then for all $\xi \in \Pi^3$ it is

$$(a\cdot\xi+b)\cdot(c\cdot\xi+d)^{-1} = \xi.$$

In particular, by choosing $\xi = \mathbf{j}t$ with $t \in \mathbb{R}^+$ one easily gets $c = 0$. Then one has $a\cdot\xi\cdot d^{-1}+b\cdot d^{-1} = \xi$ for any ξ. From this, and since $1 = ad-bc = ad$, it follows that $b = 0$, $d^{-1} = a$ and $a^2 = 1$. Therefore $\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \pm\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$ which is exactly the class of the identity in $\mathrm{PSL}(2,\mathbb{C})$.

To conclude, I check that Φ is onto. By formula (1.1), any element of $\mathrm{Isom}^+(\Pi^3)$ can be written as

$$\xi \mapsto \lambda\begin{pmatrix}A & 0 \\ 0 & 1\end{pmatrix}r(\xi)+\begin{pmatrix}b \\ 0\end{pmatrix}$$

where $\lambda \in \mathbb{R}^+$, $A \in \mathrm{SO}(2,\mathbb{R})$ and r is either the identity or the composition of an inversion w.r.t. a sphere orthogonal to $\partial\Pi^3$ with the reflection $z+\mathbf{j}t \mapsto \overline{z}+\mathbf{j}t$. Thus any orientation-preserving isometry can be written as

$$\xi \mapsto \Phi(B)(r(\xi))$$

where $B \in \mathrm{PSL}(2,\mathbb{C})$ is of the form $\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}$. It follows that it suffices to check that $r(\xi)$ lies in the image of Φ. Obviously the identity is in the image of Φ. Up to conjugating by elements in the image of Φ, one can suppose that r is the composition of the reflection $z+\mathbf{j}t \mapsto \overline{z}+\mathbf{j}t$ with the inversion w.r.t. the sphere of center 0 and radius 1. Such an inversion is given by $\xi \mapsto \xi/|\xi|^2$, so r is the map

$$\xi = z + \mathbf{j}t \mapsto \frac{\bar{z} + \mathbf{j}t}{|z|^2 + |t|^2} = \Phi \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} (\xi).$$

This completes the proof. □

The boundary at infinity. The models $\mathbb{D}^3$ and Π^3 of $\mathbb{H}^3$ suggest the presence of a *boundary* at the *infinity* of the hyperbolic space. Such a boundary actually exists and has an intrinsic meaning.

Consider the set S of all geodesics rays in $\mathbb{H}^3$, parametrized by arc length on $[0, \infty]$, and define an equivalence relation $\sim$ on S by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < +\infty.$$

Set $\partial\mathbb{H}^3 = S/\sim$ and $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \partial\mathbb{H}^3$. Is is well-known that $\overline{\mathbb{H}}^3$ has a topology and a smooth structure such that $\mathbb{H}^3$ inherits its own topology, $\partial\mathbb{H}^3$ is diffeomorphic to S^2 and $\overline{\mathbb{H}}^3$ to $\overline{D}^3$. The last diffeomorphism is realized in the disc model, *i.e.* $\overline{\mathbb{H}}^3$ is diffeomorphic to $\overline{\mathbb{D}}^3$. In the half-space model $\partial\mathbb{H}^3 = \mathbb{PC}^1 = \mathbb{C} \cup \{\infty\} = \partial\Pi^3 \cup \{\infty\}$. The point of $\partial\mathbb{H}^3$ are called points at infinity.

If p is a point at infinity of $\mathbb{H}^3$, a geodesic γ is said to start from (or to end at) p if p is in the equivalence class of $\gamma|_{(-\infty,0]}$ (or $\gamma|_{[0,\infty)}$). The point p is called an endpoint of γ. It is readily verified that all geodesics have exactly two endpoints and that for any $p \neq q \in \partial\mathbb{H}^3$ there exists a unique (up to reparametrization) geodesic having p and q as endpoints.

Proposition 1.1.2 implies that any orientation-preserving isometry of Π^3 extends to $\partial\Pi^3$ acting as a Möbius transformation on $\partial\Pi^3 \cong \mathbb{PC}^1$. From this one can easily deduce the following:

Proposition 1.1.3. *Each isometry φ of $\mathbb{H}^3$ extends to a diffeomorphism of $\overline{\mathbb{H}}^3$ and it is completely determined by its trace $\varphi|_{\partial\mathbb{H}^3}$ on $\partial\mathbb{H}^3$. Moreover* $\mathrm{Isom}(\mathbb{H}^3)$ *acts transitively on the set of triples of distinct points of $\partial\mathbb{H}^3$. More precisely, if $p_i, q_i \in \partial\mathbb{H}^3$, $i = 0, 1, 2$ with $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$, then there exists only one element $\varphi^+ \in \mathrm{Isom}^+(\mathbb{H}^3)$ and only one $\varphi^- \in \mathrm{Isom}^-(\mathbb{H}^3)$ such that $\varphi^\pm(p_i) = q_i$ for $i = 0, 1, 2$.*

Remark 1.1.4. Proposition 1.1.3 implies that given any $p_0, p_1, p_2 \in \partial\mathbb{H}^3$, it is always possible to choose a half-space model of $\mathbb{H}^3$, such that $p_0 = 0$, $p_1 = 1$ and $p_2 = \infty$, *i.e.* it is always possible to choose a diffeomorphism $\psi : \overline{\mathbb{H}}^3 \to \overline{\Pi}^3 \cup \{\infty\}$ such that $\psi(p_0) = 0$, $\psi(p_1) = 1$ and $\psi(p_2) = \infty$.

Classification of isometries. As $\overline{\mathbb{H}}^3$ is a closed disc, then each isometry $\varphi \in \mathrm{Isom}(\mathbb{H}^3)$ has at least one fixed point in $\overline{\mathbb{H}}^3$, and it is easily checked that only the following three cases are possible:

- φ has a fixed point in $\mathbb{H}^3$; in this case it is called *elliptic*.
- φ has a unique fixed point in $\overline{\mathbb{H}^3}$ that lies on $\partial\mathbb{H}^3$; in this case it is called *parabolic*.
- φ has no fixed point in $\mathbb{H}^3$ and exactly two fixed points in $\partial\mathbb{H}^3$; in this case it is called *hyperbolic*.

The prototypes, in $\mathrm{PSL}(2,\mathbb{C})$, of the orientation-preserving isometries are:

- The identity: $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Elliptic: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{C}, |\lambda| = 1,\ \lambda \neq 1$. In the half-space model, the whole geodesic from 0 to ∞ is fixed.
- Parabolic: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{C},\ b \neq 0$. The fixed point in the half-space model is ∞.
- Hyperbolic: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{C},\ |\lambda| \neq 1,\ \lambda \neq 0$. The fixed points in the half-space model are 0 and ∞.

Horospheres. Let p be a point at infinity of $\mathbb{H}^3$. A connected surface in $\mathbb{H}^3$ is called *horosphere* centered at p if it is orthogonal to all geodesics ending at p. In the disc model, a horosphere centered at p is a Euclidean sphere tangent to $\partial\mathbb{H}^3$ at p. In the half-space model, a horosphere centered at $p \in \partial\Pi^3$ is a Euclidean sphere tangent to $\partial\Pi^3$ at p, while a horosphere centered at ∞ is a horizontal Euclidean plane. Since the hyperbolic metric of Π^3 is the Euclidean one rescaled by the inverse of the height, by Remark 1.1.4 it follows that each horosphere of $\mathbb{H}^3$ inherits from the hyperbolic metric a Euclidean structure. From the above characterization of the isometries, one can see that if φ is a parabolic isometry, then the restriction of φ to any horosphere centered at the fixed point of φ is an isometry w.r.t. the Euclidean structure of the horosphere.

Ideal simplices. In $\overline{\mathbb{H}}^3$ the convex hull of a set is well-defined. A simplex in $\overline{\mathbb{H}}^3$ is said to be *straight* if it is the convex hull of its vertices, and it is said *ideal* if its vertices lie in $\partial\mathbb{H}^3$.

From Proposition 1.1.3 it follows that two straight ideal triangles are always isometric. Even if a straight ideal triangle is “infinite”, it is well-known that its area is π, which is, of course, the maximum of the areas of all straight triangles.

Two straight ideal tetrahedra in general are not isometric. It is well-known that the isometry class of a straight ideal tetrahedron depends on the dihedral angles between its faces, and it is easy to see that angles at opposite edges coincide (see also Section 2.3.2). A straight tetrahedron is said *regular* if all its dihedral angles coincide, and it turns out that all regular straight ideal tetrahedra are isometric. As above, any straight ideal tetrahedron Δ has finite volume, that can be computed by

$$\mathrm{vol}(\Delta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

where $\Lambda(x) = -\int_0^x \log|2\sin t|dt$ is the Lobachevsky function and α, β, γ are the dihedral angles of Δ.

This in particular implies that for any straight ideal tetrahedron Δ one has

$$\mathrm{vol}(\Delta) \leq V_3$$

where V_3 is the volume of a regular straight ideal tetrahedron (see [1], Section C.2 for details).

Hyperbolic manifolds. A hyperbolic 3-manifold M is a complete Riemannian manifold with constant sectional curvature -1. Thus the Riemannian universal cover of M is $\mathbb{H}^3$ and the fundamental group of M can be viewed as a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ so that $M = \mathbb{H}^3/\Gamma$. I conclude this section recalling three important theorems about hyperbolic manifolds.

Theorem 1.1.5. *Let M be a complete hyperbolic 3-manifold of finite volume. Then M is diffeomorphic to the interior of a compact manifold $\overline{M}$. Moreover $\partial\overline{M}$ is a finite (possibly empty) union of tori and Klein bottles. If C is a component of $\partial\overline{M}$ then there exists a neighborhood $U \subset \overline{M}$ of C such that $U \cap M$ is diffeomorphic to $C \times (0,\infty)$ such that for any $c \in C$ the curve $t \mapsto (c,t)$ is a geodesic parametrized by arc length and the metric on $C \times \{t\}$ is a fixed Euclidean metric on C rescaled by a factor e^{-2t}.*

Theorem 1.1.6. *Any complete hyperbolic 3-manifold of finite volume is irreducible and contains no essential tori.*

Theorem 1.1.7. (Mostow-Prasad rigidity. Mostow [19] for compact case, Prasad [23] for non-compact case) *Let M_1 and M_2 be oriented complete hyperbolic 3-manifold of finite volume. If $f : M_1 \to M_2$ is a proper homotopy equivalence, then it is properly homotopic to an isometry.*

Theorem 1.1.7 in particular implies that every 3-manifold admits at most one complete hyperbolic structure of finite volume.

1.2. (X, G)-structures, developing maps and holonomies

In this section I recall the classical notions of (X, G)-structure, developing map and holonomy. I refer the reader to Chapter 8 of [24] for further details.

(X, G)**-atlantes.** Let X be a connected, simply connected smooth manifold and let G be a subgroup of the group $\mathrm{diff}(X)$ of diffeomorphisms of X. Let M be a smooth manifold. An (X, G)-atlas for M is a family

$$\Phi = \{\varphi_i : U_i \to X,\ i \in I\}$$

where for each i the map φ_i is a diffeomorphism from an open set of M to an open set of X. Moreover, the family Φ is requested to satisfy:

- $\{U_i\}$ is an open covering of M.
- If $U_i \cap U_j \neq \emptyset$ then the map $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is the restriction of an element of G.

The U_i's are called local chart and the φ_{ji}'s changes of chart. Two atlantes are said to be equivalent if they have a common refinement. An (X, G)-structure on M is an equivalence class of (X, G)-atlantes.

Remark 1.2.1. The changes of chart satisfy the following *co-cycle* condition: whenever $U_i \cap U_j \cap U_k \neq \emptyset$ one has

$$\varphi_{ji} \circ \varphi_{ik} \circ \varphi_{kj} = \mathrm{Id}.$$

Definition 1.2.2. The pair (X, G) is said to be *rigid* if any two elements of G that coincide on an open set coincide everywhere.

If (X, G) is rigid, then every change of chart φ_{ij} determines a well-defined element of G that I still call φ_{ij}. Moreover, when (X, G) is rigid the name *co-cycle* of Remark 1.2.1 has the following interpretation. Let $N(\{U_i\})$ be the *nerve* of the covering $\{U_i\}$, *i.e.* the simplicial complex whose vertices are the elements of I and the simplex $(i_0, \cdots, i_n)$ exists if and only if $U_{i_0} \cap \cdots \cap U_{i_n} \neq \emptyset$. Then the set of changes of chart can be viewed as a G-valued 1-co-chain φ on $N(\{U_i\})$ by

$$\langle \varphi, (i, j) \rangle = \varphi_{ij}.$$

If (i, j, k) is a triangle, then, by abuse of the usual homological formalism, one gets

$$\langle d\varphi, (i, j, k) \rangle = \langle \varphi, \partial(i, j, k) \rangle = \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \mathrm{Id}$$

so one can say that the chain φ is a co-cycle (this point of view can be formalized in terms of Čech cohomology).

As examples, note that $(\mathbb{C}, \mathrm{Aut}(\mathbb{C}))$ and $(\mathbb{H}^3, \mathrm{Isom}(\mathbb{H}^3))$ are rigid because of the analyticity of the elements of $\mathrm{Aff}(\mathbb{C})$ and $\mathrm{Isom}^+(\mathbb{H}^3)$. Rigidity implies that one can think of local charts as a jigsaw puzzle-pieces and then try to glue them together. This leads to the notions of *developing map* and *holonomy*.

Let M be an n-manifold endowed with an (X, G)-structure. Choose an (X, G)-atlas such that the U_i's and their intersections are contractible and such that the nerve $N(\{U_i\})$ has dimension n. Such an atlas can be easily constructed by triangulating M and choosing suitable neighborhoods of the k-skeleta, $k = 0, \dots, n$.

Holonomy. A simplicial path in $N(\{U_i\})$ is a sequence of 1-simplices $\big((i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k)\big)$, or equivalently a sequence $(U_{i_0}, \dots, U_{i_k})$ with $U_{i_j} \cap U_{i_{j+1}} \neq \emptyset$. If γ is such a path, define

$$h(\gamma) = \varphi_{i_0 i_1} \circ \cdots \circ \varphi_{i_{k-1} i_k} \in G.$$

Let U_0 be a base-point of $N(\{U_i\})$ and let $P(M, U_0)$ be the semi-group of simplicial loops based at U_0, *i.e.* paths with $U_{i_0} = U_{i_k} = U_0$, endowed with the usual composition of paths. From the definition, it follows that the map h is a homomorphism from $P(M, U_0)$ to G.

Call *step-homotopy* one of the following moves between simplicial paths:

$$(U_{i_0}, \dots, U_{i_{m-1}}, U_{i_m}, U_{i_{m+1}}, \dots, U_{i_k}) \leftrightarrow (U_{i_0}, \dots, U_{i_{m-1}}, U_{i_{m+1}}, \dots, U_{i_k})$$

$$\text{If} \quad U_{i_{m-1}} \cap U_{i_m} \cap U_{i_{m+1}} \neq \emptyset$$

$$(U_{i_0}, \dots, U_{i_m}, U_{i_m}, \dots, U_{i_k}) \leftrightarrow (U_{i_0}, \dots, U_{i_m}, \dots, U_{i_k}).$$

Say that γ_1 is equivalent to γ_2 if it is obtained from γ_2 by a finite number of step-homotopies. Call $p_1(M, U_0)$ the set of equivalence classes of simplicial loops based at U_0. The composition of paths descends to $p_1(M, U_0)$, which becomes a group with such operation, and it is a standard fact that for any $x_0 \in U_0$ it is

$$p_1(M, U_0) \cong \pi_1(M, x_0).$$

The co-cycle condition of the changes of chart implies that the homomorphism h descends to a homomorphism

$$h : \pi_1(M, x_0) \to G$$

which is called *holonomy* of the (X, G)-structure. The holonomy depends on the chosen base-point, and, as usual, its conjugacy class is a well-defined set of representations

$$[h] : \pi_1(M) \to G.$$

For simplicity of notation, unless I specify a precise representative of the holonomy, I will write h to indicate both a generic element of the conjugacy class of the holonomy and the class itself. Note that if one changes the (X, G)-structure by composing each φ_i with an element $g \in G$, then the holonomy changes via the conjugation by g, so its conjugacy class does not change.

Developing map. Let $\pi : \widetilde{M} \to M$ be the universal cover of M. Then the (X, G)-atlas of M lifts to an (X, G)-atlas of $\widetilde{M}$

$$\Psi = \{\psi_j : V_j \to X, i \in J\}$$

such that if V_j is a lift of U_i then $\psi_j = \varphi_i \circ \pi$. Let $U_0 \in \{U_i\}$, let $x_0 \in U_0$ be a base-point in M, and let $\widetilde{x}_0 \in V_0$ with $\pi(V_0) = U_0$ and $\pi(\widetilde{x}_0) = x_0$. For any simplicial path γ in $N(\{V_j\})$ define $h(\gamma)$ as above. For any V_j fix a simplicial path γ_j from V_0 to V_j and define the map $D_j : V_j \to X$ by

$$D_j = h(\gamma_j) \circ \psi_j.$$

The fact that h is invariant under step-homotopies implies that the definition of D_j is independent from the chosen path γ_j. Moreover, it is easily checked that the D_j's glue together giving a well-defined map

$$D : \widetilde{M} \to X$$

which is called *developing map* of the (X, G)-structure.

Once the base-points x_0 and $\widetilde{x}_0$ have been fixed, the action of $\pi_1(M, x_0)$ on $\widetilde{M}$ by deck transformations is well-defined. From the constructions of the holonomy and of the developing map, it follows that D is h-equivariant for the actions of $\pi_1(M, x_0)$ on $\widetilde{M}$ by deck transformations and on X via the holonomy. More precisely, there exists a representative h of the holonomy such that

$$D(\alpha(x)) = h(\alpha)(D(x))$$

for every $x \in \widetilde{M}$ and $\alpha \in \pi_1(M, x_0)$.

Remark 1.2.3. A 3-manifold admits a hyperbolic structure if and only if it admits an $(\mathbb{H}^3, \mathrm{Isom}(\mathbb{H}^3))$-structure.

Definition 1.2.4. An (X, G)-structure on a manifold M is *complete* if the developing map is a homeomorphism of $\widetilde{M}$ onto X.

Remark 1.2.5. The metric completeness of a hyperbolic manifold is equivalent to the completeness of the correspondent $(\mathbb{H}^3, \mathrm{Isom}(\mathbb{H}^3))$-structure.

Remark 1.2.6. As an (X, G)-structure, a similarity structure on a surface corresponds to a $(\mathbb{C}, \mathrm{Aut}(\mathbb{C}))$-structure. If $E(\mathbb{C})$ is the subgroup of $\mathrm{Aut}(\mathbb{C})$ consisting of translations, a Euclidean structure on a surface corresponds to a $(\mathbb{C}, E(\mathbb{C}))$-structure. As an (X, G)-structure, a similarity structure is complete if and only if it is Euclidean, *i.e.* if the changes of chart belongs to $E(\mathbb{C})$.

Chapter 2
Triangulations and ideal triangulations: from (X, G)-structures to hyperbolic Dehn filling equations

In this chapter I extend the notion of (X, G)-atlas to the setting of triangulated objects. I focus on the cases of hyperbolic structures on cusped manifolds and similarity structures on surfaces, and I show how in these cases the existence and the properties of (X, G)-atlas translate to algebraic equations.

I start defining what I mean by triangulation and ideal triangulation. Then I extend the notions of Section 1.2 to (ideally) triangulated manifolds. The idea is to use the simplices of maximal dimension as local charts by defining on them a classical (X, G)-structure, and then to realize the face-pairing maps with elements of G so that they play the role of the changes of chart. Then I will describe a parametrization of the similarity structures on a triangle, and a parametrization of the complete, finite-volume, hyperbolic structures on an ideal tetrahedron. These parametrizations are strictly related, and in both cases the space of parameters (called moduli) will be $\mathbb{C} \setminus \{0, 1\}$. It will follow that, if τ is a triangulation of a manifold M, to chose a complex number for each simplex of maximal dimension of τ corresponds to chose a set of local charts. Then the question of whether a choice of a set of local charts actually is an (X, G)-atlas will translate to a system of algebraic equations on the moduli, called *compatibility equations*, such that a choice of moduli is a solution of the compatibility equations if and only if the corresponding set of local charts is an (X, G)-atlas for τ. I will also show how other geometric properties of an (X, G)-atlas, like completeness, translate to algebraic equations on the moduli.

The (X, G)-atlantes for triangulated objects are quite different from classical (X, G)-atlantes. Namely, in general an (X, G)-atlas for a triangulated manifold M does not define a classical (X, G)-structure on M. Because of this, I introduce here the notion of *geometric solution* of the above systems (see Definitions 2.4.28, 2.4.29, 2.5.5, 2.5.6, and 2.5.14). For example, a geometric solution of the system of compatibility equations will be a choice of moduli which is compatible with a classical (X, G)-structure on M. I will show here that any choice of moduli which is a geometric solution of a system of equations actually is also an algebraic solution of such equations.

At the end of the chapter I will discuss the so-called *equations on the angles*, and I will show how these equations are strictly related to topological aspects. These equations are related to the choices of moduli, but problems of continuity arise. Namely, one can define the angle of a complex number z as its argument $\arg(z) = \Im(\log z)$ and then write down the equations on the angles, but the imaginary part of the logarithm is a multi-valued function, and no continuous determination of $\Im(\log z)$ exists on $\mathbb{C} \setminus \{0\}$.

I refer the reader to [26] and [20] for original sources about the equations on the moduli.

2.1. Triangulations and ideal triangulations

I give here the definition of triangulation and ideal triangulation I will use in the sequel. See also [9], [21], [22], and [26] for details on ideal triangulations.

Let Δ^k denote the standard k-simplex, *i.e.* the convex hull of the standard basis $\{e_0, \ldots, e_k\}$ of $\mathbb{R}^{k+1}$ and let the *standard ideal k-simplex* be Δ^k with vertices removed.

Definition 2.1.1. (Pairing rule) Let Δ_1 and Δ_2 be two copies of Δ^k. A *pairing rule* is a bijective correspondence $r : \Delta_1 \dashrightarrow \Delta_2$ between the vertices of Δ_1 and those of Δ_2. A *realization* of r is a homeomorphism $f : \Delta_1 \to \Delta_2$ that extends r and preserves the stratifications by n-skeleta of Δ_1 and Δ_2.

Definition 2.1.2. (Triangulation) Let X be a topological space. Let $\{\Delta_i, i \in I\}$ be a set of copies of the standard k-simplex with I being a finite set of indices and let $\{r_j : F_{j1} \dashrightarrow F_{j2},\ j \in J\}$ be a set of pairing rules between $(k-1)$-dimensional faces of the Δ_i's, with $|J| = \frac{k+1}{2}|I|$ and $\cup_j\{F_{j1}, F_{j2}\} = \cup_i \partial\Delta_i$. Say that $\tau = (\{\Delta_i\}, \{r_j\})$ is a *triangulation* of X if there exists a set $\{f_j : F_{j1} \to F_{j2},\ j \in J\}$ of realizations of the rules r_j and a homeomorphism $\varphi : (\sqcup\Delta_i)/\{f_j\} \to X$. Say that $\mathcal{R} = (\{f_j\}, \varphi)$ is a *realization* of τ with gluing maps $\{f_j\}$. If X is an oriented k-manifold, I fix an orientation for Δ^k and require the r_j's to be orientation-reversing and φ to be orientation-preserving. Given a realization of τ, for each $i \in I$ the map φ_{Δ_i} is defined as the composition of φ with the inclusion $\Delta_i \to (\sqcup\Delta_i)/\{f_j\}$.

I remark that this definition of triangulation allows multiple and self-adjacencies, so cases as in Figure 2.1 (and more) possibly appear. Nevertheless, no other pathologies occur (see Remark 2.1.9).

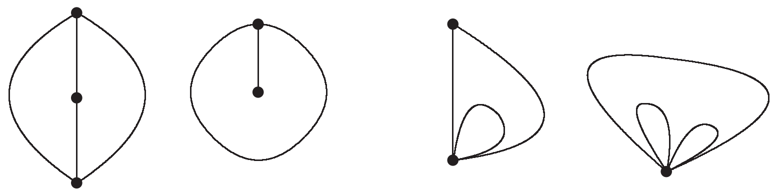


Figure 2.1. Self-adjacencies and multiple adjacencies.

I introduce now the class of manifolds I am principally interested in.

Definition 2.1.3. (Cusped manifolds) A *cusped manifold* M is a smooth manifold which is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary. A *cusp* of M is a closed regular neighborhood of a component of $\partial\overline{M}$. In the following I require M to be orientable and have dimension 3, and I require $\partial\overline{M}$ to be a union of tori. Therefore, each cusp of M is diffeomorphic to $T^2 \times [0, \infty]$, where $T^2 \times \{\infty\}$ belongs to $\partial\overline{M}$.

Define $\widehat{M}$ to be the compactification of M obtained by collapsing each component of $\partial\overline{M}$ to a point (distinct points for distinct components). If $\widetilde{\overline{M}}$ is the universal cover of $\overline{M}$, call $\widehat{\widetilde{M}}$ the space obtained by collapsing each lift of each component of $\partial\overline{M}$ to a point. The points of $\widehat{\widetilde{M}}$ corresponding to the components of $\partial\overline{M}$ are called *ideal points*.

Remark 2.1.4. Note that in general for a cusped manifold there is $\widehat{\widetilde{M}} \neq \widetilde{\widehat{M}}$. In fact, if M is for example the complement of a knot in S^3, then one can easily see that $\pi_1(\widehat{M}) = 1$, so $\widetilde{\widehat{M}} = \widehat{M} \neq \widehat{\widetilde{M}}$.

If M is a cusped manifold, each ideal point p has a neighborhood homeomorphic to the cone obtained from $T^2 \times [0, \infty]$ by collapsing $T^2 \times \{\infty\}$ to p. In particular, $\widehat{M}$ is not a manifold because it is singular at its ideal points.

Remark 2.1.5. In the sequel, I will often identify M with its image under the inclusion $M \hookrightarrow \overline{M}$ and the projection $M \hookrightarrow \overline{M} \to \widehat{M}$. I will consider a cusp of M as a subset of either M or $\overline{M}$ or $\widehat{M}$, without specifying if there are no ambiguities, so a cusp will be diffeomorphic either to $T^2 \times [0, \infty)$ or to $T^2 \times [0, \infty]$ or to the cone obtained from $T^2 \times [0, \infty]$ by collapsing $T^2 \times \{\infty\}$ to a point.

Definition 2.1.6. (Ideal triangulation 1) Let M be a cusped manifold. An *ideal triangulation* of M is a triangulation of $\widehat{M}$ whose 0-skeleton is the set of ideal points.

The above definition is equivalent to the following one.

Definition 2.1.7. (Ideal triangulation 2) Let M be a cusped manifold. Let $\{\Delta_i,\ i \in I\}$ be a finite set of copies of the standard tetrahedron and let $\{r_j : F_{j1} \dashrightarrow F_{j2},\ j \in J\}$ be a set of pairing rules between 2-dimensional faces of the Δ_i's. Say that $\tau = (\{\Delta_i\}, \{r_j\})$ is an *ideal triangulation* of M if there exists a set $\{f_j : F_{j1} \to F_{j2},\ j \in J\}$ of realizations of the rules r_j and a homeomorphism $\varphi : (\sqcup\Delta_i^*)/\{f_j\} \to M$, where Δ_i^* is Δ_i with vertices removed. Say that $\mathcal{R} = (\{f_j\}, \varphi)$ is a *realization* of τ with gluing maps $\{f_j\}$. Fix an orientation for Δ^3 and require the r_j's to be orientation-reversing and φ to be orientation-preserving. Given a realization of τ, for each $i \in I$, set φ_{Δ_i} to be the composition of φ with the inclusion $\Delta_i \to (\sqcup\Delta_i)/\{f_j\}$.

Remark 2.1.8. When speaking of an (ideal) triangulation of a (cusped) manifold M, if there are no ambiguities, I often assume that a realization has been fixed and I do not distinguish between Δ_i and its image under the map φ_{Δ_i}.

Remark 2.1.9. If τ is an (ideal) triangulation of a (cusped) manifold, then the projection $\sqcup\Delta_i \to (\sqcup\Delta_i)/\{f_j\}$ is injective when restricted to the interior of any simplex.

Proposition 2.1.10. *Any cusped manifold can be ideally triangulated.*

This is a standard fact of the theory of 3-manifolds, and depends on the fact that ideal triangulations are dual to standard spines, and any manifold has a standard spine. See for example [1] or [18] for details.

Remark 2.1.11. Let $\tau = (\{\Delta_i\},\{r_j\})$ be an ideal triangulation of a cusped manifold M and let $\mathcal{R} = (\{f_j\}, \varphi)$ be a realization of τ. It is possible to truncate each Δ_i by chopping off an open regular neighborhood U_i of its vertices in such a way that, if Δ_i^- denotes the truncated tetrahedron $\Delta_i \setminus U_i$ and $\partial^-\Delta_i = \partial U_i \cap \Delta_i^-$, then the f_j's match the faces of $\partial\Delta_i^- \setminus \partial^-\Delta_i$ and $\varphi\Big((\sqcup_i \Delta_i^-)/\{f_j\}, (\sqcup_i \partial^-\Delta_i)/\{f_j\}\Big)$ is homeomorphic to $(\overline{M}, \partial\overline{M})$. In other words, any ideal triangulation of M induces a triangulation of $\partial\overline{M}$ with the triangles of $\cup_i \partial^-\Delta_i$.

2.2. (X, G)-atlantes, developing maps and holonomies for triangulations

For this section I fix the following notations: X will be a connected, simply connected, smooth n-manifold and G a subgroup of diffeomorphisms of X; M will be either a closed n-manifold or a cusped manifold (with $n = 3$) and $\tau = (\{\Delta_i\}, \{r_j\})$ will be respectively either a triangulation or an ideal triangulation of M.

I start giving a notion of rigidity for the pair (X, G), which is the analogue for a triangulated setting of that given in Definition 1.2.2.

Definition 2.2.1. Let S be a set of singular k-simplices of X. The pair (X, G) is *rigid w.r.t.* S if for any two simplices $\sigma_1, \ \sigma_2 : \Delta^k \to X$ of S, and for any pairing rule $r : \Delta^k \dashrightarrow \Delta^k$, there exists one and only one element ϕ of G such that

$$\sigma_1^{-1} \circ \phi \circ \sigma_2$$

is a realization of r.

Example 2.2.2. The pair $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$ is rigid w.r.t. the set of affine 1-simplices

$$S = \{\sigma : [0, 1] \to \mathbb{C} : \ \sigma(t) = t\sigma(1) + (1 - t)\sigma(0), \ \sigma(0) \neq \sigma(1)\}.$$

The pair $(\overline{\mathbb{H}}^3, \mathrm{Isom}^+(\mathbb{H}^3))$ is rigid w.r.t. the set of ideal triangles

$$S = \{\sigma : \Delta^2 \to \overline{\mathbb{H}}^3 : \ \sigma \text{ is a homeomorphism, that preserves the stratification by } i\text{-skeleta, of } \Delta^2 \text{ onto a straight ideal triangle}\}.$$

I notice that the unique (X, G)-structures I will use in the sequel are $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-structures for surfaces and $(\overline{\mathbb{H}}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-structures for cusped manifolds.

Before giving the definition of (X, G)-atlas, I introduce the language for expressing the co-cycle conditions for triangulations. Define $\Gamma(\tau)$ to be the 2-skeleton of the dual cellularization of τ. Using a barycentric subdivision, it is readily checked that $\Gamma(\tau)$ embeds into M and that such embedding induces an isomorphism of the fundamental groups.

Remark 2.2.3. If τ is an ideal triangulation of M, then $\pi_1(\Gamma(\tau))$ is isomorphic to $\pi_1(M)$, which is in general different from $\pi_1(\widehat{M})$.

In the dual cellularization, the vertices correspond to the n-simplices of τ and the edges to the pairing rules. Therefore, a simplicial path in $\Gamma(\tau)$ corresponds to a sequence

$$\Delta_{i_0} \overset{r_{j_1}}{\to} \Delta_{i_1} \overset{r_{j_2}}{\to} \cdots \overset{r_{j_k}}{\to} \Delta_{i_k}$$

where the Δ_{i_m}'s are simplices of τ and each r_{j_m} is either a pairing rule of τ or its inverse between $(n-1)$-dimensional faces of $\Delta_{i_{m-1}}$ and Δ_{i_m}.

Definition 2.2.4. ((X, G)-atlas) Let S be a set of singular simplices of X and suppose that (X, G) is rigid w.r.t. S. An (X, G)-atlas relative to S for τ is a set Φ of maps, called local charts

$$\Phi = \{\varphi_i : \Delta_i \to X\}$$

such that:

- Any restriction of φ_i to an $(n-1)$-dimensional face of Δ_i belongs to S.
- If ϕ_j is the only element of G associated to the rule r_j by rigidity, and if f_j is the corresponding realization of r_j, there exists a realization of τ with gluing maps $\{f_j\}$. I call *changes of chart* the maps ϕ_j.
- The set $\{\phi_j\}$ of changes of chart, viewed as a G-valued 1-co-cycle on $\Gamma(\tau)$, satisfies the co-cycle condition "$d\phi = \mathrm{Id}$", that is, whenever $\Delta_{i_0} \overset{r_{j_1}}{\to} \Delta_{i_1} \overset{r_{j_2}}{\to} \cdots \overset{r_{j_k}}{\to} \Delta_{i_k}$ is the boundary of a 2-cell of $\Gamma(\tau)$, then
$$\phi_{j_1} \circ \cdots \circ \phi_{j_k} = \mathrm{Id}.$$

I extend now the notions of holonomy and developing map to an (X, G)-atlas relative to S. I use the notation of Definition 2.2.4. For any simplicial path γ in $\Gamma(\tau)$

$$\gamma = \Delta_{i_0} \overset{r_{j_1}}{\to} \cdots \overset{r_{j_k}}{\to} \Delta_{i_k}$$

define $h(\gamma)$ as

$$h(\gamma) = \phi_{j_1} \circ \cdots \circ \phi_{j_k} \in G.$$

Let Δ_{i_0} be a base-tetrahedron of τ and let $P(\Gamma(\tau), \Delta_{i_0})$ be the semi-group of simplicial loops based at Δ_{i_0} in $\Gamma(\tau)$, equipped with the usual composition of paths. The map h actually is a homomorphism from $P(\Gamma(\tau), \Delta_{i_0})$ to G. Call simplicial step-homotopy one of the following moves between simplicial paths:

- $\gamma_1 \circ \alpha \circ \gamma_2 \leftrightarrow \gamma_1 \circ \beta \circ \gamma_2$ if $\alpha \circ \beta^{-1}$ is the boundary of a 2-cell of $\Gamma(\tau)$.
- $$\Delta_{i_0} \overset{r_{j_1}}{\to} \cdots \overset{r_{j_{m-1}}}{\to} \Delta_{i_{m-1}} \overset{r_{j_m}}{\to} \Delta_{i_m} \overset{r_{j_{m+1}}}{\to} \Delta_{i_{m+1}} \overset{r_{j_{m+2}}}{\to} \cdots \overset{r_{j_k}}{\to} \Delta_{i_k}$$
$$\updownarrow$$
$$\Delta_{i_0} \overset{r_{j_1}}{\to} \cdots \overset{r_{j_{m-1}}}{\to} \Delta_{i_{m-1}} \overset{r_{j_{m+2}}}{\to} \cdots \overset{r_{j_k}}{\to} \Delta_{i_k}$$
if $\Delta_{i_{m-1}} = \Delta_{i_{m+1}}$ and $r_{j_{m+1}} = r_{j_m}^{-1}$, and *vice versa*.

Say that two paths γ_1 and γ_2 are equivalent if γ_2 is obtained from γ_1 by performing a finite number of simplicial step-homotopies. Call $p_1(\Gamma(\tau), \Delta_{i_0})$

the group of equivalence classes of loops based at Δ_{i_0}. As in the classical case, the co-cycle condition implies that the homomorphism h descends to a homomorphism

$$h : p_1(\Gamma(\tau), \Delta_{i_0}) \to G$$

and one can check that $p_1(\Gamma(\tau), \Delta_{i_0}) \cong \pi_1(\Gamma(\tau), x_0) \cong \pi_1(M, x_0)$, when $x_0 \in \Delta_{i_0}$. As usual, forgetting the base-points, the holonomy is a well-defined conjugacy class of representations

$$h : \pi_1(M) \to G.$$

Once one has a holonomy representation, the construction of a developing map is exactly as in the classical case (see Section 1.2), and as in the classical case a developing map is an h-equivariant map $D : \widetilde{M} \to X$.

Remark 2.2.5. In the case of cusped manifolds, a developing map is defined on $\widetilde{M}$ and not on $\widetilde{\widehat{M}}$. This is because the holonomy is defined on $\pi_1(M)$ and not on $\pi_1(\widehat{M})$. Moreover a developing map $D : \widetilde{M} \to X$ extends to $\widetilde{\widehat{M}}$. In the sequel, if there are no ambiguities, I do not distinguish between the map $D : \widetilde{M} \to X$ and its extension $D : \widetilde{\widehat{M}} \to X$.

I collect these facts in the following statement.

Theorem 2.2.6. *Let X be a connected, simply connected, smooth n-manifold and let G be a subgroup of the group of diffeomorphisms of X. Let M be either a smooth n-manifold or a cusped manifold ($n = 3$) and let $\tau = (\{\Delta_i\}, \{r_j\})$ be respectively either a triangulation or an ideal triangulation of M. Let S be set of singular simplices of X and suppose that (X, G) is rigid w.r.t. S. Call $\pi : \widetilde{M} \to M$ the universal covering and $\widetilde{\tau}$ the lift of τ to $\widetilde{M}$.*

If $\Phi = \{\varphi_i : \Delta_i \to X\}$ is an (X, G)-atlas relative to S for τ, then there exists a holonomy representation $h : \pi_1(M) \to G$ and a developing map $D : \widetilde{M} \to X$ such that for any lift $\widetilde{\Delta}_i$ of any Δ_i there exists $g \in G$ such that for every $x \in \widetilde{\Delta}_i$

$$D(x) = g(\varphi_i(\pi(x))).$$

Moreover, the map D is h-equivariant with respect to the actions of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations and on X via h, that is for every $x \in \widetilde{M}$ and $\alpha \in \pi_1(M)$

$$D(\alpha(x)) = h(\alpha)(D(x)).$$

Remark 2.2.7. In the classical case, if a manifold has an (X, G)-atlas, then it is locally modeled on X. In the present setting, two types of pathologies can occur. First of all, a local chart is not required to be a homeomorphism in the interior of a simplex. Moreover, even if the local charts are homeomorphisms, since the changes of chart involve only closed, codimension-one faces of the simplices of maximal dimension, in general one looses the property that a developing map is a local homeomorphism along these faces.

Remark 2.2.8. In the sequel, when speaking of (X, G)-atlas for triangulated manifolds with $(X, G)=(\mathbb{C},\mathrm{Aff}(\mathbb{C}))$ or $(X, G)=(\overline{\mathbb{H}}^3,\mathrm{Isom}^+(\mathbb{H}^3))$, I tacitly assume that they are relative to the sets S described in Example 2.2.2.

2.3. Moduli for triangles and ideal tetrahedra

From Theorem 1.1.5 and Remark 2.1.11 it follows that hyperbolic structures on 3-manifolds and similarity structures on surfaces are strictly related to each other. In this section I show how one can use complex numbers to parametrize both similarity structures on a triangle and finite-volume hyperbolic structures on an ideal tetrahedron. See also [26] [1], and [24].

2.3.1. Modulus of a Euclidean triangle

Let A be a straight fat (with non-aligned vertices) triangle of $\mathbb{C}$. The orientations of A as a subset of $\mathbb{C}$ are in correspondence with the orientations of A as an abstract triangle, that is cyclic orderings of its vertices. Let (v_0, v_1, v_2) be a fixed orientation of A. There exists a unique element ψ of $\mathrm{Aff}(\mathbb{C})$ such that $\psi(v_0) = 0$ and $\psi(v_1) = 1$. It follows that $z = \psi(v_2)$ is a well-defined complex number different from 0, 1. Moreover, if the chosen orientation is that induced by the positive orientation of $\mathbb{C}$, then $\Im(z) > 0$ and *vice versa*. The number z is called *modulus* of A relative to v_0. Similarly, without changing the orientation of A, define the moduli z_1 and z_2 relative to v_1 and v_2. It is easily checked that for $i \in \mathbb{Z}/3\mathbb{Z}$

$$z_{i+1} = \frac{1}{1-z_i} \qquad\qquad z_{i+2} = 1 - \frac{1}{z_i}. \tag{2.1}$$

It follows that the similarity class of an oriented triangle is completely determined by a triple of complex numbers different from 0, 1 of the form

$$\left\{z, \frac{1}{1-z}, 1-\frac{1}{z}\right\}.$$

This 3-to-1 ambiguity can be avoided be choosing a preferred vertex of A. In the following, when speaking of a modulus of a triangle, I tacitly assume that an orientation and a preferred vertex have been fixed.

The notion of modulus extends to flat (non-degenerate) triangles, *i.e.* those whose vertices are three distinct aligned points. Clearly, an abstract orientation of a flat triangle A can not correspond to an orientation of A as a subset of $\mathbb{C}$. Nevertheless, the above definition of modulus equally works, and it turns out that the modulus of a flat triangle is a real number different from 0, 1.

Unfortunately, this definition does not work for degenerate triangles, *i.e.* those having two or more coincident vertices. Actually, one could consider moduli in $\{0, 1, \infty, *\}$, with the convention that a triangle with $v_0 = v_1 = v_2$ has modulus $*$, one with $v_0 = v_2 \neq v_1$ has modulus 0, and

$$\frac{1}{1-1}=\infty=1-\frac{1}{0} \qquad 0=\frac{1}{1-\infty} \qquad 1=1-\frac{1}{\infty} \qquad *=\frac{1}{1-*}=1-\frac{1}{*}$$

but this definition is not useful for the purpose of this work (see Sections 2.3.2 and 2.4 below).

Let $\pi_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ and $\pi_- = \{z \in \mathbb{C} : \Im(z) < 0\}$. The above constructions give the following:

Proposition 2.3.1. *Let $A \subset \mathbb{C}$ be an (abstractly) oriented straight triangle, which is possibly flat but not degenerate. Once a vertex of A has been fixed, the* $\mathrm{Aff}(\mathbb{C})$*-class of A is completely determined by a complex number $z \in \mathbb{C} \setminus \{0, 1\}$, called* modulus. *Moreover $z \in \pi_+$ if and only if A is positively oriented, $z \in \pi_-$ if and only if A is negatively oriented, and $z \in \mathbb{R} \setminus \{0, 1\}$ if and only if A is flat. The modulus z is called respectively* positive, negative, *and* flat.

Now let me spend a few lines on the topic of the argument of a modulus. Let $A \subset \mathbb{C}$ be a positively oriented straight triangle. For each vertex v of A, the argument $\arg(z)$ of the modulus z relative to v is well-defined as the inner angle at v. Clearly, $\arg(z)$ is the imaginary part of the determination of $\log(z)$ with $|\Im(\log(z))| < \pi$. Moreover, since A is a Euclidean triangle, the sum of its inner angles is π. If A is a flat triangle, then the arguments of the moduli can be defined setting $\arg(z) = \pi$ if $z < 0$ and $\arg(z) = 0$ otherwise. This definition of $\arg(z)$ is continuous on $\pi_+ \cup (\mathbb{R} \setminus \{0, 1\})$, and has the property that the sum of the arguments of the moduli of a triangle is always π.

Remark 2.3.2. One can define the argument also for negatively oriented triangles, but the argument can not depend continuously on the moduli in $\mathbb{C} \setminus \{0, 1\}$. This is because no determination of $\log(z)$ is continuous on $\mathbb{C} \setminus \{0\}$.

2.3.2. Modulus of a hyperbolic ideal tetrahedron

Let $A \subset \overline{\mathbb{H}}^3$ be a straight ideal tetrahedron and suppose that A is fat, that is, the vertices of A are four distinct ideal points in $\partial\mathbb{H}^3$ whose convex hull is not contained in a hyperbolic 2-plane. The orientations of A as a subset of $\mathbb{H}^3$ are in correspondence with its orientations as an abstract tetrahedron, *i.e.* the orderings of the vertices of A up to even permutations. I work now in the half-space model of $\mathbb{H}^3$, so $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Let (v_0, v_1, v_2, v_3) be a fixed orientation of A. By Remark 1.1.4 there exists a unique isometry $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $\psi(v_0) = 0$, $\psi(v_1) = 1$, $\psi(v_2) = \infty$. It follows that $z = \psi(v_3)$ is a well-defined complex number different from $0, 1$. Moreover, if the chosen orientation is the positive one, then $\Im(z) > 0$ and *vice versa*. Note that z is exactly the complex cross-ratio

$$[v_0 : v_1 : v_2 : v_3] = \frac{v_3 - v_0}{v_3 - v_2} \cdot \frac{v_1 - v_2}{v_1 - v_0}$$

of the vertices of A. It follows that if the ordering of the vertices varies on the same orientation class, then z varies on the set

$$\left\{ z, \frac{1}{1-z}, 1 - \frac{1}{z} \right\}.$$

This ambiguity can be avoided fixing a preferred edge e of A, and arranging the vertices (v_0, v_1, v_2, v_3) in such a way that e joins v_0 and v_2. The number z is called *modulus* of A relative to e. The property $[v_0 : v_1 : v_2 : v_3] = [v_2 : v_3 : v_0 : v_1]$ of the cross-ratio implies that z depends only on e and not on its orientation. Moreover $[v_0 : v_1 : v_2 : v_3] = [v_1 : v_0 : v_3 : v_2]$ implies that the same modulus is associated to opposite edges. In the following, when speaking of a modulus of a tetrahedron, I tacitly assume that an orientation and a pair of opposite edges have been fixed.

If (v_0, v_1, v_2, v_3) is the chosen ordering and e joins v_0 and v_2, the modulus relative to e is exactly the modulus relative to v_0 of the straight triangle of $\mathbb{C}$ with vertices in v_0, v_1, v_3 (suppose $v_3 \neq \infty$). By slicing A with a sufficiently high horosphere centered at v_2, one gets a Euclidean triangle E, and the edge e intersects E in a vertex v. By mapping (v_0, v_1, v_2) to $(0, 1, \infty)$ via an element of $\mathrm{Isom}^+(\mathbb{H}^3)$, one sees that z is exactly the modulus of E relative to v. Note that the choice of a preferred pair of opposite edges of A induces by intersection a choice of a preferred vertex for any horospherical triangle of A. This implies that all the horospherical triangles of A have the same modulus, hence the same similarity structure. Conversely, the hyperbolic structure of A is completely determined by the similarity structure on any of its horospherical triangles.

As above, the notion of modulus extends to flat, non-degenerate ideal tetrahedra, *i.e.* to those whose vertices are four distinct points of $\partial\mathbb{H}^3$ belonging to the same hyperbolic 2-plane. As above, the modulus of a flat tetrahedron lies in $\mathbb{R} \setminus \{0, 1\}$. When A is a degenerate tetrahedron there is no natural way to associate a modulus to A in such a way that the relations between A and its horospherical triangles hold.

I collect these facts in the following

Proposition 2.3.3. *Let $A \subset \overline{\mathbb{H}}^3$ be an (abstractly) oriented straight ideal tetrahedron, which can be flat but not degenerate. Once a preferred pair of opposite edges has been fixed, the* $\mathrm{Isom}^+(\mathbb{H}^3)$*-class of A is completely determined by a complex number $z \in \mathbb{C}\setminus\{0, 1\}$, called modulus, such that $z \in \pi_+, \pi_-, \mathbb{R}\setminus\{0, 1\}$ if A is respectively positively, negatively oriented or flat. Moreover, the modulus of A is the same modulus of all the Euclidean triangles obtained as horospherical sections near the vertices of A.*

2.4. Compatibility equations on the moduli

In this section I describe how to use the moduli introduced in Section 2.3 to define $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$- and $(\overline{\mathbb{H}}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-atlantes on triangulated objects. The idea is to construct an atlas whose local charts are compatible with a prefixed choice of moduli. The main point is that the co-cycle conditions on the changes of chart translate to algebraic equations on the moduli, called *compatibility equations*. See [26] and [20] for details.

First of all, I fix the kind of maps I will use as local charts.

Definition 2.4.1. (Straight map) A map $\varphi : \Delta^k \to \mathbb{C}$ is said to be *straight* if it is simplicial.

Definition 2.4.2. (Straight map) A map $\varphi : \Delta^k \to \overline{\mathbb{H}}^3$ is said to be *straight* if:

1. For each subsimplex σ of Δ^k, $\varphi(\sigma)$ is contained in the hyperbolic convex hull of $\varphi(\partial\sigma)$.
2. If Q is the Euclidean convex hull of the φ-image of the 0-skeleton of Δ^k, made in a projective model of $\mathbb{H}^3$, and if $\psi : \Delta^k \to Q$ is the only simplicial map that agrees with φ on the 0-skeleton, then there exist two homeomorphisms $\eta : \mathrm{Im}(\varphi) \to Q$ and $\beta : \Delta^k \to \Delta^k$ that fix the 0-skeleta, and such that

$$\eta \circ \varphi \circ \beta = \psi.$$

Remark 2.4.3. For a map $\varphi : \Delta^k \to \overline{\mathbb{H}}^3$ to be straight does not depend on the model used to define Q. In other words, φ is straight if and only if $\gamma\varphi$ is straight for every isometry γ.

As noticed above, I am mainly interested in the cases of similarity structures on surfaces and hyperbolic structures on cusped manifold. Most of the following definitions and facts are similar for the two and three dimensional setting, and in many cases it is possible to pass from a statement in dimension two to the corresponding three-dimensional one, simply by replacing the word "triangle" by "tetrahedron". For this reason, I will deal at the same time with both cases.

Definition 2.4.4. [Map compatible with z] Let $z \in \mathbb{C} \setminus \{0, 1\}$. Let Δ be either Δ^2 or Δ^3, and let X be respectively $\mathbb{C}$ or $\overline{\mathbb{H}}^3$. A map $\varphi : \Delta \to X$ is said to be *compatible* with z if it is straight and its image is a straight triangle (resp. a straight ideal tetrahedron) of modulus z.

Notation. For the rest of this section and this chapter, M will be either a surface or a cusped manifold, $\tau = (\{\Delta_i\}, \{r_j\})$ will be respectively a triangulation or an ideal triangulation of M, and (X, G) will be respectively $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$ or $(\overline{\mathbb{H}}^3, \mathrm{Isom}^+(\mathbb{H}^3))$, S being the set of segments (resp. ideal triangles) as in Example 2.2.2 and Remark 2.2.8. For any Δ_i I fix an orientation and a choice of a preferred vertex (resp. pair of opposite edges). I require the orientations to be compatibles with a global orientation of M. Finally, I fix a choice of moduli $\mathbf{z}$ for τ, that is, a choice of a complex number $z_i \in \mathbb{C} \setminus \{0, 1\}$ for any Δ_i. The modulus z_i is referred to the preferred vertex (edges), and changing preferred vertex (edges) it changes according to relation (2.1) of Section 2.3.

Lemma 2.4.5. *Let $\Phi = \{\varphi_i : \Delta_i \to X\}$ be a set of maps, each φ_i compatible with z_i. The restriction of every φ_i to any face of Δ_i of codimension one belongs to S.*

Proof. This is because each φ_i is a straight map. $\square$

Definition 2.4.6. (Changes of chart) Let $\Phi = \{\varphi_i : \Delta_i \to X\}$ be a set of maps compatible with the moduli. By Lemma 2.4.5, for any j the map $\phi_j \in G$ is well-defined as the only element of G realizing r_j by rigidity of (X, G) w.r.t. S.

Since the set Φ is a candidate for being an (X, G)-atlas for τ, the maps ϕ_j's are the candidates for being the changes of chart. Lemma 2.4.5 tells that Φ satisfies the first condition of an (X, G)-atlas. I describe now how to express the co-cycle condition in terms of the moduli. See [26] and [20] for a complete discussion on the compatibility equations.

Proposition 2.4.7. (Compatibility equations) *Let $\Phi = \{\varphi_i : \Delta_i \to X\}$ be a set of maps compatible with* $\mathbf{z}$*. The co-cycle condition on the ϕ_j's is* *equivalent to require the moduli to satisfy a finite system $\mathcal{C}$ of algebraic equations, called* compatibility equations*, each one of the form*

$$\pm \prod z_i^{\alpha_i} (1 - z_i)^{\beta_i} = 1$$

where the same z_i possibly appears several times and α_i and β_i are in $\{0, 1, -1\}$*, depending only on the combinatorial data of τ and the choice of the preferred vertices (or edges).*

Proof. Let E be a two-cell of $\Gamma(\tau)$ and let e be its dual simplex (e is a vertex if M is a surface and it is an edge if M is a cusped manifold). Let $\gamma = (\Delta_{i_0} \stackrel{r_{j_1}}{\to} \cdots \stackrel{f_{j_k}}{\to} \Delta_{j_k})$ be the boundary of E viewed as a path of simplices. The Δ_{i_n}'s are exactly the simplices of τ containing e. Note that the same simplex can occur more than once in γ if self-adjacencies occur in τ. Put e in X in such a way that $e = 0$ if $X = \mathbb{C}$ and e is the vertical line $\overline{0\infty}$ in the half-space model if $X = \overline{\mathbb{H}}^3$. Arranging the simplices Δ_{i_n}'s around e using maps that are compatible with the moduli, one sees that the co-cycle condition

$$\phi_{j_1} \circ \cdots \circ \phi_{j_k} = \mathrm{Id}$$

holds if and only if the product of moduli of the Δ_{i_n}'s relative to e is 1 (see Figure 2.2).

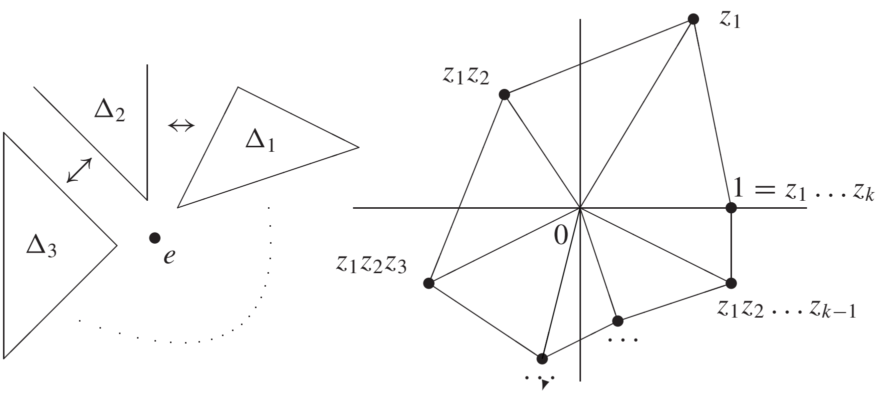


Figure 2.2. The triangulation near e and its arrangement in $\mathbb{C}$.

Thus the co-cycle condition is equivalent to require that the product of moduli around each edge e is 1. Since e may be not the preferred vertex (or edge) chosen at the beginning, the modulus of Δ_{i_n} relative to e lies in the set $\{z_{i_n}, (1 - z_{i_n})^{-1}, 1 - 1/z_{i_n}\}$. It follows that the equations have the claimed form. □

Remark 2.4.8. By Propositions 2.3.1 and 2.3.3 and Remark 2.1.11, If M is a cusped manifold, then a choice of moduli for τ induces a choice of moduli for the triangulation induced by τ on $\partial\overline{M}$. Moreover, the two system of compatibility equations for M and $\partial\overline{M}$ coincide.

Remark 2.4.9. Note that the equations $\mathcal{C}$ are equations on the moduli and do not involve the set Φ of local charts.

Suppose now that $\mathbf{z}$ is a solution of $\mathcal{C}$, and let $\Phi = \{\varphi_i : \Delta_i \to X\}$ be a set of maps compatible with $\mathbf{z}$. In order for Φ to be an (X, G)-atlas, it must be checked that a realization of τ exists which is compatible with the moduli. In the case that M is a surfaces, the simplicial realization works. In dimension three the following proposition holds.

Proposition 2.4.10. *Suppose that M is a cusped manifold and suppose that $\mathbf{z}$ is a solution of $\mathcal{C}$. For any realization $\mathcal{R} = (\{f_j\}, \varphi)$ of τ there exists a set of maps $\Phi = \{\varphi_i : \Delta_i \to X\}$, each one compatible with z_i, such that if r_j is a pairing rule between faces say of Δ_1 and Δ_2, then*

$$f_j = \varphi_2^{-1} \circ \phi_j \circ \varphi_1. \tag{2.1}$$

Proof. I define the φ_i's recursively on the $n-$skeleta of τ. On the 0-skeleton define the maps simply looking at the compatibility with the moduli. Then a set of changes of chart $\{\phi_j\}$ is well-defined. Let e be an edge of a tetrahedron Δ_{i_0} with vertices e_0 and e_1. Define φ_{i_0} on e to be a homeomorphism onto the geodesic between $\varphi_{i_0}(e_0)$ and $\varphi_{i_0}(e_1)$. Now define the φ_i's on the edges glued to e by the maps f_j using formula (2.1). Note that since $\mathcal{C}$ holds this is an unambiguous definition. Define the φ_i's on the other edges in a similar way. Once the φ_i's are defined on the 1-skeleton there are no problems to use again formula (2.1) to define them on the 2-skeleton and there are no obstructions to extend such maps to the 3-cells. □

Remark 2.4.11. From now on when speaking of an ideal triangulation of M with moduli, I suppose that a realization $\mathcal{R}$ has been fixed and that each set Φ of maps compatible with the moduli is also compatible with $\mathcal{R}$, that is, condition (2.1) holds.

Lemma 2.4.5 and Propositions 2.4.7 and 2.4.10 give the following

Theorem 2.4.12. *A choice of moduli $\mathbf{z}$ is a solution of the system $\mathcal{C}$ of the compatibility equations if and only if there exists an (X, G)-atlas $\Phi = \{\varphi_i : \Delta_i \to X\}$ for τ in which each map φ_i is compatible with the modulus z_i.*

Theorem 2.4.12 in particular implies that if $\mathbf{z}$ is a solution of $\mathcal{C}$, then a developing map and the holonomy are well-defined. A developing map clearly depends on the single local charts, while, as the following proposition shows, the holonomy depends only on the moduli.

Proposition 2.4.13. *Suppose that $\mathbf{z}$ is a solution of $\mathcal{C}$. Let $\Phi = \{\varphi_i : \Delta_i \to X\}$ and $\Phi' = \{\varphi_i' : \Delta_i \to X\}$ be two (X, G)-atlantes whose local charts are compatible with the moduli. If h and h' are the holonomies of Φ and Φ', then*

$$h = h'/\text{conjugation}.$$

Proof. By rigidity of (X, G), for every i there exists a unique element $\theta_i \in G$ such that the restriction of $\varphi_i^{-1} \circ \theta_i \circ \varphi_i'$ to $\partial \Delta_i$ realizes the pairing rules induced by the identity. It follows that if r_j is a pairing rule of τ, say between faces of Δ_1 and Δ_2, then the changes of chart ϕ_j and ϕ_j' satisfy $\phi_j' = \theta_2^{-1} \circ \phi_j \circ \theta_1$ (see Figure 2.3).

$$\begin{tikzcd}
\varphi_1(\Delta_1) \arrow[r, "\phi_j"] & \varphi_2(\Delta_2) \\
\varphi_1'(\Delta_1) \arrow[u, "\theta_1"] \arrow[r, "\phi_j'"] & \varphi_2'(\Delta_2) \arrow[u, "\theta_2"']
\end{tikzcd}$$

Figure 2.3. The relation between ϕ_j, ϕ_j', θ_1 and θ_2.

The claim follows from the definition of the holonomy via loops of simplices (see Section 2.2). □

Proposition 2.4.13 allows to give the following

Definition 2.4.14. (Holonomy and developing map for $\mathbf{z}$) Let $\mathbf{z}$ be a choice of moduli that satisfies $\mathcal{C}$. The *holonomy* $h(\mathbf{z})$ of $\mathbf{z}$ is the holonomy of any (X, G)-atlas whose local charts are compatible with $\mathbf{z}$. A map $D : \widetilde{M} \to X$ is called *developing map* for $\mathbf{z}$ if there exists an (X, G)-atlas Φ whose local charts are compatible with $\mathbf{z}$ and such that D is a developing map for Φ.

Remark 2.4.15. As above, the holonomy is well-defined as a conjugacy class of representations, and for any developing map D there exists a representative h of the holonomy such that D is h-equivariant (see Theorem 2.2.6). I recall that, as noticed in Remark 2.2.5, if M is a cusped manifold, a developing map can be viewed either as a map defined on $\widehat{\widetilde{M}}$ or as a map defined on $\widetilde{M}$ that extends to $\widehat{\widetilde{M}}$. I will often omit such a distinction.

Suppose now that M is a cusped manifold and that $\mathbf{z}$ is a solution of $\mathcal{C}$. Let T be a component of $\partial\overline{M}$, and let P be one of its lifts in $\widetilde{\overline{M}}$. Let q be the ideal point of $\widehat{\widetilde{M}}$ corresponding to P. Since P is a covering of T (maybe not the universal covering), the universal covering $\pi_T : \widetilde{T} \to T$ splits along P, that is:

$$\pi_T = \pi \circ \pi_P : \widetilde{T} \xrightarrow{\pi_P} P \xrightarrow{\pi} T.$$

Let $D_M : \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ be a developing map for $\mathbf{z}$ such that $D_M(q) = \infty$ in the half space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and let $\pi_{\mathbb{C}} : \mathbb{H}^3 \to \mathbb{C}$ be the projection to the complex component.

Proposition 2.4.16. *There exists a homeomorphism $\beta : \widetilde{T} \to \widetilde{T}$ such that the map $D_T = \pi_{\mathbb{C}} \circ D_M \circ \pi_P \circ \beta$ defined from $\widetilde{T}$ to $\mathbb{C}$*

$$D_T : \widetilde{T} \xrightarrow{\beta} \widetilde{T} \xrightarrow{\pi_P} P \subset \widetilde{\overline{M}} \xrightarrow{D_M} \overline{\mathbb{H}}^3 \xrightarrow{\pi_{\mathbb{C}}} \mathbb{C}$$

is a developing map for the moduli induced on T by $\mathbf{z}$.

Proof. From the discussion made in Section 2.3 about the relation between the moduli of an ideal tetrahedron and the horospherical triangles at its vertices, it follows that $\pi_{\mathbb{C}} \circ D_M \circ \pi_P$ maps each triangle $\widetilde{\Delta}_i$ of $\widetilde{T}$ to a straight triangle of modulus z_i. The homeomorphism β is needed only to make such maps simplicial. □

Proposition 2.4.17. *Let h_T be the holonomy of the moduli induced by* $\mathbf{z}$ *on T. Then h_T is the restriction of $h(\mathbf{z})$ to $\pi_1(T)$.*

Proof. First, I explain the use of the word "restriction". The group $\pi_1(M)$ acts on $\widehat{\widetilde{M}}$ by deck transformations. Up to conjugation, the group $\pi_1(T)$ can be viewed as a subgroup of $\pi_1(M)$ (the inclusion $\pi_1(T) \to \pi_1(M)$ maybe not injective). Let $\overline{\pi_1(T)}$ be the conjugate of $\pi_1(T)$ in $\pi_1(M)$ that fixes the point $q \in \widehat{\widetilde{M}}$. By choosing the half space model of $\mathbb{H}^3$ as above ($q \to \infty$), one sees that the restriction of $h(\mathbf{z})$ to $\overline{\pi_1(T)}$ fixes ∞. The elements of $\mathrm{PSL}(2, \mathbb{C})$ that fix ∞ are exactly those of $\mathrm{Aff}(\mathbb{C})$. Thus the restriction of $h(\mathbf{z})$ to $\overline{\pi_1(T)}$ is a representation $h' : \overline{\pi_1(T)} \to \mathrm{Aff}(\mathbb{C})$. Moreover, by Proposition 2.4.16 there exists a developing map D for the moduli on T which is h'-equivariant. Since D is also h_T equivariant, for any $\alpha \in \pi_1(T)$ the maps $h_T(\alpha)$ and $h'(\alpha)$ coincide on the image of D. Since the image of D has dimension at least one, $h_T(\alpha)$ and $h'(\alpha)$ coincide on the whole $\mathbb{C}$. □

Remark 2.4.18. In general, a developing map is not a local homeomorphism. Namely, if z_i is a real modulus, then by the definition of straight map φ_i cannot be a homeomorphism. Moreover, if two adjacent triangles (or tetrahedra) have moduli of different sign, then phenomena of overlapping occur (see Figure 2.4)

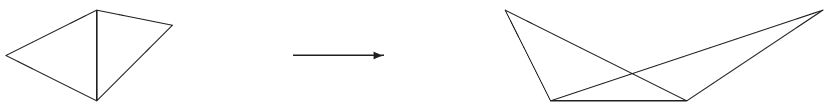


Figure 2.4. Overlapping of two triangles.

Given a solution $\mathbf{z}$ of $\mathcal{C}$ consider the diagram of Figure 2.5, where D is a developing map, f is its projection obtained by equivariance and $X/h(\mathbf{z})$ is the identification space obtained as the quotient of X under the action of the holonomy. Such a diagram always exists, but in general $X/h(\mathbf{z})$

$$\begin{tikzcd} \widetilde{M} \arrow[r, "D"] \arrow[d] & X \arrow[d] \\ M \arrow[r, "f"] & X/h(\mathbf{z}) \end{tikzcd}$$

Figure 2.5. The identification space.

is not a good topological space. A special case is when $X/h(\mathbf{z})$ is a manifold, and a very special case is when $X/h(\mathbf{z})$ is homeomorphic to M. For the following definitions I split the cases of dimension two and three.

Definition 2.4.19. (Similarity map) Suppose that M is a torus. Let T be a torus endowed with a classical $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-structure and let $D_T : \widetilde{T} \to \mathbb{C}$ be a developing map of such a structure. A map $f : M \to T$ is called *similarity* map w.r.t. $\mathbf{z}$ if, called $\widetilde{f} : \widetilde{M} \to \widetilde{T}$ its lift, the restriction of $D_T \circ \widetilde{f}$ to any triangle $\widetilde{\Delta}_i$ of $\widetilde{\tau}$ is a map compatible with z_i (see Figure 2.6).

Proposition 2.4.20. *Let M, T be as in Definition 2.4.19. If there exists a similarity map f w.r.t. $\mathbf{z}$, then $\mathbf{z}$ is a solution of $\mathcal{C}$.*

Proof. It is readily checked that $D_T \circ \widetilde{f}$ is a developing map for $\mathbf{z}$, from which one gets an $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-atlas. The claim follows from Theorem 2.4.12. □

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{T} \arrow[r, "D_T"] \arrow[d] & \mathbb{C} \\ M \arrow[r, "f"] & T & \end{tikzcd}$$

Figure 2.6. Similarity map.

Definition 2.4.21. (Hyperbolic map) Suppose that M is a cusped manifold. Let N be an oriented hyperbolic 3-manifold and let $D_N : \widetilde{N} \to \mathbb{H}^3$ be a developing map of its hyperbolic structure. A map $f : M \to N$ is called *hyperbolic* w.r.t. $\mathbf{z}$ if, called $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ its lift, the restriction of $D_N \circ \widetilde{f}$ to any tetrahedron $\widetilde{\Delta}_i$ of $\widetilde{\tau}$ is a map compatible with z_i (see Figure 2.7).

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{N} \arrow[r, "D_N"] \arrow[d] & \mathbb{H}^3 \\ M \arrow[r, "f"] & N & \end{tikzcd}$$

Figure 2.7. Hyperbolic map.

Proposition 2.4.22. *Let M, N as in Definition 2.4.21. If there exists a map $f : M \to N$ hyperbolic w.r.t.* $\mathbf{z}$*, then* $\mathbf{z}$ *is a solution of* $\mathcal{C}$.

Proof. It is readily checked that $D_N \circ \widetilde{f}$ is a developing map for $\mathbf{z}$, from which one gets an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-atlas. The claim follows from Theorem 2.4.12. □

Proposition 2.4.23. *Suppose that M is a cusped manifold and that* $\mathbf{z}$ *is a solution of* $\mathcal{C}$*. Let N be an oriented hyperbolic* 3*-manifold and let h_N be the holonomy of N. If $f : M \to N$ is a hyperbolic map w.r.t.* $\mathbf{z}$*, then the holonomy $h(\mathbf{z})$ of* $\mathbf{z}$ *is given by*

$$h(\mathbf{z}) = h_N \circ f_*$$

where $f_* : \pi_1(M) \to \pi_1(N)$.

Proof. Consider the diagram of Figure 2.7. The groups $\pi_1(M)$ and $\pi_1(N)$ act respectively on $\widetilde{M}$ and $\widetilde{N}$, and it is possible to choose base-points in such a way that for any $\alpha \in \pi_1(M)$ and $x \in \widetilde{M}$

$$\widetilde{f}(\alpha(x)) = f_*(\alpha)\widetilde{f}(x).$$

Since D_N is a developing map for N, and since $D_N \circ \widetilde{f}$ is a developing map for $\mathbf{z}$, for any $\alpha \in \pi_1(M)$ and $x \in \widetilde{M}$

$$\begin{aligned} h(\mathbf{z})(\alpha)\Big(D_N \circ \widetilde{f}\Big)(x) &= \Big(D_N \circ \widetilde{f}\Big)\big(\alpha(x)\big) = D_N\Big(f_*(\alpha)\widetilde{f}(x)\Big) \\ &= \big(h_N \circ f_*\big)(\alpha)\Big(D_N \circ \widetilde{f}\Big)(x). \end{aligned}$$

It follows that for any $\alpha \in \pi_1(M)$, $h(\mathbf{z})(\alpha)$ and $h_N \circ f_*(\alpha)$ coincide on the image of $D_N \circ \widetilde{f}$. Since $D_N \circ \widetilde{f}$ is a developing map, the dimension of its image is at least two. Since both $h(\mathbf{z})(\alpha)$ and $h_N \circ f_*$ are orientation preserving isometries, they coincide on the whole $\mathbb{H}^3$. □

The same statement holds for similarity tori.

Proposition 2.4.24. *Suppose that M is a torus and that $\mathbf{z}$ is a solution of $\mathcal{C}$. Let T be an oriented torus endowed with a similarity structure and let h_T be the holonomy of T. If $f : M \to T$ is a similarity map w.r.t. $\mathbf{z}$, then the holonomy $h(\mathbf{z})$ of $\mathbf{z}$ is given by*

$$h(\mathbf{z}) = h_T \circ f_*$$

where $f_* : \pi_1(M) \to \pi_1(T)$.

Proof. As in Proposition 2.4.23, *mutatis mutandis*. □

Since a solution $\mathbf{z}$ of $\mathcal{C}$ determines an (X, G)-atlas for τ, the first natural question is whether $\mathbf{z}$ leads to a classical (X, G)-structure for M. As I will show in the next chapters, the situations in dimension 2 and 3 are quite different. As a first answer, I state the following fact (see [26]).

Proposition 2.4.25. *If $\mathbf{z}$ is a solution of $\mathcal{C}$ such that the imaginary part of each z_i is strictly positive, then each (X, G)-atlas for τ whose local charts are compatible with the moduli defines a classical (X, G)-structure on M, with holonomy $h(\mathbf{z})$.*

This is because if the imaginary part of each modulus is positive, then no pathologies as in Remark 2.4.18 occur.

Definition 2.4.26. Let $\mathbf{z}$ be a choice of moduli for τ. I call $\mathbf{z}$ *positive* (resp. *negative*) if for every i, $\Im(z_i) > 0$ (resp. < 0). I call $\mathbf{z}$ *partially flat* if for every i, $\Im(z_i) \geq 0$ and there exists i such that $\Im(z_i) > 0$. I call $\mathbf{z}$ *flat* if each z_i is a real number. Otherwise I call $\mathbf{z}$ *mixed*.

Remark 2.4.27. Suppose M is a cusped manifold and $\mathbf{z}$ is a positive solution of $\mathcal{C}$. By Remark 2.4.8 τ induces a triangulation on ∂M and $\mathbf{z}$ induces a positive solution of the compatibility equations on the boundary. Thus $\mathbf{z}$ defines both a hyperbolic structure on M and a similarity structure on ∂M. In general if $\mathbf{z}$ induces a hyperbolic structure on M then it also induces a similarity structure on ∂M, but the converse is false.

When $\mathbf{z}$ is positive it is clear what is the meaning of the sentence "$\mathbf{z}$ induces an (X, G)-structure on M." In general it is not clear what is the geometric interpretation of a solution $\mathbf{z}$ of $\mathcal{C}$. I introduce here the notion of *geometric solution* of $\mathcal{C}$.

Definition 2.4.28. (Geometric solution of $\mathcal{C}$) Suppose M is a torus. I say that $\mathbf{z}$ is a *geometric solution* of $\mathcal{C}$ if there exist a torus T equipped with a similarity structure and a similarity map $f : M \to T$ of degree one.

Definition 2.4.29. (Geometric solution of $\mathcal{C}$) Suppose M is a cusped manifold. I say that $\mathbf{z}$ is a *geometric solution* of $\mathcal{C}$ if there exist a hyperbolic structure $\mathfrak{S}$ on M and a proper degree-one map $f : M \to M_{\mathfrak{S}}$ which is hyperbolic w.r.t. $\mathbf{z}$ (where $M_{\mathfrak{S}}$ means M with the structure $\mathfrak{S}$).

In Definitions 2.4.28 and 2.4.29 I did not require the map f to be a homeomorphism because in general one cannot avoid the phenomena described in Remark 2.4.18.

Proposition 2.4.30. *Any geometric solution of $\mathcal{C}$ is also an algebraic solution of the system $\mathcal{C}$.*

Proof. It follows from Proposition 2.4.20 if M is a torus and from Proposition 2.4.22 if M is a cusped manifold. $\square$

I will show in the next chapters that not all algebraic solutions are geometric.

2.5. Completeness and hyperbolic Dehn filling equations

For this section I keep the notation fixed at the beginning of Section 2.4. Suppose that M is a cusped manifold and that $\mathbf{z}$ is a positive solution of $\mathcal{C}$. Then by Proposition 2.4.25 and Remark 2.4.27 $\mathbf{z}$ induces a hyperbolic structure on M and a similarity structure on ∂M. Then one can ask for the completeness of such structures. Note that the hyperbolic volume of the structure of M is finite because τ is finite. From the characterization of complete hyperbolic manifolds of finite volume (Theorem 1.1.5) and Remarks 1.2.5 and 1.2.6 one gets the following

Proposition 2.5.1. *Suppose M is a cusped manifold and $\mathbf{z}$ is a positive solution of $\mathcal{C}$. The hyperbolic structure of M induced by $\mathbf{z}$ is complete if and only if all the similarity structures of the components of ∂M are complete (*i.e. *Euclidean).*

Proof. If the structure of M is complete, the thesis follows from Theorem 1.1.5 and Proposition 2.4.17. Now suppose that all the structures of the boundary tori are Euclidean. By Proposition 2.4.17 it follows that each cusp has a complete structure. Let $\{x_n\}$ be a Cauchy sequence (w.r.t. the hyperbolic metric) in M. Since $\widehat{M}$ is compact, up to subsequences x_n has a limit $x \in \widehat{M}$. Moreover, since the cusps are complete, x is not an ideal point. Then the whole sequence converges to x. Hence M is complete. $\square$

Lemma 2.5.2. *Suppose M is a torus. Let $\mathbf{z}$ be a solution of $\mathcal{C}$. Either the holonomy $h(\mathbf{z})$ consists of translations or there exists a unique point $x \in \mathbb{C}$ which is fixed under the action of $h(\mathbf{z})$.*

Proof. This immediately follows from the Abelianity of the fundamental group of M. $\square$

Definition 2.5.3. (Axis of the holonomy) Suppose M is a torus and $\mathbf{z}$ is a solution of $\mathcal{C}$. If the holonomy has a unique fixed point in $\mathbb{C}$, I call such a point *axis* of the holonomy.

If M is a torus, then for each $\alpha \in \pi_1(M)$ the map $h(\mathbf{z})(\alpha)$ is of the form

$$z \mapsto az + b$$

with $a, b \in \mathbb{C}$. The number a is called dilation component of $h(\mathbf{z})(\alpha)$. Since the holonomy is well-defined up to conjugation, its *dilation component* $\overline{h}$ is a well-defined representation

$$\overline{h}(\mathbf{z}) : \pi_1(M) \to \mathbb{C}^*.$$

Proposition 2.5.4. (Completeness equations) *Let $\mathbf{z}$ be a positive solution of $\mathcal{C}$. Then the (X, G)-structure induced by $\mathbf{z}$ on M is complete if and only if the moduli satisfy a finite system $\mathcal{M}$ of algebraic equations, called* completeness equations, *each one of the form*

$$\pm \prod z_i^{\alpha_i}(1 - z_i)^{\beta_i} = 1$$

where each z_i possibly appears several times and α_i and β_i are in $\{0, 1, -1\}$, depending on the combinatorial data of τ. Moreover, such equations can be written down even without the hypothesis that $\mathbf{z}$ is positive.

I refer to [26] and [20] for a detailed discussion on the equations.

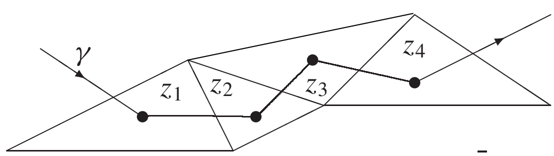
Proof of 2.5.4. By proposition 2.5.1 it suffices to consider the case in which M is a torus. In this case the completeness condition is equivalent to require the dilation component of the holonomy to be the trivial representation

$$\overline{h}(\mathbf{z}) \equiv 1.$$

The dilation component of the holonomy can be computed from the moduli as follows. Let $\gamma = \Delta_{i_0} \to \cdots \to \Delta_{i_k}$ be a loop of triangles. Then

$$\overline{h}(\mathbf{z})(\gamma) = \prod z_{i_s}^{n_s}$$

where the moduli z_{i_s} and the coefficients n_s are as in Figure 2.8.



$$\bar{h}(\mathbf{z})(\gamma) = \cdots z_1 \cdot z_2 \cdot \frac{1}{z_3} \cdot z_4 \cdots$$

Figure 2.8. Moduli along the path γ.

Moreover, since $\mathcal{C}$ holds, such a product is invariant under step-homotopies, so it does not depend on the representative of $[\gamma] \in \pi_1(M)$. As in the case of the compatibility equations, the ambiguity z, $1-1/z$, $(1-z)^{-1}$ on each modulus leads to the coefficients α_i's and β_i's. Since $\overline{h}(\mathbf{z})$ is a representation, then it suffices to require that $\overline{h}(\mathbf{z})(\gamma_1) = \overline{h}(\mathbf{z})(\gamma_2) = 1$ for a basis (γ_1, γ_2) of $\pi_1(M)$. Then the system $\mathcal{M}$ is finite (note that if M is a cusped manifold, then ∂M is a finite union of tori). Finally, the construction of $\overline{h}$ needs only that $\mathbf{z}$ is a solution of $\mathcal{C}$, so the system $\mathcal{M}$ can be written down whenever $\mathbf{z}$ satisfies $\mathcal{C}$. $\square$

Note that when M is a cusped manifold, a picture as in Figure 2.8 is obtained in a suitable half-space model of $\mathbb{H}^3$. I remark that the holonomy and both the equations $\mathcal{C}$ and $\mathcal{M}$ can be read on the boundary of M. This seems to say that to know what happens to M it suffices to control the geometry of ∂M. As in the case of completeness equations, in general the geometric meaning of a choice of moduli $\mathbf{z}$ that algebraically solves the systems $\mathcal{C}$ and $\mathcal{M}$ in general is not clear.

Definition 2.5.5. (Geometric solution of $\mathcal{C} + \mathcal{M}$) Suppose M is an oriented torus. A choice of moduli $\mathbf{z}$ is called *geometric solution* of $\mathcal{C} + \mathcal{M}$

if there exist a Euclidean structure $\mathfrak{E}$ on M and a similarity map (w.r.t. $\mathbf{z}$) $f : M \to M_{\mathfrak{E}}$ of degree one, where $M_{\mathfrak{E}}$ is M endowed with the structure $\mathfrak{E}$.

Definition 2.5.6. (Geometric solution of $\mathcal{C} + \mathcal{M}$) Suppose M is an oriented cusped manifold. A choice of moduli $\mathbf{z}$ is called *geometric solution* of $\mathcal{C} + \mathcal{M}$ if there exist a complete hyperbolic structure $\mathfrak{S}$ on M and a proper map $f : M \to M_{\mathfrak{S}}$ of degree one which is hyperbolic w.r.t $\mathbf{z}$, where $M_{\mathfrak{E}}$ is M endowed with the structure $\mathfrak{S}$.

Proposition 2.5.7. *Any geometric solution of $\mathcal{C} + \mathcal{M}$ is also an algebraic solution of the system $\mathcal{C} + \mathcal{M}$ of the union of the compatibility and completeness equations.*

Proof. If M is a torus the thesis follows from Proposition 2.4.24. If M is a cusped manifold note that, since f is proper, it maps cusps to cusps. Then the thesis follows from Propositions 2.4.17, 2.4.23, and Theorem 1.1.5. □

For the rest of the section M will be a cusped manifold. I introduce now the system of so-called *hyperbolic Dehn filling equations*. If $\mathbf{z}$ is a positive solution of $\mathcal{C}$, then such a system express the fact that the completion of the hyperbolic structure induced on M is a prefixed Dehn filling of M. I start recalling the definition of Dehn filling of a cusped manifold.

Definition 2.5.8. (Dehn filling) Let M be a cusped oriented manifold and set $\partial\overline{M} = \sqcup_n T_n$. For each torus T_n let (μ_n, λ_n) be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n)\}$ where (p_n, q_n) is either a pair of coprime integers or the symbol ∞. For each n such that $(p_n, q_n) \neq \infty$, let L_n be an oriented solid torus, m_n be a meridian of $T_n' = \partial L_n$, l_n be a loop in T_n such that $[l_n] = p_n\mu_n + q_n\lambda_n$ and $\varphi_n : T_n \to T_n'$ be an orientation reversing homeomorphism such that $\varphi_n(l_n) = m_n$. The Dehn filling of M with coefficients (p, q) is the manifold

$$M_{(p,q)} = \mathrm{int}\Big(\overline{M} \sqcup \{L_n\}\big/\{\varphi_n\}\Big)$$

The tori L_n are called filling tori.

Remark 2.5.9. The resulting manifold $M_{(p,q)}$ actually depends only on the coefficients (p, q) and not on the maps φ_n.

Remark 2.5.10. Not all the boundary tori are filled in $M_{(p,q)}$. Namely, a torus T_n is filled if and only if $(p_n, q_n) \neq \infty$. If $(p_n, q_n) = \infty$ for all n, then $M_{(p,q)} = M$.

The principal condition expressed by the hyperbolic Dehn filling equations is that the holonomy of each loop l_n, killed by the filling, is trivial.

For any torus $T \subset \partial\overline{M}$ let h_T and $\overline{h}_T$ be the holonomy of T and its dilation component. Recall that h_T is the restriction to $\pi_1(T)$ of the holonomy of $\mathbf{z}$. If h_T consists of translations then $\overline{h}_T \equiv 1$. Otherwise, the holonomy of T has an axis. Up to change coordinates of $\mathbb{C}$, one can always suppose that the axis is 0, so that for all $\alpha \in \pi_1(T)$ and $\zeta \in \mathbb{C}$, $h_T(\alpha)(\zeta) = \overline{h}_T(\alpha) \cdot \zeta$.

Remark 2.5.11. In the following, if there are no ambiguities, by writing $\overline{h}_T \equiv 1$, I mean that $h_T(\pi_1(T))$ consists of translations and by $h_T = \overline{h}_T$, I mean that $h_T(\pi_1(T))$ consists of maps which fix 0. I recall that h, h_T and $\overline{h}_T$ depend on $\mathbf{z}$. When I need to emphasize this, I write $h(\mathbf{z})$, $h_T(\mathbf{z})$ and $\overline{h}_T(\mathbf{z})$.

To write the hyperbolic Dehn filling equations, I need to work with $\log(\overline{h}_T)$, which is not a single-valued function. In the following definition I fix a suitable determination of the logarithm of $\overline{h}_T$.

Definition 2.5.12. (Logarithm of the dilation component) Let $\mathbf{z}$ be a solution of $\mathcal{C}$ and let D_M be a developing map for $\mathbf{z}$. Let T be a torus contained in $\partial\overline{M}$, let $\mathbf{z}_T$ be the solution of $\mathcal{C}$ induced by $\mathbf{z}$ on T, and let $D_T : \widetilde{T} \to \mathbb{C}$ be the developing map for $\mathbf{z}_T$ described in Proposition 2.4.16. Suppose that $h_T = \overline{h}_T$ and suppose that the following condition holds:

$$\text{The image of } D_T \text{ does not contain the axis } 0. \tag{2.1}$$

Then I choose a determination of $\log(\overline{h}_T)$ as follows: let $\exp : \mathbb{C} \to \mathbb{C}^*$ be the universal cover of $\mathbb{C}$. Let x_0 and $\widetilde{x}_0$ be base-points in T and $\widetilde{T}$. Let $\gamma : [0,1] \to T$ be a loop based at x_0 and $\widetilde{\gamma}$ be its lift starting from $\widetilde{x}_0$. Let

$$\alpha = D_T \circ \widetilde{\gamma} : [0,1] \to \mathbb{C}^*$$

and let $\widetilde{\alpha} : [0,1] \to \mathbb{C}$ be one of its lift via exp. Since $h_T = \overline{h}_T$, $\alpha(1) = \overline{h}_T([\gamma]) \cdot \alpha(0)$, and then

$$\widetilde{\alpha}(1) = \log(\overline{h}_T([\gamma])) + \widetilde{\alpha}(0).$$

The points $\widetilde{\alpha}(0)$ and $\widetilde{\alpha}(1)$ depend only on the homotopy class of γ and on the choice of the base-points. If one changes base-points, then the determination of $\log(\overline{h}_T([\gamma]))$ changes by a conjugation by translations, and so it is well-defined.

Suppose that T is a torus in $\partial\overline{M}$, let (μ, λ) a basis for $H_1(T, \mathbb{Z})$, and let (a, b) be a pair of coprime integers. Consider the Dehn filling of M with parameters (a, b), *i.e.* the filling in which an oriented loop homotopic to $a\mu + b\lambda$ is mapped to the meridian m of a solid torus L. The coefficient (a, b) induces an orientation of m. Since the gluing map inverts the orientations of the boundary tori, then the core γ of the filling tours is canonically oriented by requiring that m turns around γ by following the right-hand rule in L.

Definition 2.5.13. (To spiral around) Suppose $\mathbf{z}$ is a solution of $\mathcal{C}$. Let N be an oriented hyperbolic 3-manifold and let $f : M \to N$ be a map which is hyperbolic w.r.t. $\mathbf{z}$. Let γ be an oriented geodesic in N and let v be a vertex of τ. Consider $\widetilde{N} = \mathbb{H}^3$, and use a half-space model of $\mathbb{H}^3$ in which the oriented line $(0, \infty)$ is a lift of γ. Let $\widetilde{f} : \widetilde{M} \to \mathbb{H}^3$ be a lift of f. Since $\widetilde{f}$ is a developing map, it extends to $\widetilde{\widetilde{M}}$. I say that f *spirals around* γ near v if there exists a lift $\widetilde{v}$ of v such that $\widetilde{f}(\widetilde{v}) = \infty$.

Definition 2.5.14. (Hyperbolic Dehn filling equations) Let $\mathbf{z}$ be a solution of $\mathcal{C}$. For each boundary torus T_n let (μ_n, λ_n) be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n)\}$ be a set of coefficients such that (p_n, q_n) is either a pair of coprime integers or the symbol ∞. Let $\overline{h}_n(\mathbf{z})$ be the dilation component of the holonomy of T_n. I say that $\mathbf{z}$ is an *algebraic solution* of the (p, q)-equations if for every n:

- If $(p_n, q_n) = \infty$, then $\overline{h}_n(\mathbf{z}) \equiv 1$.
- If $(p_n, q_n) \neq \infty$, then $h_{T_n}(\mathbf{z}) = \overline{h}_n(\mathbf{z})$, condition (2.1) of Definition 2.5.12 holds, and

$$p_n \log(\overline{h}_n(\mathbf{z})[\mu_n]) + q_n \log(\overline{h}_n(\mathbf{z})[\lambda_n]) = 2\pi i.$$

I say that $\mathbf{z}$ is a *geometric solution* of the (p, q)-equations if, called $N = M_{(p,q)}$ the Dehn filling of M with parameters (p_n, q_n), then:

a) N is complete hyperbolic and the cores of the filling tori are disjoint geodesics $\{\gamma_n\}$.
b) There exists a proper map $f : M \to N \setminus \{\gamma_n\} \subset N$ of degree 1, which is hyperbolic w.r.t. $\mathbf{z}$.
c) For each boundary torus T_n with $(p_n, q_n) \neq \infty$, if v_n is the vertex corresponding to T_n, then f spirals around γ_n near v_n, where γ_n has the orientation induced by the Dehn filling coefficient (p_n, q_n).

Remark 2.5.15. When all the coefficients (p_n, q_n) are ∞, then the system of the (p, q)-equations is exactly the classical system $\mathcal{M}$ of the *completeness equations*.

Theorem 2.5.16. *Let* $\{(\mu_n, \lambda_n)\}$ *and* (p, q) *be as in Definition* 2.5.14. *Each geometric solution* $\mathbf{z}$ *of the* (p, q)*-equations is also algebraic.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equations. By Proposition 2.4.23 the holonomy of $\mathbf{z}$ is the composition of f_* with the holonomy of N.

The map f is surjective on $N \setminus \{\gamma_n\}$ because it has degree one. Moreover, since f is proper and spirals around γ_n near v_n, it maps the unfilled cusps of M to the cusps of N. This implies that if $(p_n, q_n) = \infty$ the holonomy of T_n consists of translations. Similarly, if $(p_n, q_n) \neq \infty$, then the image of h_{T_n} is contained in the subgroup of $\pi_1(N)$ generated by γ_n, and then $h_{T_n} = \overline{h}_n$, because for every n the holonomy of γ_n as an element of $\pi_1(N)$ is a hyperbolic isometry.

The fact that $\mathrm{Im}(f) = N \setminus \{\gamma_n\}$ implies condition (2.1) of Definition 2.5.14. Finally, using the determination of the logarithm of Definition 2.5.12, since $N = M_{(p,q)}$, one has $p_n \log(\overline{h}_n(\mathbf{z})[\mu_n]) + q_n \log(\overline{h}_n(\mathbf{z})[\lambda_n]) = 2\pi i$. □

2.6. Equations on the angles

For this section I keep the notations fixed in Section 2.4. Here I discuss the system $\mathcal{C}^*$ of the so-called equations on the angles.

Let $\mathbf{z}$ be a positive choice of moduli for τ. Let Δ_i be a simplex of maximal dimension of τ and let e be a codimension-two subsimplex of Δ_i. The argument $\arg_e(z_i)$ of z_i at e is well defined (see Section 2.3). The system $\mathcal{C}^*$ of the equations on the angles is given by the equations

$$\sum_{\Delta_i \supset e} \arg_e(z_i) = 2\pi$$

where e varies on the set of the simplices of codimension two of τ.

Proposition 2.6.1. *If* $\mathbf{z}$ *is a positive solution of* $\mathcal{C}$*, then the equations on the angles are satisfied.*

Proof. This is the usual characteristic calculation. Clearly, it suffices to prove this in the case that M is a torus. Let $\mathcal{V}$, $\mathcal{L}$, $\mathcal{T}$ be the number respectively of vertices, edges, triangles of τ. Since the sum of inner angles of a Euclidean triangle is π,

$$\sum_{e \text{ vertex}} \sum_{\Delta_i \supset e} \arg_e(z_i) = \pi \cdot \mathcal{T}.$$

Since $\mathcal{C}$ holds, for every vertex e the sum of the angles around e is $2\pi \cdot K_e$, where K_e is a positive integer depending on e, so the claim becomes that $K_e = 1$ for all e. It holds

$$\pi \cdot \mathcal{T} = \sum \arg_e(z_i) = \sum_{e \text{ vertex}} 2\pi \cdot K_e.$$

The characteristic of a torus is zero, then

$$0 = \chi(M) = \mathcal{V} - \mathcal{L} + \mathcal{T}.$$

Moreover, since the simplices of maximal dimension of τ are triangles,

$$2\mathcal{L} = 3\mathcal{T}.$$

It follows that $\mathcal{T} = 2\mathcal{V}$, so

$$2\pi \cdot \mathcal{V} = \pi \cdot \mathcal{T} = 2\pi \sum_{e \text{ vertex}} K_e$$

whence $K_e = 1$ for all e. □

Suppose M is a cusped manifold. A choice of arguments for τ is a choice of a real number $\arg_e(\Delta_i)$ for every edge e of every tetrahedron Δ_i of τ, in such a way that opposite edges have the same argument and such that for any Δ_i the sum of the arguments of all edges of Δ_i is 2π. The equations on the angles can be written down for any choice of arguments. These equations give important informations about the topology of M. The same technique of Proposition 2.6.1, combined with the theory of normal surfaces (see [13]), can be used to prove the following result (compare with Theorem 1.1.6 and see [17] for a proof).

Theorem 2.6.2. *Suppose M is a cusped manifold. Suppose that there exists a choice of strictly positive arguments for τ that satisfies the equations on the angles. Then M is incompressible and atoroidal.*

Proof. Let S be either an essential sphere or tours. Then S can be put in a normal position w.r.t. τ. So τ induces a cellularization of S with triangles and quadrilaterals. Calculations of characteristic as in Proposition 2.6.1 exclude the presence of quadrilaterals. So S is parallel to a boundary torus. I notice that the hypothesis that the arguments are strictly positive is crucial for the proof. □

Petronio and Weeks in [22] proved the following

Theorem 2.6.3. *If $\mathbf{z}$ is a partially flat algebraic solution of $\mathcal{C} + \mathcal{M} + \mathcal{C}^*$, then it is a geometric solution of $\mathcal{C} + \mathcal{M}$.*

In Subsection 4.2.3 I will show an example of a partially flat solution of $\mathcal{C} + \mathcal{M}$ which is not geometric.

Even if the conditions at the angles seems to be fundamental in order for a solution of $\mathcal{C}$ to be geometric, when the moduli are allowed to be negative, there is no natural continuous way to define its argument (see Section 2.3). The following examples show some pathologies that can occur when the moduli are not positive.

Example 2.6.4. Suppose M is a torus. Suppose that around a vertex v there are only positive moduli and that the co-cycle condition around v is satisfied. Then a geometric picture near v looks like Figure 2.9a).

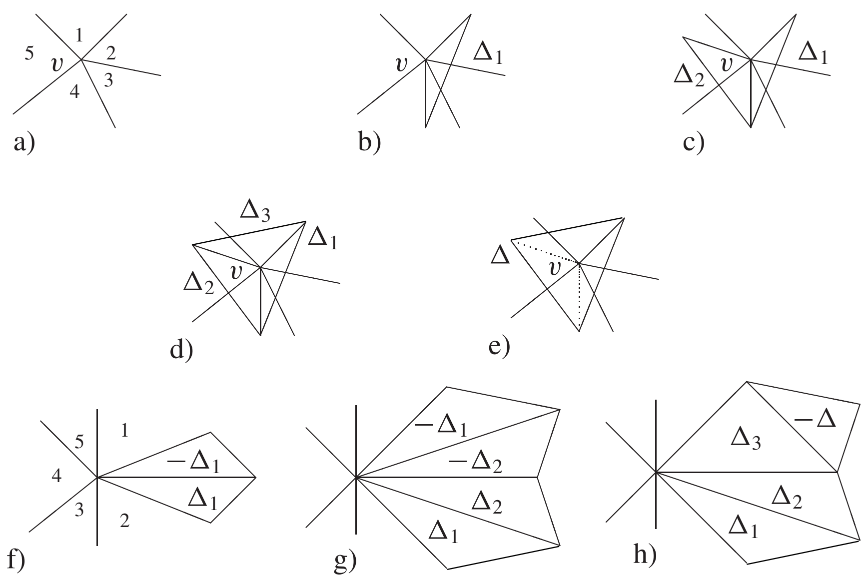


Figure 2.9. Geometric and topological situations around v.

Now add and remove near v a triangle Δ_1, *i.e.* add two copies of Δ_1 with opposite moduli. The topological picture is like in Figure 2.9f), but the geometric result is the same as before (Figure 2.9b)). Add and remove a second triangle Δ_2 (Figures 2.9c) and 2.9g)). Now observe that $(-\Delta_1) \cup (-\Delta_2)$ is geometrically equivalent to a big negative triangle $-\Delta$ plus a positive triangle Δ_3 (Figure 2.9d)–e)). By replacing $(-\Delta_1) \cup (-\Delta_2)$ with $(-\Delta) \cup (+\Delta_3)$, the topological picture looks like Figure 2.9h), the geometric situation is not changed, around v there are only positive triangles, but now the sum of arguments around v is 4π. Note that there is only one negative triangle. As this is a local picture, this phenomenon can occur in any triangulation.

Example 2.6.5. Suppose that M is a torus and suppose that $\mathbf{z}$ is a positive solution of $\mathcal{C}$. Then the arguments of moduli are defined. Consider two triangles glued together and change the triangulation by adding Δ_1 and Δ_2 as in Figure 2.10.

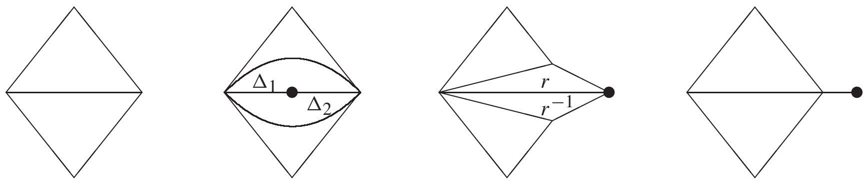


Figure 2.10. Equations $\mathcal{C}^*$ are not necessary.

Assign now moduli r and r^{-1} to the $\bullet$-vertex of Δ_1 and Δ_2, with $r \in \mathbb{R}^+$. It is easy checked that such a choice of moduli is geometric, but equations $\mathcal{C}^*$ are not satisfied (if one uses the convention of Section 2.3 for the argument of a flat modulus).

Chapter 3
Geometric solutions of $\mathcal{C}$ and $\mathcal{C}+\mathcal{M}$ in dimension two

In this chapter I study the difference between algebraic and geometric solutions of $\mathcal{C}$ and $\mathcal{C}+\mathcal{M}$ for a triangulated torus. I will give a precise characterization of geometric solutions, showing that "almost any" algebraic solution is also geometric. In particular it will follow that there is no uniqueness of geometric solutions of $\mathcal{C}+\mathcal{M}$. I study the problem from two different viewpoints. In the first section I look at the combinatorial data of the triangulation and I show how the problem can be reduced to a simpler one, which is easy to solve. In the second section I study the problem more algebraically, looking at the properties of the holonomy of a solution, and I give a complete algebraic condition for a solution to be geometric. In the last section I treat the case of the Klein bottle.

Notation. For the whole chapter T will be an oriented torus and $\widetilde{T}$ will be its universal covering, $\tau = (\{\Delta_i\}, \{r_j\})$ will be a triangulation of T, $\widetilde{\tau}$ will be its lift to $\widetilde{T}$, and $\mathcal{R}$ will be the simplicial realization of τ. The *dual graph* of τ is the 1-skeleton $\Gamma^{(1)}$ of the dual cellularization $\Gamma(\tau)$ of τ. I identify the paths of triangles with the simplicial paths in $\Gamma^{(1)}$. The symbol $\mathbf{z}$ will denote a choice of moduli $\mathbf{z} = \{z_i\}$ for τ. I call triangulation with moduli a pair $(\tau, \mathbf{z})$ where τ is a triangulation of T and $\mathbf{z}$ is a choice of moduli for τ. Since $\pi_1(T)$ is Abelian, it is isomorphic to $H_1(T) = H_1(T, \mathbb{Z})$. In the following I do not distinguish between $\pi_1(T)$ and $H_1(T)$.

3.1. Simplifying triangulations

In this section I develop an algorithmic method to manipulate triangulations with moduli, based on a geometric version of the topological diagonal swap. Under a supplementary hypothesis, this algorithm produces a triangulation with moduli of T, equivalent to $(\tau, \mathbf{z})$ (in a sense that will be clear in the following) and with only two triangles. For such a triangulation it is easy to check if a choice of moduli is geometric or not.

The dual graph of τ is a trivalent graph whose vertices correspond to the triangles of τ, and a modulus for a triangles corresponds to a modulus for a vertex of $\Gamma^{(1)}$ as pictured in Figure 3.1.

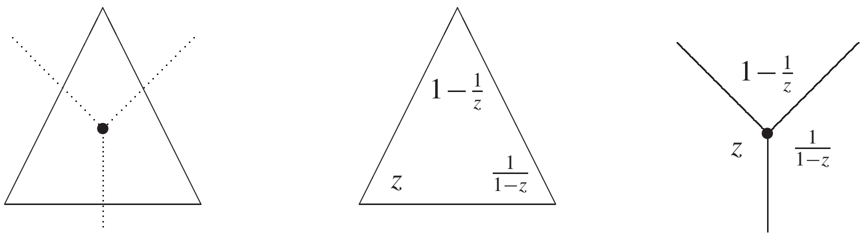


Figure 3.1. Correspondence between vertices of $\Gamma^{(1)}$ and triangles of τ.

3.1.1. The moves

First of all I state two supplementary hypotheses which will be crucial in the sequel. Let $\mathbf{z}$ be a solution of $\mathcal{C}$.

H3.1.1 Suppose the image of the holonomy has rank 2, *i.e.* it is not cyclic.
H3.1.2 Suppose that if the holonomy has an axis, then it lies outside the image of a developing map.

Remark 3.1.3. It is easily checked that Hypotheses H3.1.1 and H3.1.2 do not depend on the choice of the representatives of holonomy and on the developing map.

I use the classical move of *topological diagonal swap* (TDS) to manipulate triangulations (Figure 3.2). This move extends in an obvious way in a geometric setting to give a *geometric diagonal swap* (GDS) as in Figures 3.2 and 3.3. Any GDS can be viewed as a function from the space of triangulations with moduli of the torus to itself.

Figure 3.2. Topological and geometric diagonal swap.

Remark 3.1.4. With notation as in Figure 3.2, since only moduli in $\mathbb{C} \setminus \{0, 1\}$ are allowed, one can apply the GDS only if $z' \neq z^{-1}$. In this case the GDS is continuous as a function from the space of moduli on τ to the space of moduli on the resulting triangulation.

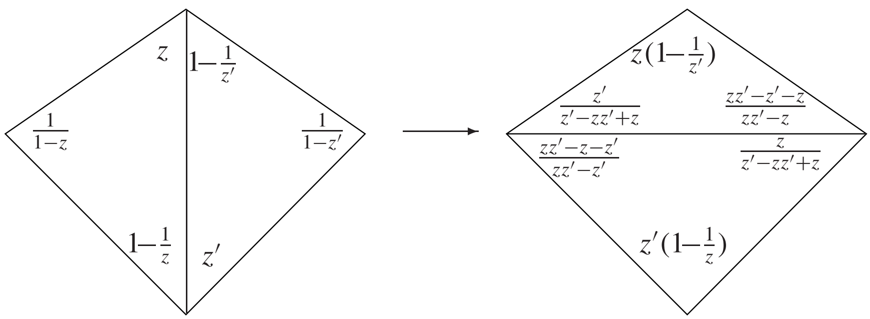


Figure 3.3. The moduli in the GDS.

Let γ be a path (loop) in the dual graph of τ. Then by applying a GDS to two consecutive vertices of γ we obtain a path (loop) γ' as Figure 3.4 shows.

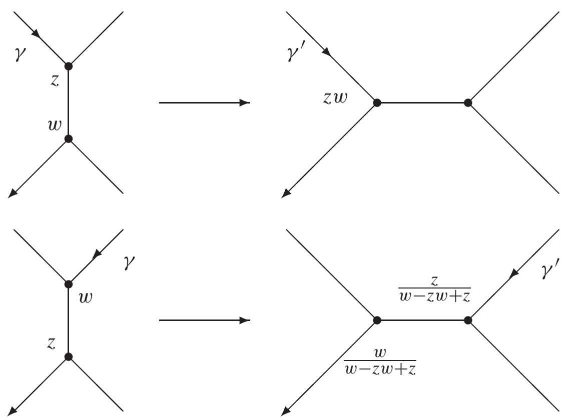


Figure 3.4. Effects of a GDS on a path in $\Gamma^{(1)}$.

Simple calculations lead to the following facts:

Lemma 3.1.5. *The* GDS *does not change the products of moduli along paths in the dual graph used to write equations $\mathcal{C}$ and $\mathcal{M}$. So* $\mathbf{z}$ *satisfies equations $\mathcal{C}$ (or $\mathcal{C}+\mathcal{M}$) if and only if the resulting moduli after the move do.*

Lemma 3.1.6. *Any* GDS *lifts to a set of moves on the universal covering $\widetilde{T} \to T$.*

Lemma 3.1.7. *Let* $\mathbf{z}$ *be a solution of $\mathcal{C}$ and let D be a developing map for* $\mathbf{z}$*. The application of a* GDS *does not change the restriction of D to the* 0*-skeleton.*

Corollary 3.1.8. *Let* $\mathbf{z}$ *be a solution of* $\mathcal{C}$*. Then the holonomy does not change under the* GDS*, and in particular a* GDS *preserves Hypothesis* H3.1.1.

Remark 3.1.9. A GDS-move in general does not preserve Hypothesis H3.1.2. Figure 3.5 shows a particular case of GDS, in which the axis is being incorporated into the image of the developing map.

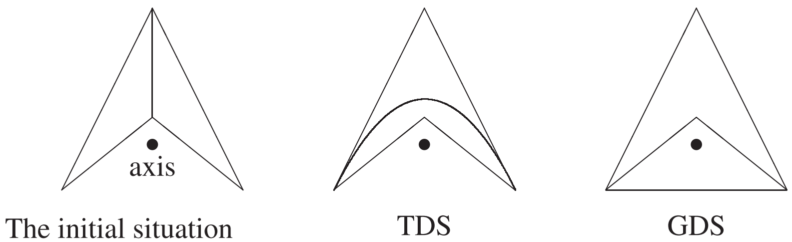


Figure 3.5. The developed image of a GDS.

Remark 3.1.10. The moves do not change the number of simplices of the triangulations.

3.1.2. Loops and e-loops

For this subsection I fix a solution $\mathbf{z}$ of the compatibility equations and I suppose that Hypothesis H3.1.1 holds. I call h the holonomy of $\mathbf{z}$.

Definition 3.1.11. I call *e-loop*, an edge of the triangulation which starts and ends at the same vertex v and I say that v *has an e-loop*.

Definition 3.1.12. I set $S(v)$ to be the number of triangles (with multiplicity) having v as a vertex.

Remark 3.1.13. Since an e-loop is an edge of the triangulation, it is an embedded loop.

Lemma 3.1.14. *For each vertex* v*,* $S(v) > 1$.

Proof. Since equations $\mathcal{C}$ hold, if $S(v) = 1$ then the modulus of the only triangle around v must be 1. But this contradicts the fact that the moduli lie in $\mathbb{C} \setminus \{0, 1\}$. □

Proposition 3.1.15. *If* l *is an e-loop, then* $[l] \neq 0$ *as an element of* $H_1(T)$.

Proof. Suppose the contrary. Then l bounds a sub-complex B homeomorphic to a disc, so l lifts to a loop in $\widetilde{T}$. But the developed image of l is a straight segment which cannot be a loop. □

Remark 3.1.16. The notion of parallelism between loops is well-defined for disjoint loops. Since I deal with loops that can share at most one point, I say that two loops on a surface are parallel if they jointly bound either an embedded annulus or an embedded pinched annulus.

Lemma 3.1.17. *In a torus, the relation of parallelism between non-contractible loops sharing at most one point is transitive.*

Proof. Let α, β, γ be non-contractible loops so that α is parallel to β which is parallel to γ. Cutting the torus along β we obtain a cylinder in which α and γ are parallel to the boundary. Since α and γ share at most one point, it follows that $\alpha \cup \gamma$ bounds either an annulus or a pinched annulus, so they are parallel. □

Remark 3.1.18. Two e-loops at different vertices are disjoint. By Proposition 3.1.15 and an argument as in Lemma 3.1.17, two disjoint e-loops are topologically parallel. Therefore, e-loops at different vertices are parallel.

Proposition 3.1.19. *If there exist more than one vertex with e-loops, then e-loops at the same vertex are parallel.*

Proof. Let l_1 and l_2 be two different e-loop at a vertex v. Let $v' \neq v$ be a vertex which has an e-loop l. By Remark 3.1.18, the e-loop l is parallel to both l_1 and l_2. By Lemma 3.1.17 it follows that l_1 is parallel to l_2. □

Proposition 3.1.20. *Suppose that each vertex has an e-loop. Then two different e-loops l_1 and l_2 at the same vertex v are not parallel.*

Proof. Suppose the contrary. Then l_1 and l_2 jointly bound a region R whose fundamental group is isomorphic to $\mathbb{Z}$ and generated by $[l_1] = [l_2]$. So R cannot contain any vertex in its interior because each vertex has an e-loop and e-loops are not contractible. It follows that R is a bigon, but this cannot happen in a triangulation. □

Corollary 3.1.21. *Suppose that each vertex has an e-loop. If a vertex v has two different e-loops then there exists only one vertex. Equivalently, if there exists more than one vertex, then each vertex has exactly one e-loop.*

Lemma 3.1.22. *In the current hypotheses, suppose moreover that Hypothesis* H3.1.2 *holds. Let* Δ_1 *and* Δ_2 *be two triangles, glued along an edge and let* v *be a vertex of such edge. Suppose that* Δ_1 *and* Δ_2 *have inverse moduli at* v*, and that* v *has no e-loop. Suppose moreover, with notation as in Figure 3.6, that* $v_1 = v_2$*. Then* $[\gamma_1\gamma_2^{-1}] = 0$ *as an element of* $H_1(T)$*.*

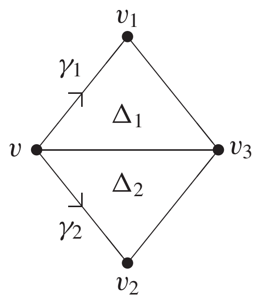


Figure 3.6. The triangles Δ_1 and Δ_2.

Proof. Suppose $[\gamma_1\gamma_2^{-1}] \neq 0$. Since v has no e-loops, then $\gamma_1\gamma_2^{-1}$ is a loop embedded in T. Choose $\gamma_1\gamma_2^{-1}$ as an element of a $\mathbb{Z}$-basis of $H_1(T)$. Let $\widetilde{v}_1$ and $\widetilde{v}_2$ be lifts of v_1 and v_2 such that they are the endpoints of a lift of $\gamma_1\gamma_2^{-1}$. Now, the fact that Δ_1 and Δ_2 have inverse moduli at v implies that the developed images of $\widetilde{v}_1$ and $\widetilde{v}_2$ coincide and are a fixed point of $h(\gamma_1\gamma_2^{-1})$. So either the image of the holonomy is cyclic or the axis lies in the image of a developing map, but both cases are impossible because of Hypotheses H3.1.1 and H3.1.2. □

Remark 3.1.23. In the proof of Lemma 3.1.22, I used Hypotheses H3.1.1 and H3.1.2. This is not only a technical trick but it has relevant topological aspects, see Subsection 3.1.4 for more details.

3.1.3. The strategy

Let $\mathbf{z}$ be a solution of $\mathcal{C}$ and suppose that Hypothesis H3.1.1 holds. In this subsection I describe a recursive algorithm based on six steps. The algorithm, that stops in a finite time, will either get a triangulation with moduli of T with two triangles and the same holonomy of $\mathbf{z}$, or stop saying that Hypothesis H3.1.2 has been violated. I call *strategy* a complete application of the algorithm.

Remark 3.1.24. Suppose that a vertex v has no e-loops. Then, performing a TDS to two triangles having v as a vertex and that are consecutive around v, one gets:

1. $S(v)$ is decreased by 1.
2. v remains without e-loops.

Remark 3.1.25. If $S(v) = 2$ then the compatibility equations imply that the two triangles have inverse moduli at v.

Step 1. If Hypothesis H3.1.2 does not hold, then stop here. If Hypothesis H3.1.2 is satisfied, and each vertex has an e-loop, then go to Step 6. Otherwise go to Step 2.

Step 2. Let v be a vertex without e-loops, and let Δ_1 and Δ_2 be two consecutive triangles around v. If Δ_1 and Δ_2 have inverse moduli, then go to Step 3. Otherwise perform a GDS to Δ_1 and Δ_2. Then, if Hypothesis H3.1.2 does not hold stop here, otherwise repeat this Step keeping the vertex v fixed.

Remark 3.1.26. Note that by Remarks 3.1.24 and 3.1.25 one has to repeat Step 2 only a finite number of times.

Remark 3.1.27. If one reaches Step 3, then Hypothesis H3.1.2 holds. Moreover, two adjacent triangles Δ_1 and Δ_2 around v have inverse moduli.

Remark 3.1.28. With notation as in Figure 3.6, if $v_1 = v_2$ then the hypotheses of the Lemma 3.1.22 are satisfied. Then $\gamma_1\gamma_2^{-1}$ bounds an embedded disc B inside which there are no e-loops because e-loops are not contractible. Moreover if Δ_1 and Δ_2 are glued along only one edge, then B contains a vertex $w \neq v, v_1, v_3$.

Step 3. If either $v_1 \neq v_2$ or Δ_1 and Δ_2 are glued along two edges, then go to Step 4. Otherwise look only at vertices $w \in B$ with $w \neq v, v_1, v_3$ as described in Remark 3.1.28 and repeat Steps 1–3.

Remark 3.1.29. Note that performing a GDS around vertices different from v, v_1, v_3 does not change Δ_1 and Δ_2. Then, since τ is finite, one has to repeat Steps 1–3 only a finite number of times.

Recall that to have inverse moduli, geometrically means that Δ_1 and Δ_2 completely overlap with inverse orientations.
Step 4. Delete Δ_1 and Δ_2 from τ and change the pairing rules of τ in the natural way (see Figure 3.7). Then go to Step 5.

In terms of the dual graph the cancellation of Step 4 corresponds to Figure 3.7 *a*) if Δ_1 and Δ_2 are glued along one edge, and to Figure 3.7 *b*) if they are glued along two edges.

Remark 3.1.30. The choice made in Step 3 is necessary in order to avoid changes of topology of the torus. Namely, suppose that instead to repeat Steps 1-3 we go directly to the Step 4. Then the cancellation can disconnect the dual graph (see again Figure 3.7 *a*)). I will show in the next subsection that all the cancellations done in a strategy actually do not change the topology of the torus.

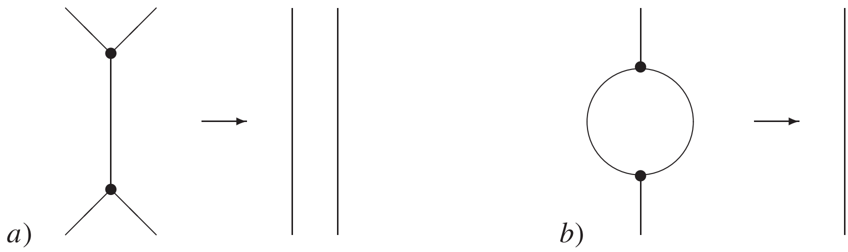


Figure 3.7. Cancellations in the dual graph.

Step 5. Repeat Steps 1–4 until each vertex has an e-loop. Then go to Step 6.

Remark 3.1.31. Note that by performing a GDS around vertices without e-loops, the number of e-loops of τ does not decrease. Then, by induction on the number of triangles and on the number of vertices without e-loops, one has to repeat Steps 1–4 only a finite number of times.

Remark 3.1.32. It is easy to see that the cancellations of Step 4 preserve the equations $\mathcal{C}$, $\mathcal{M}$ and the conjugacy class of the holonomy. Moreover a cancellation preserves also Hypothesis H3.1.2.

I call *minimal* a triangulation that has only one vertex (and then only two triangles). The last part of the algorithm consists in reducing the triangulation to a minimal one.

Step 6. Recall that when one reaches this step, Hypothesis H3.1.2 holds and each vertex has an e-loop. If the triangulation is minimal, stop here. Otherwise choose an e-loop l and an embedded closed simplicial path α so that α is a generator of $H_1(T)$ that meets l once. It follows that α meets once any e-loop parallel to l and, by Remark 3.1.18 and Corollary 3.1.21, it meets any e-loop. Since each e-loop contains only one vertex, it follows that α contains all vertices. Cutting the torus along l and α one gets a disc. The triangulation of the disc looks like the one of Figure 3.8.
Since the triangulation is not minimal, one can choose an e-loop $l' \neq l$ at a vertex v. Now try to perform a GDS to the two triangles having l' as an edge. If the move is not possible, *i.e.* if the triangles have inverse moduli at v, then cancel the triangles as in Step 4 and restart from Step 1. If the GDS is possible, then perform it. After the GDS the vertex v has no e-loops. Then restart with Step 1.

Remark 3.1.33. Note that Steps 4 and 6 involve a cancellation of two triangles. Then by induction on the number of triangles it follows that one has to apply only a finite number of steps of the algorithm.

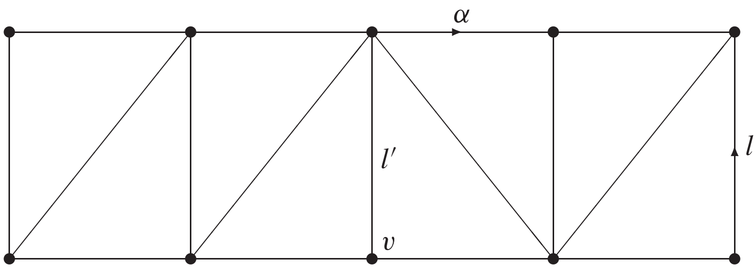


Figure 3.8. Cutting along α and l.

Remark 3.1.34. A strategy following the rules of Steps 1–6 as described stops only when either one loses Hypothesis H3.1.2 (Steps 1 and 2) or the triangulation is reduced to a minimal one (Step 6). Moreover, if one considers a strategy S as a map between triangulations with moduli, then we have that $(\tau, \mathbf{z})$ satisfies $\mathcal{C}$ (or $\mathcal{C} + \mathcal{M}$) if and only if $S(\tau, \mathbf{z})$ does.

Definition 3.1.35. I say that a strategy works if it leads to a minimal triangulation.

3.1.4. The effects of cancellations

In general, a cancellation can produce a degeneration of the topology of the torus (see Proposition 3.1.37). In this subsection I show that the cancellations that occur in a strategy do not change the topology of the torus. To prove this I simply check all possible cases. Let $\mathbf{z}$ be a solution of $\mathcal{C}$ and suppose that Hypothesis H3.1.1 holds. Suppose that during a strategy, a cancellation occurs.

Let Δ_1 and Δ_2 be the triangles that are going to be canceled. For this subsection I fix the notation of Figure 3.9. If the triangles have two common edges, then they are either both embedded or both not embedded in the torus.

If they are embedded in the torus, then the cancellation corresponds to the collapse of an embedded disc to its diameter and this does not change the topology. See Figure 3.9a).

If Δ_1 and Δ_2 have two common edges and are not embedded in the torus, then the unique possibility is that one vertex is in the interior of $\Delta_1 \cup \Delta_2$ and the other two coincide. Since the e-loops are not contractible, in this case the cancellation corresponds to the collapse of a pinched annulus to a loop, and this does not change the topology of the torus.

Now suppose that Δ_1 and Δ_2 have only one common edge. If $\Delta_1 \cup \Delta_2$ is embedded in the torus, then the cancellation corresponds again the

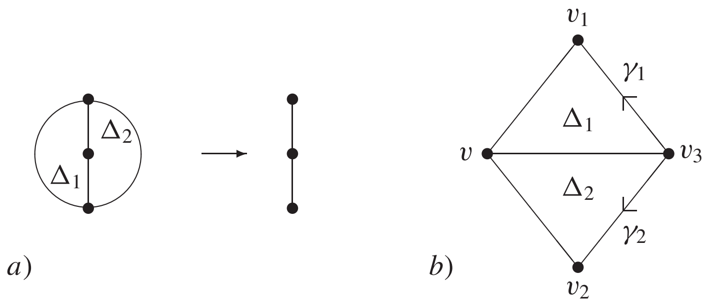


Figure 3.9. The triangles Δ_1 and Δ_2.

collapse of an embedded disc and there are no problems. If $\Delta_1 \cup \Delta_2$ is not embedded, then there are two cases. Either the cancellation occurs in a Step 4 or in a Step 6.

If the cancellation is performed in a Step 4, then v has no e-loops and one can easily see that the only possible case is that $v_1 = v_3 \neq v_2$ (or $v_2 = v_3 \neq v_1$). Then γ_1 is an e-loop so $[\gamma_1] \neq 0 \in H_1(T)$. To see that this cancellation does not change the topology of the torus see Figure 3.10.

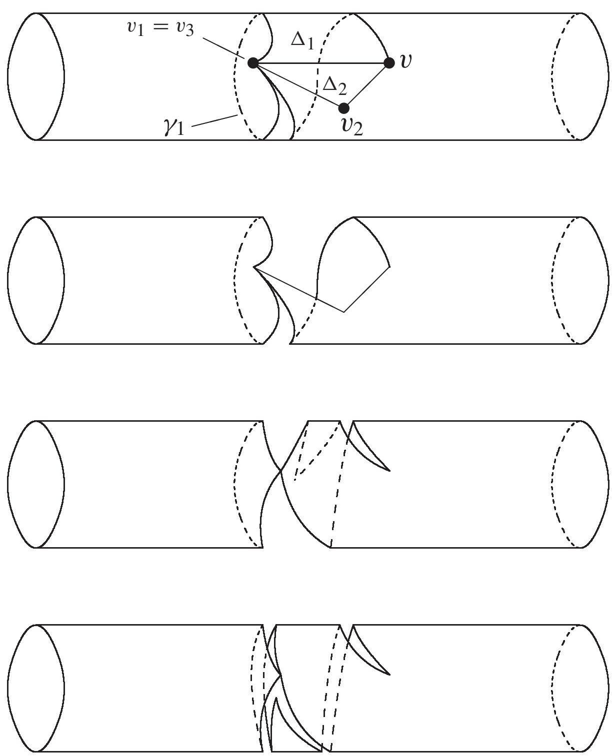


Figure 3.10. The cancellation in the case $v_1 = v_3$.

If the cancellation is performed in a Step 6 then, using hypotheses H3.1.1 and H3.1.2 as in Lemma 3.1.22, one can see that $v_1 \neq v_2$. In this case the cancellations corresponds to the collapse of a pinched annulus to a loop and this does not change the topology of the torus.

Remark 3.1.36. I used hypotheses H3.1.1 and H3.1.2 to say that no cancellations with $v_1 = v_2$ and $[\gamma_1\gamma_2^{-1}] \neq 0 \in H_1(T)$ occur. Actually this is a very bad case.

Proposition 3.1.37. *Suppose that two triangles Δ_1 and Δ_2 have inverse moduli at v and are glued along only one edge. Suppose moreover that $v_1 = v_2$ and $[\gamma_1\gamma_2^{-1}] \neq 0 \in H_1(T)$. Then a cancellation of Δ_1 and Δ_2 as in Step 4 produces a degeneration of the topology of the torus.*

Proof. Call P the space obtained from T by removing $\Delta_1 \cup \Delta_2$ and changing the gluing rules as in Figure 3.7a) (so P is the resulting space after the cancellation). Since $[\gamma_1\gamma_2^{-1}] \neq 0$, by cutting T along $\gamma_1\gamma_2^{-1}$ one obtains a cylinder. By removing $\Delta_1 \cup \Delta_2$ and changing the gluing rules, one obtains a sphere. Now in order to reconstruct P we have only to glue v_1 to v_2, so P is not a torus. □

3.1.5. Existence of similarity maps

In the previous subsection it is shown that a strategy preserves the topology of the torus. Here I show that it preserves also similarity structures. A strategy S can be viewed as a finite sequence $\{(\tau_n, \mathbf{z}_n)\}$ of triangulations with moduli of T with $(\tau_0, \mathbf{z}_0) = (\tau, \mathbf{z})$, each one obtained from the preceding via a GDS or a cancellation, depending on the steps of S.

Theorem 3.1.38. *Let $\mathbf{z}$ be a solution of $\mathcal{C}$ and suppose Hypotheses* H3.1.1 *and* H3.1.2 *hold. Suppose that one follows a strategy $S = \{(\tau_n, \mathbf{z}_n)\}$ which works and let $(\overline{\tau}, \overline{\mathbf{z}})$ be the minimal triangulation with moduli obtained via S. If there exists a torus T' endowed with a similarity structure and a degree-one map $\overline{\varphi} : T \to T'$ which is a similarity map w.r.t. $\overline{\mathbf{z}}$, then for each n there exists a degree-one map $\varphi_n : T \to T'$ which is a similarity map w.r.t. $\mathbf{z}_n$ and such that φ_{n+1} agrees with φ_n on the simplices that are not changed during the n-th step. In particular $\mathbf{z}$ is a geometric solution of $\mathcal{C}$. Moreover, the structure of T' does not depend on the strategy used.*

Proof. I construct the similarity maps by following backward the steps of the strategy. Let $\widetilde{T}'$ be the universal covering of T' and let $D' : \widetilde{T}' \to \mathbb{C}$ be a developing map for its similarity structure. The map $\overline{\varphi}$ exists by hypothesis. Suppose that φ_{n+1} exists.

First, suppose that τ_{n+1} is obtained from τ_n via a cancellation, and let Δ_1 and Δ_2 be the canceled triangles. Define $\varphi_n = \varphi_{n+1}$ outside Δ_1 and Δ_2. If $A \subset \widetilde{T}$ is the set of the lifts of Δ_1 and Δ_2, the map φ_n lifts to a map

$$\widetilde{\varphi}_n : \widetilde{T} \setminus A \to \widetilde{T}'$$

such that the restriction of $D' \circ \widetilde{\varphi}$ to each triangle $\widetilde{\Delta}_i$ of $\widetilde{\tau}$ is compatible with z_i, and it is readily checked that $D' \circ \widetilde{\varphi}_n$ extends to a map

$$D_n : \widetilde{T} \to \mathbb{C}$$

which is a developing map for $(\tau_n, \mathbf{z}_n)$. Since Hypothesis H3.1.2 holds during the strategy, the image of D_n does not contain the axis of the holonomy, thus the map D_n splits along a map $\widetilde{\varphi}_n : \widetilde{T} \to \widetilde{T}'$ such that $D' \circ \widetilde{\varphi}_n = D_n$

$$D_n : \widetilde{T} \xrightarrow{\widetilde{\varphi}_n} \widetilde{T}' \xrightarrow{D'} \mathbb{C}.$$

Such a $\widetilde{\varphi}_n$ projects to the requested similarity map $\varphi_n : T \to T'$.

Now suppose that τ_{n+1} is obtained from τ_n via a GDS replacing two triangles, say Δ_1 and Δ_2, with two new triangles Δ_1' and Δ_2'. Define $\varphi_n = \varphi_{n+1}$ outside $\Delta_1 \cup \Delta_2$. Note that $\Delta_1 \cup \Delta_2 = \Delta_1' \cup \Delta_2'$ and proceed exactly as above. By induction, for any n there exists a similarity map $\varphi_n : T \to T'$ with the requested properties. Regarding the degree of such maps, note that neither a cancellation, nor a GDS can affect the degree of φ_n.

Now I prove the second assertion. Let $\mathbb{X}$ be the universal covering of the image of D', equipped with the pull-back similarity structure. The map D' lifts to a map $\widetilde{D}' : \widetilde{T}' \to \mathbb{X}$. It turns out that $\widetilde{D}'$ is a global homeomorphism. Moreover, since D' is a developing map for T', the holonomy h' of T' lifts to a representation

$$\widetilde{h}' : H_1(T) \to \mathrm{Aff}(\mathbb{X})$$

such that $\widetilde{D}'$ is $\widetilde{h}'$-equivariant. As a $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-space, the torus T' is isomorphic to the quotient of $\mathbb{X}$ under the action of the image of $\widetilde{h}'$. Similarly, $h(\overline{\mathbf{z}})$ lifts to a representation

$$\widetilde{h}(\overline{\mathbf{z}}) : H_1(T) \to \mathrm{Aff}(\mathbb{X}).$$

By Proposition 2.4.24 the holonomy $h(\overline{\mathbf{z}})$ for $\overline{\mathbf{z}}$ is the composition of $\overline{\varphi}_*$ with h', so $\widetilde{h}(\overline{\mathbf{z}}) = \widetilde{h}' \circ \overline{\varphi}_*$. Since $\deg(\varphi) = 1$, the map $\overline{\varphi}_*$ is an isomorphism. It follows that the image of $\widetilde{h}'$ is the same as that of $\widetilde{h}(\overline{\mathbf{z}})$. Finally, observe that a cancellation does not change the image of the holonomy, and by Corollary 3.1.8 also a GDS does not change the image of the holonomy. So

$$\widetilde{h}(\overline{\mathbf{z}}) = \widetilde{h}(\mathbf{z}_n)$$

for any n, and in particular for $n = 0$, *i.e.* for the initial triangulation with moduli. It follows that the structure of T' is independent on the strategy used. □

Remark 3.1.39. One can easily obtain a partial converse of this theorem. Namely, if there exists a similarity map $\varphi : T \to T'$ where T' is a torus with a similarity structure, then $\mathcal{C}$ and Hypothesis H3.1.2 hold, but in general Hypothesis H3.1.1 can be violated (see Example 3.1.43).

Remark 3.1.40. If $\mathbf{z}$ is a solution of equations $\mathcal{C}$ then in general each triangle of τ only has a well-defined similarity structure, so its size is not well-defined. If $\mathbf{z}$ is a solution of $\mathcal{C} + \mathcal{M}$, then one can coherently choose the sizes of the triangles of $\widetilde{\tau}$. It follows that the algebraic sum A of the areas of the triangles of τ with moduli $\mathbf{z}$ is well-defined up to multiplication by a positive factor.

Corollary 3.1.41. *Suppose* $\mathbf{z}$ *is a solution of equations* $\mathcal{C}$ *and* $\mathcal{M}$ *and let* A *be as in Remark* 3.1.40. *If* $A \neq 0$, *then either* $\mathbf{z}$ *or its conjugate is a geometric solution of* $\mathcal{C} + \mathcal{M}$.

Proof. Up to conjugating all the z_i's, I can suppose that $A > 0$. Since $\mathbf{z}$ is a solution of $\mathcal{M}$, then the holonomy has no axis, so Hypothesis H3.1.2 is always satisfied. Suppose now that Hypothesis H3.1.1 is not satisfied. Since the image $\mathrm{Im}(h)$ of the holonomy consists of translations, it is isomorphic either to $\mathbb{Z}$ or to the trivial group. It follows that $\mathrm{Im}(h)$ acts freely and properly discontinuously on $\mathbb{C}$. Let C be the quotient of $\mathbb{C}$ under the action of $\mathrm{Im}(h)$. Because of equivariance, any developing map $D : \widetilde{T} \to \mathbb{C}$ projects to a map $\mathcal{D} : T \to C$. If ω is the area-form on C induced by $\mathbb{C}$, then

$$A = \int_T \mathcal{D}^*\omega = \int_{\mathcal{D}(T)} \deg(\mathcal{D})\omega.$$

As T is compact and C is not compact, $\deg(\mathcal{D}) = 0$. Thus $A = 0$: a contradiction. It follows that Hypothesis H3.1.1 is satisfied, so any strategy works. Moreover, for a minimal triangulation obtained via a strategy, equations $\mathcal{M}$ imply that the moduli of the two triangles lie both in π_+. Then, by Proposition 2.4.25, $\mathbf{z}$ is a geometric solution of $\mathcal{C}+\mathcal{M}$. □

Remark 3.1.42. To prove these results I used Hypotheses H3.1.1 and H3.1.2 in a crucial way. As seen in the proof of Theorem 3.1.38, Hypothesis H3.1.2 is necessary in order to have similarity maps, while Hypothesis H3.1.1 is necessary only in order to apply a strategy. Namely, if Hypothesis H3.1.1 is not satisfied, then it may be that **z** defines a similarity structure on T, but one cannot use a strategy to find similarity maps.

Example 3.1.43. Take $\mathbb{C} \setminus \{0\}$ and make the quotient by the multiplication by 2. This quotient is a torus equipped with a similarity structure. Triangulate a fundamental domain as in Figure 3.11 and choose the moduli **z** for the triangles in the obvious way. It is clear that **z** is a solution of $\mathcal{C}$ that does not satisfy Hypothesis H3.1.1.

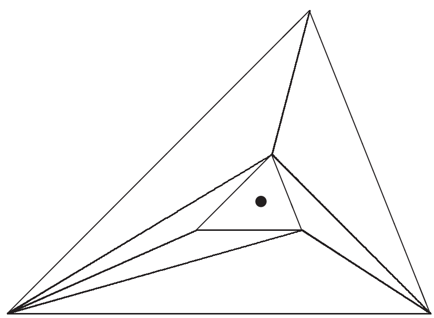


Figure 3.11. A similarity structure without Hypothesis H3.1.1.

If one tries to apply a strategy, one loses Hypothesis H3.1.2 at the first steps.

3.2. Algebraic conditions on the moduli

In this section I give a complete characterization of the geometric solutions of $\mathcal{C}$ and $\mathcal{C} + \mathcal{M}$ for the torus T. Let **z** be a solution of $\mathcal{C}$ and let $D : \widetilde{T} \to \mathbb{C}$ be a developing map for **z**. Let $h : H_1(T) \to \mathrm{Aff}(\mathbb{C})$ be a representative of the holonomy such that D is h-equivariant. I fix also a $\mathbb{Z}$-basis (γ_1, γ_2) of $H_1(T)$. By Lemma 2.5.2 there are two cases:

1) $h(\gamma_1)$ and $h(\gamma_2)$ are both translations.
2) The holonomy has an axis. In this case I always suppose that the axis is the point 0, so Hypothesis H3.1.2 translates to “$0 \in \mathrm{Im}(D)$”.

Definition 3.2.1. Let $\mathbb{X}$ be defined as follows. In case 1) let $\mathbb{X} = \mathbb{C}$, considered as the universal covering of itself. In case 2) let $\mathbb{X}$ be the universal covering of $\mathbb{C}_*$. The space $\mathbb{X}$ is equipped with the pull-back similarity structure and the group of the similarities of $\mathbb{X}$ is denoted by $\mathrm{Aff}(\mathbb{X})$.

Remark 3.2.2. In both cases $\mathbb{X} = \mathbb{C}$. In case 1) the covering map is the identity and the similarity structure on $\mathbb{X}$ is the usual one. In case 2) the covering map $\mathbb{X} \to \mathbb{C}_*$ is the usual exponential map $\exp : \mathbb{C} \to \mathbb{C}_*$, the similarity structure on $\mathbb{C}_*$ is the usual one, while the similarity structure I consider on $\mathbb{X}$ is not the usual one.

Proposition 3.2.3. *In both cases* 1) *and* 2) *the translations of* $\mathbb{X}$ *belong to* $\mathrm{Aff}(\mathbb{X})$.

Proof. There is nothing to say in case 1). In case 2) the thesis follows because, a translation $\eta(x) = x + \xi$ projects to the map $\theta(y) = ye^{\xi}$ that is a similarity of $\mathbb{C}_*$. □

Proposition 3.2.4. *Suppose that Hypothesis* H3.1.2 *holds. Then the map* D *lifts to a map* $\widetilde{D} : \widetilde{T} \to \mathbb{X}$ *and the representation* h *to a representation* $\widetilde{h} : H_1(T) \to \mathrm{Aff}(\mathbb{X})$ *such that* $\widetilde{D}$ *is* $\widetilde{h}$*-equivariant. Moreover the image of* $\widetilde{h}$ *consists of translations.*

Proof. This is tautological in case 1). In case 2), the map $\widetilde{D}$ exists because $0 \notin \mathrm{Im}(D)$, and $\widetilde{h}$ is defined as follows. For each $\gamma \in H_1(T)$ and $x \in \widetilde{T}$

$$\begin{aligned}\exp(\widetilde{D}(\gamma x)) &= D(\gamma x) = h(\gamma)(D(x)) = h(\gamma)(1) \cdot D(x)\\ &= h(\gamma)(1) \cdot \exp(\widetilde{D}(x))\end{aligned}$$

Thus $\widetilde{D}(\gamma x) = \widetilde{D}(x) + \xi(\gamma, x)$ with $\exp(\xi(\gamma, x)) = h(\gamma)(1)$. The function $\xi(\gamma, x)$, as a function of x, is continuous from a connected set to a discrete set and then it is constant. It follows that $\xi(\gamma, x) = \xi(\gamma)$. The function $\gamma \mapsto \widetilde{h}(\gamma)$ defined by $\widetilde{h}(\gamma)(x) = x + \xi(\gamma)$ is the requested representation. The second claim immediately follows. □

Definition 3.2.5. When Hypothesis H 3.1.2 holds, with the notation used in the proof of Proposition 3.2.4, I set

$$\widetilde{h}(\gamma_i)(x) = x + \xi_i, \quad i = 1, 2.$$

I say that $\widetilde{h}$ has rank 2 over $\mathbb{R}$ if ξ_1 and ξ_2 are linearly independent over $\mathbb{R}$.

Remark 3.2.6. The previous definition is equivalent to saying that $\xi_1/\xi_2 \notin \mathbb{R} \cup \{\infty\}$ or that, if one sets $\xi_k = x_k + iy_k$, $k = 1, 2$ then $x_1 y_2 \neq x_2 y_1$.

Remark 3.2.7. One can easily see that the conditions that $0 \notin \mathrm{Im}(D)$ and that $\widetilde{h}$ has rank 2 over $\mathbb{R}$ do not depend on the choice of the developing map D.

Theorem 3.2.8. *Let* $\mathbf{z}$ *be a solution of* $\mathcal{C}$ *and suppose Hypothesis* H 3.1.2 *holds. If* $\widetilde{h}$ *has rank* 2 *over* $\mathbb{R}$ *then either* $\mathbf{z}$ *or its conjugate is a geometric solution of* $\mathcal{C}$*. Moreover, in case* 1)*,* $\mathbf{z}$ *or its conjugate is a geometric solution of* $\mathcal{C}+\mathcal{M}$*.*

Proof. Since ξ_1 and ξ_2 are linearly independent, the action of $H_1(T)$ on $\mathbb{X}$ via $\widetilde{h}$ is free and properly discontinuous. So $X=\mathbb{X}/\widetilde{h}$ is well-defined and is a torus with a similarity structure, and such a structure is Euclidean if and only if case 1) holds. Moreover the map $\widetilde{D}$ projects to a well-defined map $f : T \to X$. Obviously f is a similarity map. Moreover, the homotopy class of f is completely determined by f_*. Since f_* is an isomorphism between $\pi_1(T)$ and $\pi_1(X)$, it is easy to construct a map $g : T \to X$ of degree ± 1 such that $g_* = f_*$. Then f is homotopic to g and thus has degree ± 1.

If f has degree one, then $\mathbf{z}$ is a geometric solution of $\mathcal{C}$ (or $\mathcal{C}+\mathcal{M}$ in case 1)). If f has degree -1 then by changing each z_i with $\overline{z}_i$ one gets a geometric solution of $\mathcal{C}$ ($\mathcal{C}+\mathcal{M}$ in case 1)). □

The converse of Theorem 3.2.8 is also true, so its hypotheses are necessary and sufficient conditions for $\mathbf{z}$ to be a geometric solution of $\mathcal{C}$ (or $\mathcal{C}+\mathcal{M}$).

Theorem 3.2.9. *Suppose that* $\mathbf{z}$ *is a geometric solution of* $\mathcal{C}$*. Then Hypothesis* H3.1.2 *holds and* $\widetilde{h}$ *has rank* 2 *over* $\mathbb{R}$*. If in addiction either* $\mathbf{z}$ *or its conjugate is a geometric solution of* $\mathcal{C}+\mathcal{M}$*, then case* 1) *holds.*

Proof. Since either $\mathbf{z}$ or its conjugate is a geometric solution of $\mathcal{C}$ (or $\mathcal{C}+\mathcal{M}$), and since to change each z_i with $\overline{z}_i$ is equivalent to change the orientation of T, there exists a torus T' endowed with a similarity structure and a similarity map w.r.t. $\mathbf{z}$ $f : T \to T'$ which has degree ± 1.

Let h' be the holonomy of T' and let $D' : \widetilde{T}' \to \mathbb{C}$ be a developing map for T'. If h' has an axis, suppose that it is the point 0. As in Proposition 3.2.4 the map D' lifts to a map $\widetilde{D}' : \widetilde{T}' \to \mathbb{X}$ and the holonomy h' to a map $\widetilde{h}'$ such that $\widetilde{D}'$ is $\widetilde{h}'$-equivariant (see Figure 3.12), and one can easily check that $T' = \mathbb{X}/\widetilde{h}'$. Since $D' \circ \widetilde{f}$ is a developing map for $\mathbf{z}$, it is no restrictive to suppose that $D = D' \circ \widetilde{f}$, so that $\widetilde{D} = \widetilde{D}' \circ \widetilde{f}$. By Proposition 2.4.24, $h(\mathbf{z}) = h' \circ f_*$, and from the constructions of $\widetilde{h}$ and $\widetilde{h}'$ it follows that

$$\widetilde{h} = \widetilde{h}' \circ f_*.$$

If T' is a Euclidean torus, then case 1) holds, otherwise case 2) holds and it is readily checked that Hypothesis H3.1.2 is satisfied. Since $\deg(f) = \pm 1$, f_* is an isomorphism, so the image of $\widetilde{h}'$ and $\widetilde{h}$ coincide. This implies

$$
\begin{tikzcd}
 & & \mathbb{X} \arrow[d] \\
\widetilde{T} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{T}' \arrow[r, "D'"] \arrow[ur, "\widetilde{D}'"] \arrow[d] & \mathbb{C} \\
T \arrow[r, "f"] & T' &
\end{tikzcd}
$$

Figure 3.12. The lifts of f and D'.

that T' is the quotient of $\mathbb{X}$ under the image of $\widetilde{h}$. Suppose that ξ_1 and ξ_2 are linearly dependent over $\mathbb{R}$. Then the image of $\widetilde{h}$ is isomorphic to a subgroup of $\mathbb{R}$, therefore it is either cyclic or dense in $\mathbb{R}$. In both cases T' cannot be a torus. $\square$

Theorem 3.2.8 and its converse 3.2.9 completely characterize the geometric solutions of $\mathcal{C}$ or $\mathcal{C} + \mathcal{M}$. Proposition 2.4.30 says that the set of geometric solution of $\mathcal{C}$ is a subset of the set of the algebraic solutions of $\mathcal{C}$. The following proposition refines this statement.

Proposition 3.2.10. *The set of geometric solutions of $\mathcal{C}$ is open in the set of algebraic solutions of the system $\mathcal{C}$.*

Proof. Let $\mathbf{z}$ be a geometric solution of $\mathcal{C}$ and let $\mathbf{z}'$ be a solution of $\mathcal{C}$ sufficiently close to $\mathbf{z}$. Let D and D' be the corresponding developing maps, let h and h' be the holonomies, let $\widetilde{D}$, $\widetilde{D}'$, $\widetilde{h}$ and $\widetilde{h}'$ be their lifts as in Proposition 3.2.4, and let ξ_1, ξ_2, ξ_1', ξ_2' be as in Definition 3.2.5.

First, suppose that the holonomy relative to $\mathbf{z}$ has an axis. Since to have an axis that lies outside the image of a developing map is an open condition, it follows that also h' has an axis outside the image of a developing map. Moreover, also the map $\widetilde{D}'$ is close to the map $\widetilde{D}$, so ξ_1' and ξ_2' are close to ξ_1 and ξ_2. Since the condition on the ξ_i's is an open one, then also $\widetilde{h}'$ has rank 2 over $\mathbb{R}$, and the thesis follows from Theorem 3.2.8.

Now suppose that h consists of translations. If also h' consists of translations, then as above $\widetilde{h}'$ has rank 2 over $\mathbb{R}$. Suppose that h' has an axis. I have to check that the axis lies outside the image of D' and that $\widetilde{h}'$ has rank 2 over $\mathbb{R}$. Since h consists of translations, if $\mathbf{z}'$ is sufficiently close to $\mathbf{z}$ then the axis of h' is sufficiently far from 0. Then the axis of h' lies outside the developed image of a fundamental domain and then it lies outside the image of D'. I check now that $\widetilde{h}'$ has rank 2 over $\mathbb{R}$. Since $\mathbf{z}'$

is close to $\mathbf{z}$

$$h'(\gamma_1)(x) = (1+\beta_1)x + \eta_1$$
$$h'(\gamma_2)(x) = (1+\beta_2)x + \eta_2$$

with $\beta_i \sim 0$ and $\eta_i \sim \xi_i$. From Abelianity it follows that $\beta_1/\beta_2 = \eta_1/\eta_2$. With this notation $e^{\xi_i} = 1+\beta_i$. Using the determination of the logarithm such that $\log(x)$ has imaginary part in $(-\pi,\pi)$ it turns out that $\xi_i = \log(1+\beta_i)$.
Thus for $\mathbf{z}'$ sufficiently close to $\mathbf{z}$ one has

$$\frac{\xi_1'}{\xi_2'} = \frac{\log(1+\beta_1)}{\log(1+\beta_2)} \sim \frac{\beta_1}{\beta_2} = \frac{\eta_1}{\eta_2} \sim \frac{\xi_1}{\xi_2} \notin \mathbb{R} \cup \{\infty\}.$$

Therefore $\widetilde{h}'$ has rank 2 over $\mathbb{R}$, and the thesis follows from Theorem 3.2.8. □

If $\mathbf{z}$ is an algebraic solution of $\mathcal{C}+\mathcal{M}$, then the hypothesis that $\widetilde{h}$ has rank 2 over $\mathbb{R}$ can be easily checked as the following proposition shows. I recall that if $\mathcal{M}$ holds then the algebraic area A as in Remark 3.1.40 is well-defined up to multiplication by a positive factor.

Proposition 3.2.11. *Suppose that* $\mathbf{z}$ *is a solution of* $\mathcal{C}+\mathcal{M}$. *Then* $\widetilde{h}$ *has rank 2 over* $\mathbb{R}$ *if and only if* $A \neq 0$.

Proof. The *if* part follows from Corollary 3.1.41 and Theorem 3.2.9. The *only if* part follows from Theorem 3.2.8 and from the fact that if $f : T \to T'$ is a degree-one similarity map from T to a Euclidean torus T', then A can be calculated as

$$A = \int_T f^*\omega = \int_{\mathrm{Im}(f)} \deg(f)\omega = \mathrm{area}(T') \neq 0$$

where ω is the area-form of T'. □

3.3. Similarity structures on the Klein bottle

In this section I show how the existence of similarity structures on the Klein bottle can be reduced to a problem on the torus.

Let K be the Klein bottle and let $\pi : T \to K$ be the orienting double covering. Let θ be a triangulation of K and let τ be the lift of θ, where I fix a global orientation, that is an orientation for each triangle which is compatible with a global orientation of T. I fix an orientation for each triangle of θ (clearly such orientations cannot be coherent because K is not orientable). Each triangle Δ of θ is covered by exactly two triangles of τ, mapped to Δ one with the opposite orientation to the other.

Each choice of moduli $\mathbf{z}_K$ for θ induces a choice of moduli $\mathbf{z}$ on τ in a natural way. Namely, if $\pi^{-1}(\Delta) = \Delta_1 \cup \Delta_2$, if π preserves the orientation of Δ_1 and z is the modulus of Δ, then the modulus of Δ_1 is z and that of Δ_2 is $\bar{z}$.

Let J be a Klein bottle endowed with a similarity structure and let $p : Y \to J$ be the orienting double covering. The structure of J lifts to a structure of the torus Y. As Lemma 3.3.2 shows, any continuous map $f : K \to J$ lifts to a map φ between the orienting tori.

I say that a map $f : K \to J$ is a *similarity map* if its lift φ is a similarity map (I use this definition because the definition of similarity map I previously used involves the integer degree, which is not defined in a non-oriented setting). I say that $\mathbf{z}_K$ is a *geometric solution* of $\mathcal{C}$ if there exists a Klein bottle J endowed with a similarity structure and a similarity map $f : K \to J$ such that its lift φ has degree one, and I say that $\mathbf{z}_K$ is a *geometric solution* of $\mathcal{C}+\mathcal{M}$ if the structure of J is Euclidean. The following theorem tautologically follows from Lemma 3.3.2 and the definition of similarity map for a Klein bottle.

Theorem 3.3.1. *The choice of moduli $\mathbf{z}_K$ is a geometric solution of $\mathcal{C}$ (or $(\mathcal{C}+\mathcal{M})$ for K if and only if $\mathbf{z}$ is a geometric solution of $\mathcal{C}$ (or $\mathcal{C}+\mathcal{M}$) for T.*

Lemma 3.3.2. *Let $f : K \to J$ be a continuous map between two Klein bottles and let $\pi : T \to K$ and $p : Y \to J$ be their double orienting coverings. Then f lifts to a map φ such that the following diagram is commutative.*

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Y \arrow[d, "p"] \\ K \arrow[r, "f"] & J \end{tikzcd}$$

Proof. Since there are no ambiguities, I denote both $\pi_1(K)$ and $\pi_1(J)$ by $\langle a, b; abab^{-1}\rangle$. So one has the commutation rule

$$ab = ba^{-1}.$$

Each element in $\pi_1(K)$ can be written in a unique way in the form $b^\beta a^\alpha$. Now $\pi_*(\pi_1(T)) \subset \pi_1(K)$ is the set of elements of the form $b^{2k}a^x$ and the same holds for $p_*(\pi_1(Y)) \subset \pi_1(J)$. The claimed map φ exists if and only if $f_*\pi_*(\pi_1(T)) \subset p_*(\pi_1(Y))$. Then I only have to check that $f_*\pi_*(\gamma)$ is of the form $b^{2k}a^x$ for all $\gamma \in \pi_1(T)$.

Let $f_*(a) = b^m a^n$ and $f_*(b) = b^s a^t$. Since f_* is a homomorphism

$$1 = f(abab^{-1}) = b^m a^n b^s a^t b^m a^n b^{-s} a^{(-1)^{s+1}t}$$

and it easily follows that $m = 0$. Finally, since $(b^p a^q)^r = b^{rp} a^{\text{something}}$, one has $f_*(b^{2k}a^x) = b^{2ks}a^{\text{something}}$ which is of the requested form for all $k, x \in \mathbb{Z}$. $\square$

Chapter 4
Geometric solutions vs algebraic ones in dimension three

In this chapter I deal with the geometric solutions of $\mathcal{C}$ and $\mathcal{C} + \mathcal{M}$ in dimension 3. In the first section I show that there is a duality between the representations of the fundamental group of a given ideally triangulated 3-manifold and the (algebraic) solutions of $\mathcal{C}$ for such a manifold. Then I compare the geometric solutions with the algebraic ones. I show that the set of geometric solutions of $\mathcal{C}$ is an open subset of the set of algebraic ones, that is

$$\text{Algebraic close to geometric} \;\Rightarrow\; \text{geometric.}$$

This means that if M is a hyperbolic manifold, one can think of the space of deformations of the structure of M as the set of the algebraic solutions of $\mathcal{C}$. This also gives another way to see the space of generalized Dehn filling coefficients. Then I show that the geometric solutions of $\mathcal{C} + \mathcal{M}$ (or $\mathcal{C}$ + hyperbolic Dehn filling equations) are unique.

In the second section of this chapter I do explicit calculations for three interesting examples, showing that in general an algebraic solution of $\mathcal{C} + \mathcal{M}$ is not geometric. I first study two one-cusped manifolds, namely two bundles over S^1 called LR^3 and L^2R^3 with a punctured torus as a fiber. These manifolds admit non-unique algebraic solutions and a (unique) geometric one. I notice that some of these "bad" solutions do not involve flat tetrahedra and have a good behavior on the boundary. Namely, the boundary torus inherits an intrinsic Euclidean structure (up to scaling). This shows that there is a deep difference between the two-dimensional case and the three-dimensional one. Then I study a manifold with non-trivial JSJ decomposition, obtained by gluing a Seifert manifold to the complement of the figure-eight knot. This manifold is not hyperbolic but it admits a partially flat solution of the compatibility and completeness equations. Such a solution cannot be geometric as the manifold is not hyperbolic. This shows that the equations on the angles are necessary in Theorem 2.6.3.

4.1. Geometric solutions of $\mathcal{C}$ and hyperbolic Dehn filling equations

Notation. For this section M will be a cusped manifold, equipped with an ideal triangulation $\tau = (\{\Delta_i\}, \{r_j\})$. Let $\partial \overline{M} = \{T_1, \dots, T_k\}$, where the T_n's are the boundary tori. The symbol $\mathbf{z}$ will denote a choice of moduli for τ and when $\mathbf{z}$ is a solution of $\mathcal{C}$, $h(\mathbf{z})$ will denote its holonomy. For each isometry $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3)$ let $\mathrm{Fix}(\gamma)$ denote the set of the points of $\overline{\mathbb{H}}^3$ fixed by γ. For a subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ set $\mathrm{Fix}(\Gamma) = \cap_{\gamma\in\Gamma}\mathrm{Fix}(\gamma)$. To simplify notations, I often omit to indicate the base-points for the fundamental groups. For any boundary torus T_n, I assume that a representative $\pi_1(T_n) < \pi_1(M)$ of the conjugacy class of its fundamental group has been fixed.

For any boundary torus T_n I fix a basis (μ_n, λ_n) for $H_1(T_n, \mathbb{Z})$. The symbol (p, q) will denote a set $\{(p_n, q_n)\}$ of Dehn filling coefficients as in Definition 2.5.8. The manifold $M_{(p,q)}$ will be the Dehn filling of M with coefficients (p, q) and γ_n will be the core of the n-th filling torus.

In this section first I prove that the set of geometric solutions of $\mathcal{C}$ is open in the set of algebraic solutions of $\mathcal{C}$, then I prove that there exists at most one geometric solution of the (p, q)-equations. This will follow from the fact that a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ determines a choice of moduli for τ, and that such a choice is essentially unique. The geometric solutions will be unique because of the rigidity of hyperbolic manifolds. I will need the following strong statement of the rigidity (compare with Theorem 1.1.7).

Theorem 4.1.1. (Strong statement of Mostow-Prasad rigidity) *Let M_1 and M_2 be two complete connected hyperbolic 3-manifolds of finite volume. Let $f : M_1 \to M_2$ be a continuous proper map such that*

$$\mathrm{vol}(M_1) = |\deg(f)|\mathrm{vol}(M_2).$$

Then f is properly homotopic to a locally isometric covering of degree $\deg(f)$ *of M_1 onto M_2.*

A proof of this result can be found in [3], and a different proof in [10]. In Chapter 5 below I will give a proof using the techniques of [10].

Lemma 4.1.2. *If $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is Abelian, then* $\mathrm{Fix}(\Gamma)$ *is not empty. Moreover,*

1. $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *is infinite if and only if* $\Gamma = \{\mathrm{Id}\}$.
2. $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3 = \emptyset$ *if and only if Γ is a dihedral group generated by two rotations of angle π around orthogonal axes.*

3. $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *contains a single point if and only* Γ *contains only parabolic isometries.*
4. *Otherwise* $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *contains exactly two points.*

Proof. If $\Gamma = \{\mathrm{Id}\}$, then $\mathrm{Fix}(\Gamma) = \overline{\mathbb{H}}^3$. From the classification of the hyperbolic isometries (see Section 1.1) it follows that if $\varphi \in \mathrm{Isom}^+(\mathbb{H}^3)$ is different from the identity, then $\mathrm{Fix}(\varphi) \cap \partial\mathbb{H}^3$ consists of either one or two points. Therefore if Γ is not trivial, then $\mathrm{Fix}(\Gamma)$ is finite, and this proves point (1) of the second claim.

Now suppose that Γ is not trivial. For any $\varphi_1, \varphi_2 \in \Gamma$, from the Abelianity of Γ it follows that $\mathrm{Fix}(\varphi_1)$ is φ_2-invariant. So $\mathrm{Fix}(\varphi)$ is Γ-invariant for every $\varphi \in \Gamma$. Therefore, if Γ contains a parabolic element φ, then all the element of Γ are parabolic and $\mathrm{Fix}(\Gamma) = \mathrm{Fix}(\varphi) \subset \partial\mathbb{H}^3$ and conversely. This proves point (3) of the second claim.

Now suppose that Γ contains no parabolic isometries. Let $\mathrm{Id} \neq \gamma \in \Gamma$ and let $x_0 \neq x_1$ be its unique fixed points in $\partial\mathbb{H}^3$. For every $\varphi \in \Gamma$ and $i \in \mathbb{Z}/2\mathbb{Z}$, either $\varphi(x_i) = x_i$ or $\varphi(x_i) = x_{i+1}$.

Suppose that $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3 = \emptyset$. Then there exists $\varphi \in \Gamma$ such that

$$\varphi(x_0) = x_1 \qquad \varphi(x_1) = x_0.$$

Since φ is an isometry, the geodesic $\overline{x_0 x_1}$ is φ-invariant, so there exists a point $x \in \mathrm{int}(\overline{x_0 x_1})$ such that

$$\varphi(x) = x.$$

Using the Abelianity, by induction one gets that for every n

$$\varphi(\gamma^n(x)) = \gamma^n(x).$$

By continuity this implies that, if $\gamma(x) \neq x$, then $\varphi(x_0) = x_0$ and $\varphi(x_1) = x_1$. Therefore $\gamma(x) = x$, so γ is elliptic, and more precisely it is a rotation around the geodesic $\overline{x_0 x_1}$. Interchanging φ and γ, one sees that also φ is a rotation, with an axis orthogonal to $\overline{x_0 x_1}$, and that both φ and γ are rotations of angle π. This proves that if $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3 = \emptyset$, then Γ consists of rotations of angle π around orthogonal axes. So $\gamma^2 = \mathrm{Id}$ for each element of Γ and it is easily checked that Γ is the dihedral group generated by two rotations. In particular all the axes intersect in a point which is the unique fixed point of Γ. This proves the first claim and points (2) and (4) of the second claim, and the proof is complete. □

Proposition 4.1.3. *Let* $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ *be a representation. Suppose that for any boundary torus* T_n*,* $\rho(\pi_1(T_n))$ *is not dihedral. Then the set* $\mathcal{D}_\rho$ *of* ρ*-equivariant maps from the ideal points of* $\widehat{\widetilde{M}}$ *to* $\partial\mathbb{H}^3$ *is not empty.*

Proof. I show how to construct an element D of $\mathcal{D}_\rho$. Let q be an ideal point of $\widehat{\widetilde{M}}$. The stabilizer Stab(q) of q in $\pi_1(M)$ is conjugated to the fundamental group of some boundary torus. It follows that $\rho(\mathrm{Stab}(q))$ is not dihedral, so by Lemma 4.1.2 it has at least one fixed point x in $\partial\mathbb{H}^3$. Define $D(q) = x$ and extend D to the $\pi_1(M)$-orbit of q by equivariance. Do the same for the remaining ideal points. $\square$

Lemma 4.1.4. *All the elements of $\mathcal{D}_\rho$ are obtained as in the proof of Proposition 4.1.3.*

Proof. Because of equivariance, $D(q) \in \mathrm{Fix}\big(\rho(\mathrm{Stab}(q))\big)$ for any ideal point q. $\square$

Proposition 4.1.5. *In the hypotheses of Proposition* 4.1.3*, suppose in addition that the ρ-images of the fundamental groups of all the boundary tori are not trivial. Then $\mathcal{D}_\rho$ is finite. Moreover, $\mathcal{D}_\rho$ consists of one element if and only if the ρ-images of the fundamental groups of all the boundary tori are parabolic.*

Proof. Let T_n be a boundary torus. Since $\pi_1(T_n)$ is Abelian, by Lemma 4.1.2 if $\rho(\pi_1(T_n))$ is not trivial then it has one or two fixed points in $\partial\mathbb{H}^3$. Thus, when one has to choose the image of an ideal point, one has at most two possibilities. Since the ideal points of $\widehat{M}$ are finite in number, then in $\widehat{\widetilde{M}}$ there is only a finite number of $\pi_1(M)$-orbits of ideal points, so one has to make only a finite number of choices. The second claim directly follows from point (3) of Lemma 4.1.2. $\square$

In the sequel, let the symbol $*$ denote the *degenerate* modulus, with the meaning that an ideal tetrahedron has modulo $*$ if and only if it is a degenerate tetrahedron (it has two ore more coincident vertices).

Theorem 4.1.6. (Representations determine moduli) *In the hypotheses of Proposition* 4.1.3*, each element D of $\mathcal{D}_\rho$ naturally induces a choice of moduli $\mathbf{z}_D$ in* $(\mathbb{C} \setminus \{0, 1\}) \cup \{*\}$*. If $\mathbf{z}_D$ contains no $*$-moduli, then it is an algebraic solution of $\mathcal{C}$ with holonomy ρ.*

Proof. The moduli $\mathbf{z}_D$ are defined simply by choosing, for each Δ_i of τ, the modulus of the convex hull of the D-image of the vertices of any lift $\widetilde{\Delta}_i$ of Δ_i, setting the modulus to $*$ if D is not injective on the vertices of $\widetilde{\Delta}_i$. This definition is unambiguous because of the equivariance of D. If $\mathbf{z}_D$ contains no $*$-moduli then, by induction on the n-skeleta, one can easily construct a developing map for $\mathbf{z}_D$ that extends D. Thus by Theorem 2.4.12, $\mathbf{z}_D$ is a solution of $\mathcal{C}$. The holonomy of $\mathbf{z}_D$ is ρ because of the ρ-equivariance of D. □

Remark 4.1.7. If $\varphi \in \mathrm{Isom}^+(\mathbb{H}^3)$ and $\rho' = \varphi \circ \rho \circ \varphi^{-1}$, then a natural correspondence between $\mathcal{D}_\rho$ and $\mathcal{D}_{\rho'}$ is defined by mapping $D \in \mathcal{D}_\rho$ to the element $\varphi \circ D \in \mathcal{D}_{\rho'}$. Note that $\mathbf{z}_D = \mathbf{z}_{\varphi \circ D}$.

I give now a topological description of the sets $\mathcal{D}_\rho$ when ρ varies in $\mathrm{Hom}\big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big)$ (here I write the basepoint because ρ denotes a representation, not a conjugacy class of representations).

Let $p_1, \ldots, p_k$ be the ideal points of M, and for all $n = 1, \ldots, k$ let q_n be a lift of p_n. Let $\mathcal{D}$ be the fiber-space whose basis is $\mathrm{Hom}\big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big)$ and such that the fiber over ρ is the set

$$\mathrm{Fix}\big(\rho(\mathrm{Stab}(q_1))\big) \cap \partial\mathbb{H}^3 \times \cdots \times \mathrm{Fix}\big(\rho(\mathrm{Stab}(q_k))\big) \cap \partial\mathbb{H}^3.$$

The space $\mathcal{D}$ is not a fiber-space in the usual meaning, because the fibers are not diffeomorphic to each other. Nevertheless, it is a well-defined topological sub space of $\mathrm{Hom}\big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big) \times (\partial\mathbb{H}^3)^k$ with a well-defined projection

$$p : \mathcal{D} \to \mathrm{Hom}\big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big)$$

such that

$$p^{-1}(\rho) = \{\rho\} \times \mathrm{Fix}\big(\rho(\mathrm{Stab}(q_1))\big) \times \cdots \times \mathrm{Fix}\big(\rho(\mathrm{Stab}(q_k))\big).$$

By Proposition 4.1.3 and Lemma 4.1.4, for any representation ρ, a natural bijection between $p^{-1}(\rho)$ and $\mathcal{D}_\rho$ is well-defined by mapping $(\rho, x_1, \ldots, x_k)$ to the element D of $\mathcal{D}_\rho$ such that $D(q_n) = x_n$. In the following I identify $\mathcal{D}_\rho$ with $p^{-1}(\rho)$.

The space $\mathcal{D}$ is strictly related to the space of generalized Dehn filling coefficients. I briefly recall some results in this field, referring the reader to [26], [4] and [2] for a detailed discussion.

Let $R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2, \mathbb{C}))$ be the variety of representations of $\pi_1(M)$ into $\mathrm{SL}(2, \mathbb{C})$ and let $\chi(M) = R(M)//\mathrm{SL}(2, \mathbb{C})$ be its variety of characters. For $\rho \in R(M)$, its character χ_ρ is its projection to $\chi(M)$ and can be viewed as the map $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$.

For each $j = 1, \ldots, k$ let s_j be a slope in T_j. If χ_0 is the character of the holonomy of the complete structure of M (if any), then (see for example [2]) there exists a branched covering

$$\overline{p} : V \subset \mathbb{C}^k \to W \subset \chi(M) \tag{4.1}$$

where V and W are neighborhoods respectively of 0 and χ_0, such that if $\chi_\rho = \overline{p}(u_1, \dots, u_k)$ then

$$2\cosh(u_j/2) = \pm\operatorname{trace}(\rho(s_j)).$$

Since $\cosh(u_j/2) = \cosh(-u_j/2)$, the $\overline{p}$-fiber of a point is a finite set with a 2-to-1 choice for each $u_j \neq 0$.
I show now that also the projection $p : \mathcal{D} \to \operatorname{Hom}\big(\pi_1(M, x_0), \operatorname{Isom}^+(\mathbb{H}^3)\big)$ has a branched covering structure which is strictly related to the one of $\overline{p}$. I denote by *parabolic order* of ρ the number $P(\rho)$ of boundary tori where ρ is parabolic:

$$P(\rho) = \#\{n \in \{1, \dots, k\} : \rho(\pi_1(T_n)) \text{ is parabolic}\}.$$

The parabolic order naturally stratifies $\operatorname{Hom}\big(\pi_1(M, x_0), \operatorname{Isom}^+(\mathbb{H}^3)\big)$ as follows. Let

$$\operatorname{Par}^{(l)}(M) = \{\rho \in \operatorname{Hom}\big(\pi_1(M, x_0), \operatorname{Isom}^+(\mathbb{H}^3)\big) : P(\rho) \leq l\}$$

then

$$\operatorname{Hom}\big(\pi_1(M, x_0), \operatorname{Isom}^+(\mathbb{H}^3)\big) = \bigcup_{l=0}^{k} \operatorname{Par}^{(l)}(M).$$

Proposition 4.1.8. *Let $\rho_0 : \pi_1(M, x_0) \to \operatorname{Isom}^+(\mathbb{H}^3)$ be a representation such that $\rho_0(T_n)$ is not dihedral nor trivial for all the tori T_n's. Then there exists a neighborhood U of ρ_0 in $\operatorname{Hom}\big(\pi_1(M, x_0), \operatorname{Isom}^+(\mathbb{H}^3)\big)$ such that the restriction of p to $p^{-1}(U)$ is a branched covering whose branched locus is stratified by the parabolic order. More precisely, if $U^{(l)}$ denotes $U \cap \operatorname{Par}^{(l)}(M)$, then for each l*

$$p : p^{-1}\Big(U^{(l)} \setminus U^{(l-1)}\Big) \longrightarrow U^{(l)} \setminus U^{(l-1)}$$

is a finite covering which branches at $U^{(l-1)}$.

Proof. The fact that $\rho_0(\pi_1(T_n))$ is trivial is a closed condition, so there exists a neighborhood U of ρ_0 such that $\rho(\pi_1(T_n))$ is not trivial for any $\rho \in U$ and $n = 1, \dots, k$. Moreover, for $n = 1, \dots, k$ the condition that $\rho_0(\pi_1(T_n))$ is dihedral is a closed condition, so U can be chosen in such a way that if $\rho_0(\pi_1(T_n))$ is not dihedral, then the same holds for any $\rho \in U$.

Suppose that $\rho_0 \in U^{(l)} \setminus U^{(l-1)}$. It is not restrictive to assume that $\rho_0(\pi_1(T_n))$ is parabolic for $n = 1, \dots, l$. Thus, since $\rho_0 \in U^{(l)}$, $\rho_0(\pi_1(T_n))$ is not parabolic for $n > l$, and the same holds for any $\rho \in U^{(l)}$.

By Proposition 4.1.5, for $\rho \in U^{(l)}$ the set $\mathcal{D}_\rho$ consists of a finite number of points. Let now $\alpha : [0, 1] \to U^{(l)} \setminus U^{(l-1)}$ be a continuous path with $\alpha(0) = \rho_0$. The sets $\mathrm{Fix}(\rho(\mathrm{Stab}(q_n)))$ depend continuously on ρ. Moreover, since $\alpha(t) \in U^{(l)} \setminus U^{(l-1)}$, the cardinality of the sets $\mathrm{Fix}\big(\alpha(t)(\mathrm{Stab}(q_n))\big)$ depends continuously on t. It follows that for any $D_0 \in \mathcal{D}_{\rho_0}$ there exists a unique lift $\widetilde{\alpha} : [0,1] \to \mathcal{D}$ with $\widetilde{\alpha}(0) = D_0$ and $p(\widetilde{\alpha}(t)) = \alpha(t)$. Finally, it is easy to see that when $\rho \in U^{(l)} \setminus U^{(l-1)}$ approaches $U^{(l-1)} \setminus U^{(l-2)}$, two fibers glue together, and this shows that there is an effective branch at $U^{(l-1)}$. □

Proposition 4.1.9. *In the hypotheses of Proposition* 4.1.8*, suppose moreover that there exists $D_0 \in \mathcal{D}_{\rho_0}$ such that $\mathbf{z}_{D_0}$ contains no $*$-moduli. Then there exists a neighborhood U of ρ_0 such that the thesis of Proposition* 4.1.8 *holds for U and, for each path $\alpha : [0,1] \to U$ with $\alpha(0) = \rho_0$ and each lift $\widetilde{\alpha} : [0,1] \to \mathcal{D}$ with $\widetilde{\alpha}(0) = D_0$, $\mathbf{z}_{\widetilde{\alpha}(t)}$ has no $*$-moduli for $t \in [0,1]$.*

Proof. This is because $\mathbf{z}_{\widetilde{\alpha}(t)}$ depends continuously on t. □

I consider now the character-map defined as follows:

$$\begin{aligned}\chi : \mathrm{Hom}\big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big) &\to \chi(M)\\ \rho &\mapsto \chi_\rho.\end{aligned}$$

Suppose that ρ_0 is the holonomy of the complete hyperbolic structure of M (if any). Let U be a neighborhood as in Proposition 4.1.8 and let V, W be as in (4.1). It is not restrictive to assume $W = \chi(U)$. Then one can prove the following fact.

Proposition 4.1.10. *With the above notation, the map χ lifts to a map*

$$\widetilde{\chi} : p^{-1}(U) \subset \mathcal{D} \to V \subset \mathbb{C}^k$$

such that $\chi \circ p = \overline{p} \circ \widetilde{\chi}$.

Idea of the Proof. This is because the coverings p and $\overline{p}$ have the same behavior at the branch locus. □

For each cusp C_n, I fix a product structure on the lift $N_n \cong P_n \times [0,\infty]$ of C_n, where P_n covers T_n and $P_n \times \{\infty\} \sim q_n$.

Lemma 4.1.11. *Let h_0 be the holonomy of a geometric solution of $\mathcal{C}$. Then there exists a neighborhood U of h_0 in $\mathrm{Hom}\big(\pi_1(M,x_0), \mathrm{Isom}^+(\mathbb{H}^3)\big)$ such that $p|_{p^{-1}(U)}$ is a branched covering and, for each $\rho \in U$ and $D \in \mathcal{D}_\rho$, there exists a local diffeomorphism $D_\rho : \widetilde{M} \to \mathbb{H}^3$ such that:*

1. *D_ρ is a developing map for a (incomplete) hyperbolic structure $\mathfrak{S}_\rho$ on M with holonomy ρ.*
2. *The map D_ρ "extends" D. Namely, in each N_n, D_ρ maps all the sets of the form $\{x\} \times [0, \infty]$ to geodesic rays ending at $D(q_n)$.*
3. *The maps D_ρ can be chosen continuously in $\mathcal{D}$ w.r.t. the compact C^1-topology of maps $\widetilde{M} \to \mathbb{H}^3$.*

Proof. This is nothing but Lemma 1.7.2 of [4] or Lemma B.1.10 of [2]. These Lemmas are stated and proved starting from the holonomy of a complete hyperbolic structure of M, but it is not hard to see that they hold if one starts from the holonomy of a geometric solution of $\mathcal{C}$, the proofs remaining substantially the same. □

Theorem 4.1.12. (Geometric solutions are open in algebraic) *The set of geometric solutions of $\mathcal{C}$ is open in the set of algebraic solution of $\mathcal{C}$.*

Proof. Let $\mathbf{z}_0$ be a geometric solution of $\mathcal{C}$ and let h_0 be its holonomy. Since $\mathbf{z}_0$ is geometric, there exists a hyperbolic structure $\mathfrak{S}_0$ on M with holonomy h_0, a developing map D_0 for $\mathfrak{S}_0$ and a map $f : M \to M$ such that, if $\widetilde{f}$ is a lift of f, $D_0 \circ \widetilde{f}$ is a developing map for $\mathbf{z}_0$ (Figure 4.1).

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{M} \arrow[r, "D_0"] \arrow[d] & \mathbb{H}^3 \\
M \arrow[r, "f"] & M &
\end{tikzcd}$$

Figure 4.1. The hyperbolic map f.

Let U be a neighborhood of h_0 such that the theses of Proposition 4.1.9 and Lemma 4.1.11 hold for U. Then for any algebraic solution $\mathbf{z}$ of $\mathcal{C}$ such that $h(\mathbf{z}) \in U$ there exists a hyperbolic structure $\mathfrak{S}_\mathbf{z}$ on M and a developing map $D_\mathbf{z}$ for $\mathfrak{S}_\mathbf{z}$ such that, if $g_z = D_z \circ \widetilde{f}$ (see Figure 4.2), then

$$\mathbf{z}_{g_\mathbf{z}} = \mathbf{z}$$

where I used the symbol $g_\mathbf{z}$ also for the restriction of $g_\mathbf{z}$ to the ideal points. Moreover, since $h(\mathbf{z})$ depends continuously on $\mathbf{z}$, $D_\mathbf{z}$ depends continuously on $\mathbf{z}$. To show that $\mathbf{z}$ is a geometric solution of $\mathcal{C}$, I construct a hyperbolic map $f_\mathbf{z}$ from M to $(M, \mathfrak{S}_\mathbf{z})$ by perturbing the initial hyperbolic map f.

Let $\varphi_\mathbf{z} : \widehat{\widetilde{M}} \to \mathbb{H}^3$ be a developing map for $\mathbf{z}$ which coincides with $g_\mathbf{z}$ on the ideal points and depends continuously on $\mathbf{z}$ (Figure 4.2). Moreover, I require $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f}$.

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "D_{\mathbf{z}}"] \\ \widetilde{M} \arrow[ur, "\widetilde{f}"] \arrow[r, shift left, "g_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}"] \arrow[r, shift right, "\varphi_{\mathbf{z}}"'] & \mathbb{H}^3 \end{tikzcd}$$

Figure 4.2. The maps $g_{\mathbf{z}}$ and $\varphi_{\mathbf{z}}$.

Such a $\varphi_{\mathbf{z}}$ can be easily constructed by straightening $g_{\mathbf{z}}$ (see Section 5.1 for details on the straightening process). Moreover, using convex combinations in $\mathbb{H}^3$ (see Section 5.2, page 106 for details), an $h(\mathbf{z})$-equivariant homotopy $H_{\mathbf{z}} : \widetilde{M} \times [0, 1] \to \mathbb{H}^3$ can be constructed such that

$$H_{\mathbf{z}}(x, 0) = g_{\mathbf{z}}(x) \qquad H_{\mathbf{z}}(x, 1) = \varphi_{\mathbf{z}}(x).$$

The fact that $\varphi_{\mathbf{z}}$ is a developing map does not imply in general that $\mathbf{z}$ is geometric. The problem is that $\varphi_{\mathbf{z}}$ should be the lift of a map $M \to M$, and this may not happen if, for example, looking at the restriction of $\varphi_{\mathbf{z}}$ to a cusp, one sees that its image intersects the axis of the holonomy of the cusp.

With Figure 4.2 in mind, the idea to rule out pathologies is to try to lift the homotopy $H_{\mathbf{z}}$ to a homotopy of $\widetilde{f}$, namely, I try to construct a map $F_{\mathbf{z}} : \widetilde{M} \times [0, 1] \to \widetilde{M}$ such that

$$F_{\mathbf{z}}(x, 0) = \widetilde{f}(x) \qquad \text{and} \qquad H_{\mathbf{z}}(x, t) = D_{\mathbf{z}} \circ F_{\mathbf{z}}(x, t).$$

At the 0-level, clearly I set $F_{\mathbf{z}}(x, 0) = \widetilde{f}(x)$. Since $D_{\mathbf{z}}$ is a local diffeomorphism, $H_{\mathbf{z}}$ can be locally lifted a little near the 0-level. Since $\widetilde{M}$ is not compact, it is not clear a priori how long $H_{\mathbf{z}}$ lifts, and how this depends on the point x.

For any x, $\mathbf{z}$ define

$$\varepsilon_{x,\mathbf{z}} = \sup \big\{ s \in [0, 1] : H_{\mathbf{z}} \text{ continuously lifts if restricted to } \{x\} \times [0, s] \big\}.$$

Since $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f}$, the homotopy $H_{\mathbf{z}_0}$ is constant in t, that is $H_{\mathbf{z}_0}(x, t) = \varphi_{\mathbf{z}_0}(x)$. Therefore $\varepsilon_{x,\mathbf{z}_0} = 1$.

Since the local diffeomorphisms $D_{\mathbf{z}}$ converge to D_0 when $\mathbf{z}$ goes to $\mathbf{z}_0$, for any $y \in \widetilde{M}$ there exists a neighborhood $A(y)$ of y in $\widetilde{M}$ and a neighborhood B_y of $\mathbf{z}_0$ such that for any $\mathbf{z} \in B_y$ the map $D_{\mathbf{z}}$ is a diffeomorphism with the image when restricted to $A(y)$. Moreover, the neighborhoods B_y's can be chosen in such a way that they are intersection of the space of solutions of $\mathcal{C}$ with balls of $\mathbb{C}^k$ centered at $\mathbf{z}_0$.

Lemma 4.1.13. *The neighborhoods $A(y)$'s and B_y's can be chosen in such a way that radii of the balls B_y are lower semicontinuous in y.*

Proof. For any open, regular neighborhood A of y with compact closure, let $r(A)$ be the biggest radius such that if $\mathbf{z} \in \overline{B(\mathbf{z}_0, r(A))}$, then $D_{\mathbf{z}}|_{\overline{A}}$ is a diffeomorphism with its image. The map $A \mapsto r(A)$ is monotone, that is

$$A \subset A' \Rightarrow r(A) \geq r(A').$$

Then, for any nested sequence $\{A_j\}$ converging to A:

$$\left(A_j \nearrow A\right) \Rightarrow \left(r(A_j) \searrow \overline{L} \geq r(A)\right)$$

$$\left(A_j \searrow A\right) \Rightarrow \left(r(A_j) \nearrow \underline{L} \leq r(A)\right).$$

I claim that $\underline{L} = r(A)$. Suppose the contrary, then $\underline{L} < r(A)$. Let $\underline{L} < L < r(A)$. For any A_j there exists $\mathbf{z}_j \in \overline{B(\mathbf{z}_0, L)}$ such that $D_{\mathbf{z}_j}|_{\overline{A_j}}$ is a local but not global diffeomorphism. Thus there exists $a_j, b_j \in \overline{A_j}$ such that $D_{\mathbf{z}_j}(a_j) = D_{\mathbf{z}_j}(b_j)$ and $a_j \neq b_j$. Up to pass to subsequences, $\mathbf{z}_j \to \mathbf{z} \in \overline{B(\mathbf{z}_0, L)}$, $a_j \to a \in \overline{A}$, and $b_j \to b \in \overline{A}$. Since $D_{\mathbf{z}_j} \to D_{\mathbf{z}}$ uniformly, then $D_{\mathbf{z}}(a) = D_{\mathbf{z}}(b)$. If $a \neq b$ then $D_{\mathbf{z}}$ is not injective on $\overline{A}$, if $a = b$ it follows that $D_{\mathbf{z}}$ is not a local diffeomorphism on $\overline{A}$. In both cases, one has $r(A) \leq L < r(A)$, a contradiction.

Now, for any y, choose $A(y)$ in such a way that whenever $y_n \to y$

$$\left(A(y) \bigcup_{n \geq m} A(y_n)\right) \searrow A(y) \qquad \text{as } m \to \infty.$$

Then the radii of the balls $B_y = B(\mathbf{z}_0, r(A(y)))$ have the requested property. Indeed, the function $y \mapsto r(A(y))$ is lower semicontinuous because, if $y_n \to y$, then

$$r(A(y_m)) \geq r\left(A(y) \bigcup_{n \geq m} A(y_n)\right) \nearrow r(A(y))$$

so $\liminf_{m \to \infty} r(A(y_m)) \geq r(A(y))$. □

Define now

$$R(x) = \sup\{s \in \mathbb{R} : |\mathbf{z} - \mathbf{z}_0| < s \Rightarrow \varepsilon_{x,\mathbf{z}} = 1\}$$

Since $D_{\mathbf{z}} \to D_0$ as $\mathbf{z} \to \mathbf{z}_0$, and since $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f} = g_{\mathbf{z}_0}$, the maps $\varphi_{\mathbf{z}}$ and $g_{\mathbf{z}}$ become closer and closer as $\mathbf{z} \to \mathbf{z}_0$. It follows that for every $x \in \widetilde{M}$ if $|\mathbf{z} - \mathbf{z}_0|$ is small enough, then the whole geodesic segment joining $\varphi_{\mathbf{z}}(x)$ to $g_{\mathbf{z}}(x)$ is completely contained in $D_{\mathbf{z}}(A(\widetilde{f}(x)))$. It follows that for all $x \in \widetilde{M}$, $R(x) > 0$.

Lemma 4.1.14. *There is no converging sequence* $(x_n) \subset \widetilde{M}$ *such that*

$$\lim R(x_n) = 0.$$

Proof. Let $x_n \to x \in \widetilde{M}$. Since $D_{\mathbf{z}} \to D_{\mathbf{z}_0}$, in particular $D_{\mathbf{z}}|_{A(\widetilde{f}(x))} \to D_{\mathbf{z}_0}|_{A(\widetilde{f}(x))}$ uniformly. Therefore, there exists a neighborhood V of $\varphi_{\mathbf{z}_0}(x)$ such that $V \subset D_{\mathbf{z}}(A(\widetilde{f}(x)))$ eventually for $\mathbf{z} \to \mathbf{z}_0$. Since φ_z and $g_{\mathbf{z}}$ both converge to $\varphi_{\mathbf{z}_0} = g_{\mathbf{z}_0}$, eventually in n the whole geodesic segment joining $\varphi_{\mathbf{z}}(x)$ to $g_{\mathbf{z}}(x)$ lies in V. It follows that there exists $a > 0$ such that if $|\mathbf{z} - \mathbf{z}_0| < a$, then $\varepsilon_{x_n,\mathbf{z}} = 1$ eventually in n. So $R(x_n) \geq a > 0$ eventually in n. □

Let now

$$\underline{R}(x) = \sup\left\{\xi : \widetilde{M} \to \mathbb{R} \text{ lower semicontinuous s.t. } \xi(x) \leq R(x)\right\}$$

$\underline{R}(x)$ is lower semicontinuous, and by Lemma 4.1.14,

$$\underline{R}(x) > 0. \tag{4.2}$$

Now, let M_0 be the closure of M minus the cusps (so $M_0 \simeq \overline{M}$), let $\widetilde{M}_0$ be its lift and let E be a fundamental domain of $\widetilde{M}_0$ for the action of $\pi_1(M)$. Since E is compact and by lower semicontinuity, the function $\underline{R}$ has a minimum in E, which is strictly positive because of (4.2). It follows that there exists a neighborhood B of $\mathbf{z}_0$ such that for all $\mathbf{z} \in B$ and $x \in E$

$$\varepsilon_{x,\mathbf{z}} = 1.$$

Thus for $\mathbf{z} \in B$ the homotopy $H_{\mathbf{z}}$ lifts to $F_{\mathbf{z}}$ on the points of E, and $F_{\mathbf{z}}$ extends to the whole $\widetilde{M}_0$ by equivariance. For any $x \in \widetilde{M}_0$ I set

$$\widetilde{f}_{\mathbf{z}}(x) = F_{\mathbf{z}}(x, 1).$$

Clearly $\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}_{\mathbf{z}}$, and I will show in Lemma 4.1.15 that $\widetilde{f}_{\mathbf{z}}$ extends to the whole $\widehat{\widetilde{M}}$, keeping the property that

$$\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}_{\mathbf{z}}.$$

By equivariance, $\widetilde{f}_{\mathbf{z}}$ projects to a map $f_{\mathbf{z}} : M \to M$ which is hyperbolic w.r.t. $\mathbf{z}$ because $\varphi_{\mathbf{z}}$ is a developing map for $\mathbf{z}$. Moreover the degree of $f_{\mathbf{z}}$ continuously depends on $\mathbf{z}$, so it is constant 1. Then each $\mathbf{z} \in B$ is a geometric solution of $\mathcal{C}$.

Lemma 4.1.15. *The map $\widetilde{f_{\mathbf{z}}}$ extends to the whole $\widehat{\widetilde{M}}$, keeping the property that*

$$\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f_{\mathbf{z}}}.$$

Proof. For each $n = 1, \ldots, k$, the map $\widetilde{f_{\mathbf{z}}}$ is defined on $N_n \times \{0\}$. Moreover, since $\varphi_{\mathbf{z}}$ is a developing map for $\mathbf{z}$, it is not restrictive to suppose that it maps sets of the form $\{x\} \times [0, \infty] \subset N_n$ to geodesic rays ending at $g_z(q_n)$. By Property 2 of Lemma 4.1.11, such rays lift to $\widetilde{M}$. It follows that φ_z lifts on the cusps to a map extending $\widetilde{f_{\mathbf{z}}}$. □

This completes the proof of Theorem 4.1.12. □

Proposition 4.1.16. *Suppose that the Dehn filling $N = M_{(p,q)}$ is hyperbolic. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two finite-volume, complete hyperbolic structures on N such that the cores γ_n of the filling tori are geodesics for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Then there exists an orientation-preserving isometry $\alpha : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ such that $\alpha(\gamma_n) = \gamma_n$ for all n.*

Proof. By rigidity (Theorem 1.1.7), the identity $\mathrm{Id} : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ is homotopic to an isometry α. Thus for each n the loop γ_n is freely homotopic to $\alpha(\gamma_n)$. By hypothesis γ_n is geodesic for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Since α is an isometry it follows that $\alpha(\gamma_n)$ is a geodesic for $\mathfrak{S}_2$. Hence γ_n and $\alpha(\gamma_n)$ are geodesics for $\mathfrak{S}_2$ and they are freely homotopic, so they must coincide. □

Lemma 4.1.17. *If the Dehn filling coefficients (p, q) are such that there exists a geometric solution of the (p, q)-equations, then $M_{(p,q)}$ has finite volume.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equations. By definition, $M_{(p,q)}$ is complete hyperbolic. Let $\mathrm{vol}(z_i)$ be the volume of a hyperbolic ideal tetrahedron of modulus z_i, with $\mathrm{vol}(z_i) < 0$ is $\Im(z_i) < 0$. Since by definition of geometric solution there exists a proper degree-one map $f : M \to M_{(p,q)} \setminus \{\gamma_n\}$ which is hyperbolic w.r.t. $\mathbf{z}$, then

$$\mathrm{vol}(M_{(p,q)}) = \mathrm{vol}(\mathrm{Im}(f)) \leq \sum |\mathrm{vol}(z_i)| < \infty. \qquad \square$$

Lemma 4.1.18. *Let (p, q) be a set of Dehn filling coefficients and let $\mathbf{z}$ and $\mathbf{w}$ be two geometric solutions of the (p, q)-equations. Then there exists $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $h(\mathbf{w}) = \psi \circ h(\mathbf{z}) \circ \psi^{-1}$.*

Proof. This lemma easily follows from the rigidity theorem for representations (Theorem 5.4.1) but I give here an alternative proof that uses only the rigidity of manifolds. Let $N = M_{(p,q)}$ be the (p,q)-Dehn filling of M endowed with its hyperbolic structure, so $\widetilde{N} = \mathbb{H}^3$. The universal cover $\widetilde{M} \to M$ splits as

$$\widetilde{M} \to \widetilde{N} \setminus \widetilde{\{\gamma_n\}} = \mathbb{H}^3 \setminus \widetilde{\{\gamma_n\}} \to M$$

in such a way that the deck transformations of $\widetilde{N} \setminus \widetilde{\{\gamma_n\}} \to M$ are $\pi_1(N)$. The ideal triangulation τ lifts to an ideal triangulation τ_N of $\widetilde{N} \setminus \widetilde{\{\gamma_n\}}$. I embed $\widehat{\widetilde{N} \setminus \widetilde{\{\gamma_n\}}}$ into $\mathbb{H}^3 \setminus \widetilde{\{\gamma_n\}}$ in $\overline{\mathbb{H}}^3$ following condition c) of Definition 2.5.14. More precisely, remove from N a tubular neighborhood U_n of each γ_n in such a way that the resulting manifold $N \setminus \{U_n\}$ is diffeomorphic to $\overline{M}$. Then the triangulation of $(\overline{M}, \partial\overline{M})$ with truncated tetrahedra (see Remark 2.1.11) lifts to a triangulation of $\mathbb{H}^3 \setminus \widetilde{\{U_n\}}$. For any lift $\widetilde{\gamma}_n$ of any γ_n do the following. If V_n is the neighborhood of $\widetilde{\gamma}_n$ that projects to U_n, choose the half-space model of $\mathbb{H}^3$ in which $\widetilde{\gamma}_n$ is the oriented line $\overline{0\infty}$. Here, ∂V_n is a triangulated cone with axis $\overline{0\infty}$. Extend such a triangulation to $V_n \setminus \overline{0\infty}$ by coning each simplex to ∞. The resulting triangulation of $\widehat{\widetilde{N} \setminus \widetilde{\{\gamma_n\}}}$ is τ_N, embedded in $\overline{\mathbb{H}}^3$ in such such a way that the ideal point corresponding to ∂V_n is the positive end-point of $\widetilde{\gamma}_n$.

Now, let $\mathbf{z}$ be a geometric solution of the (p,q)-equations and let $f : M \to N$ be a hyperbolic map as in Definition 2.5.14, and let F be its lift to $\widetilde{N} \setminus \widetilde{\{\gamma_n\}}$ (Figure 4.3).

$$\begin{tikzcd}
\mathbb{H}^3 \supset \widetilde{N} \setminus \widetilde{\{\gamma_n\}} \arrow[r, "F"] \arrow[d] & \widetilde{N} = \mathbb{H}^3 \arrow[d] \\
M \arrow[r, "f"] & N \setminus \{\gamma_n\}
\end{tikzcd}$$

Figure 4.3. The lift F of f.

Such a lift exists because, since $\mathbf{z}$ is a geometric solution of the (p,q)-equations, the image of $\pi_1(\widetilde{N} \setminus \widetilde{\{\gamma_n\}})$ is contained in $\ker f_*$. Moreover, the holonomy $h(\mathbf{z})$ induces a representation $h(\mathbf{z}) : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that F is $h(\mathbf{z})$-equivariant. Up to changing a little F near the lifts of the γ_n's, one sees that the map F extends to a map

$$F : \mathbb{H}^3 \to \mathbb{H}^3$$

which is $h(\mathbf{z})$-equivariant. Therefore F projects to a degree-one map $\varphi : N \to N$ that coincides with f outside a neighborhood of the γ_n's (recall that f has degree one by hypothesis). By Lemma 4.1.17 Theorem 4.1.1 applies, so φ is homotopic to an orientation-preserving isometry, and that homotopy lifts to an $h(\mathbf{z})$-equivariant homotopy between F and an isometry $\psi_{\mathbf{z}} \in \mathrm{Isom}^+(\mathbb{H}^3)$. It follows that

$$h(\mathbf{z}) : \gamma \mapsto \psi_{\mathbf{z}} \circ \gamma \circ \psi_{\mathbf{z}}^{-1}$$

Similarly $h(\mathbf{w})$ is the conjugation by an element $\psi_{\mathbf{w}} \in \mathrm{Isom}^+(\mathbb{H}^3)$. Then $h(\mathbf{w}) = (\psi_{\mathbf{w}}\psi_{\mathbf{z}}^{-1}) \circ h(\mathbf{z}) \circ (\psi_{\mathbf{w}}\psi_{\mathbf{z}}^{-1})^{-1}$. □

Theorem 4.1.19. *For any Dehn filling coefficient (p, q) there exists at most one geometric solution of the (p, q)-equations.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equations. By Theorem 2.5.16 $\mathbf{z}$ is also an algebraic solution of the (p, q)-equations. In particular $h(\mathbf{z})(\pi_1(T_n))$ is not dihedral for any boundary torus T_n. If $D_{\mathbf{z}}$ is the restriction of a developing map for $\mathbf{z}$ to the ideal points, then $D_{\mathbf{z}} \in \mathcal{D}_{h(\mathbf{z})}$ and $\mathbf{z} = \mathbf{z}_{D_{\mathbf{z}}}$. If $(p_n, q_n) = \infty$ for all n, then by Proposition 4.1.5 D is the unique element of $\mathcal{D}_{h(\mathbf{z})}$, otherwise D is the unique element of $\mathcal{D}_{h(\mathbf{z})}$ that satisfies condition c) of Definition 2.5.14. If $\mathbf{w}$ is another geometric solution of the (p, q)-equations, then by Lemma 4.1.18 there exists $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $h(\mathbf{w}) = \psi \circ h(\mathbf{z}) \circ \psi^{-1}$. As above, and by Proposition 4.1.16, $D_{\mathbf{w}}$ is completely determined as an element of $\mathcal{D}_{h(\mathbf{w})}$, and $D_{\mathbf{w}} = \psi \circ D_{\mathbf{z}} \in \mathcal{D}_{\psi \circ h(\mathbf{z}) \circ \psi^{-1}} = \mathcal{D}_{h(\mathbf{w})}$. Finally, by Remark 4.1.7

$$\mathbf{z} = \mathbf{z}_{D_{\mathbf{z}}} = \mathbf{z}_{\psi \circ D_{\mathbf{z}}} = \mathbf{z}_{D_{\mathbf{w}}} = \mathbf{w}.$$ □

Remark 4.1.20. Theorem 4.1.19 in particular implies the uniqueness of geometric solutions of $\mathcal{C}$+$\mathcal{M}$.

4.2. Examples

In this section I explicitly compute the solutions of the compatibility and completeness equations for some particular one-cusped 3-manifolds.

To begin I fix some notation. Let L and R be the following matrices of $\mathrm{SL}(2, \mathbb{Z})$:

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Each element A of $\mathrm{SL}(2, \mathbb{Z})$ can be written as a product $A = \prod_{i=1}^{n} A_i^{n_i}$, with $A_i \in \{L, R\}$ and $n_i \in \mathbb{Z}$.

Let S be the punctured torus $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$. Then each element $A \in \mathrm{SL}(2, \mathbb{Z})$ induces a homeomorphism φ_A of S. Given $A = \prod A_i^{n_i}$, I call $\prod A_i^{n_i}$ the manifold obtained from $S \times [0,1]$ by gluing $(x,0)$ to $(\varphi_A(x),1)$. For such a manifold, using the algorithm described in [8], one easily obtains an ideal triangulation with $\sum n_i$ tetrahedra.

I notice that the complement of the figure-eight knot is the manifold LR, and its standard ideal triangulation with two tetrahedra is the one obtained according to [8].

I use the following notation to label simplices. For each vertex v of a tetrahedron X, I call X_v the triangle obtained by chopping off the vertex v from X and X^v the face of X opposite to v. Given a tetrahedron X and two vertices v, w of X, by abuse of notation, I use the label w also for the edge of the triangle X_v corresponding to the face X^w. A modulus for a tetrahedron X is named z_X and I will specify the edge to which it is referred.

4.2.1. The manifold LR^3

Let M be the manifold LR^3, *i.e.* the manifold obtained as described above by using the element $LR^3 = \left(\begin{smallmatrix} 4 & 1 \\ 3 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$. Using the algorithm described in [8], one gets the ideal triangulation τ of M with four tetrahedra, labeled A, B, C, D, pictured in Figure 4.4.

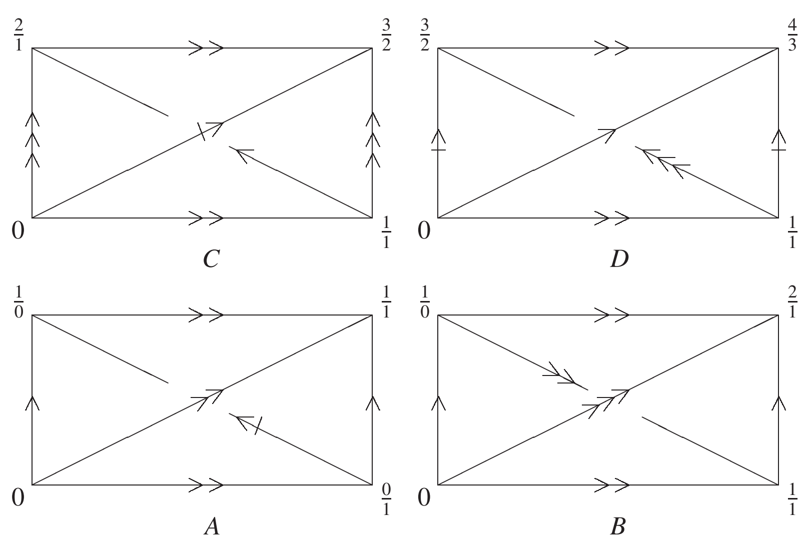


Figure 4.4. Ideal triangulation of M.

I label the vertices of the tetrahedra as in Figure 4.4 (I use such labels because they are natural using the algorithm of [8]). The moduli are referred to the edge $\overline{0\,\frac{1}{1}}$ (note that this edge is common to all the tetrahedra).

The face-pairing rules of τ are, according to the arrows in the picture:

$$
\begin{array}{cccc}
A^{\frac{0}{1}} \longleftrightarrow B^{\frac{2}{1}} & B^{\frac{1}{0}} \longleftrightarrow C^{\frac{3}{2}} & C^{\frac{2}{1}} \longleftrightarrow D^{\frac{4}{3}} & D^{\frac{3}{2}} \longleftrightarrow A^{\frac{1}{1}} \\
A^{\frac{1}{0}} \longleftrightarrow B^{0} & B^{\frac{1}{1}} \longleftrightarrow C^{0} & C^{\frac{1}{1}} \longleftrightarrow D^{0} & D^{\frac{1}{1}} \longleftrightarrow A^{0}
\end{array}
$$

The induced triangulation on the boundary torus is described in Figure 4.5.

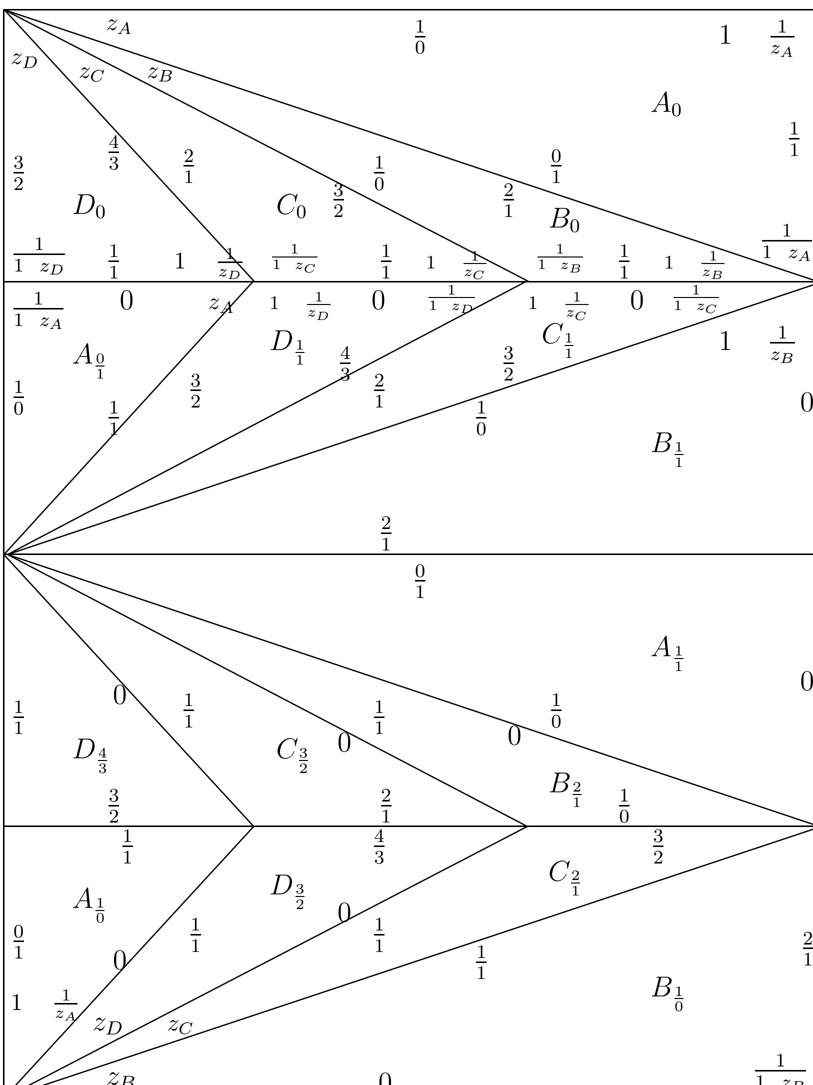


Figure 4.5. The triangulation of the boundary torus.

Now, the compatibility and completeness equations can be written down. It is easy to check that $\mathcal{C} + \mathcal{M}$ is equivalent to the system (4.1).

$$\mathcal{C}\begin{cases} \mathcal{C}_1. & z_A\left(1-\dfrac{1}{z_A}\right)^2 z_D^2 z_C^2 z_B^2 \dfrac{1}{1-z_B} = 1 \\[2ex] \mathcal{C}_2. & \left(\dfrac{1}{1-z_A}\right)^2 \dfrac{1}{1-z_D}\left(1-\dfrac{1}{z_B}\right)^2 \dfrac{1}{1-z_C} = 1 \\[2ex] \mathcal{C}_3. & \left(1-\dfrac{1}{z_D}\right)^2 \dfrac{1}{1-z_C} z_A = 1 \\[2ex] \mathcal{C}_4. & \left(1-\dfrac{1}{z_C}\right)^2 \dfrac{1}{1-z_D}\dfrac{1}{1-z_B} = 1 \end{cases} \tag{4.1}$$

$$\mathcal{M}. \qquad z_D z_C z_B (1-z_A) = 1$$

Moreover, the product of the four equations $\mathcal{C}$ is exactly the square of the product of all the moduli, so it is 1. Thus if three equations are satisfied, then the remaining one must be. It follows that one can discard one of the $\mathcal{C}$'s. I discard $\mathcal{C}_2$. Using $\mathcal{M}$ in $\mathcal{C}_1$ and then $\mathcal{C}_1$ in $\mathcal{C}_4$ and $\mathcal{M}$ one obtains the following system of equations, equivalent to $\mathcal{C} + \mathcal{M}$:

$$\begin{cases} \mathcal{M}. & z_D z_C (1-z_A)^2 = -z_A \\[2ex] \mathcal{C}_1. & z_A(1-z_B) = 1 \\[2ex] \mathcal{C}_3. & \left(\dfrac{z_D-1}{z_D}\right)^2 \dfrac{z_A}{1-z_C} = 1 \\[2ex] \mathcal{C}_4. & \left(\dfrac{z_C-1}{z_C}\right)^2 \dfrac{z_A}{1-z_D} = 1 \end{cases} \tag{4.2}$$

Solving the system, one finds four non-degenerate solutions; one completely positive, giving the hyperbolic structure of M, one with two negative tetrahedra, and their conjugates (i.e. the same situations but with reversed orientation). The following table contains numerical approximations of the solutions. Note that even if the modulus z_B is different from the modulus z_A, equation $\mathcal{C}_1$ implies that the geometric versions of A and B are isometric to each other.

Note that for Solution 2, the total volume is particularly small, which implies that, even if the identification space is defined (see Figure 2.5), it cannot be a hyperbolic manifold. More precisely, the smallest volume of an oriented cusped hyperbolic manifolds is known to be $2V_3$, where $V_3 = 1.01494...$ is the volume of a regular ideal tetrahedron of $\mathbb{H}^3$ (see [5]).

Solution 1		Volumes
z_A	$0.4275047 + i1.5755666$	0.9158907
z_B	$0.8395957 + i0.5911691$	0.9158907
z_C	$0.7271548 + i0.2284421$	0.5786694
z_D	$0.7271548 + i0.2284421$	0.5786694
Solution 2		Volumes
z_A	$1.0724942 + i0.5921114$	0.8144270
z_B	$0.2854042 + i0.3945194$	0.8144270
z_C	$-1.7271548 - i0.6779619$	-0.2398640
z_D	$-1.7271548 - i0.6779619$	-0.2398640

Thus gluing together the tetrahedra of Solution 2 one can not obtain an oriented cusped hyperbolic manifold.

In Figures 4.6 and 4.7 I describe what the triangulation of the boundary torus of M looks like when one chooses the moduli of Solution 2. There are two types of triangles, the positive ones, relative to the tetrahedra A and B and the negative ones, relative to C and D. In Figure 4.6 the four triangles of the top quarter of the triangulation of Figure 4.5 are pictured (compare with Figure 2.2).

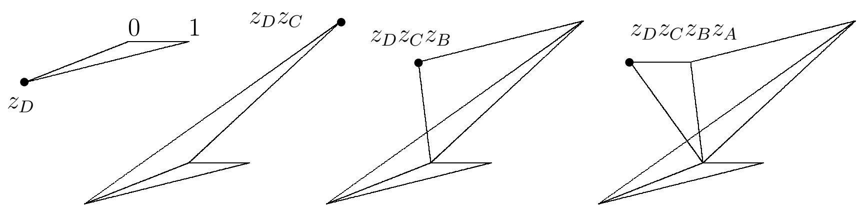


Figure 4.6. The triangles $D_0,\ C_0,\ B_0,\ A_0$ with the moduli of Solution 2.

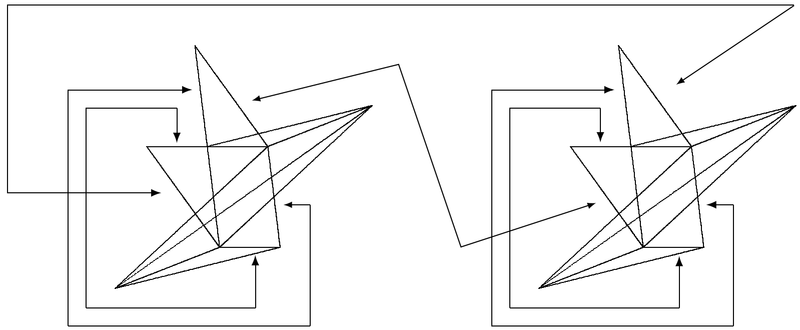


Figure 4.7. Geometric triangulation of the boundary torus, Solution 2.

The two parts of Figure 4.7 are the top and bottom part of the triangulation of Figure 4.5.

Now I look at the algebraic expression of the solutions. A simple calculation shows that the moduli can be expressed by equations (4.3):

$$\begin{cases} z_C = z_D = w \\ z_A = \dfrac{w^2}{1-w} \\ z_B = 1 - \dfrac{1}{z_A} = \dfrac{w^2+w-1}{w^2} \\ w^4 + 2w^3 - w^2 - 3w + 2 = 0 \end{cases} \tag{4.3}$$

The four solutions correspond to the four roots $w_1, \overline{w_1}, w_2, \overline{w_2}$ of the polynomial $P(w) = w^4 + 2w^3 - w^2 - 3w + 2$. Note that looking at the reduction (mod 2) of P, one can see that P is irreducible over $\mathbb{Z}$, and then also over $\mathbb{Q}$.

The holonomy representation can be explicitly calculated as a function of w. Let me fix a fundamental domain F for M obtained by taking one copy of each tetrahedron and then performing the gluings:

$$A^{\frac{1}{0}} \longleftrightarrow B^0 \qquad B^{\frac{1}{1}} \longleftrightarrow C^0 \qquad C^{\frac{1}{1}} \longleftrightarrow D^0$$

Consider now the geometric version of F, i.e. a developed image of F. The holonomy is generated by the isometries corresponding to the remaining face-pairing rules. I consider the upper half-space model of $\mathbb{H}^3$ with coordinates in which the points $0, 1, \infty$ of $\partial\mathbb{H}^3$ are the vertices of D labeled respectively $\frac{3}{2}, 0, \frac{4}{3}$. Calculations show that in this model the holonomy is generated by the elements of $\mathrm{PSL}(2, \mathbb{C})$ represented by the matrices:

$$\begin{pmatrix} 1 & \dfrac{w^2}{w^2+w-1} \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -w \\ \dfrac{1}{w} & -w-1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -w^2 \\ -1 & w^2+w-1 \end{pmatrix}$$

that respectively correspond to the face-pairing rules

$$A^0 \longrightarrow D^{\frac{1}{1}} \qquad C^{\frac{2}{1}} \longrightarrow D^{\frac{4}{3}} \qquad B^{\frac{2}{1}} \longrightarrow A^{\frac{0}{1}}$$

What is important is that the entries of such matrices are numbers belonging to $\mathbb{Q}(w)$ (and this can be proved even without the explicit calculations).

Proposition 4.2.1. *Solution* 2 *is not geometric.*

Proof. This obviously follows from the uniqueness of geometric solutions, but I also give an alternative proof. Let w_1 (resp. w_2) be the root of P relative to Solution 1 (resp. 2) of $\mathcal{C}+\mathcal{M}$. So w_1 gives the hyperbolic structure of M. Let $h_j : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be the holonomy representation relative to w_j for $j = 1, 2$. Since P is irreducible and the entries of the holonomy-matrices are in $\mathbb{Q}(w)$, it follows that a relation between elements holds for h_1 if and only if it holds for h_2. Since h_1 is the holonomy of the complete hyperbolic structure of M, it is faithful. Whence also h_2 is faithful.

Since $\pi_1(M)$ has no torsion, then the image of any discrete and faithful representation of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$ does not contain elliptic elements. Thus, if the image of h_2 is discrete, then $\mathbb{H}^3/h_2$ is a well-defined hyperbolic manifold M', and this cannot happen because in that case the manifold M' should have a too small a volume. I notice that by the rigidity of representations (see Corollary 5.3.12 and Theorem 5.4.1) it follows that to obtain an absurd it suffices that $\mathrm{vol}(h_2) \neq \mathrm{vol}(h_1)$. By Proposition 2.4.23 the holonomy of any geometric solution is discrete, so Solution 2 cannot be geometric. □

From the fact that h_2 is not discrete and Proposition 2.4.23 it follows that there is no map, which is hyperbolic w.r.t. Solution 2, from LR^3 to any hyperbolic manifold. I show now that the image of h_2 is dense in $\mathrm{PSL}(2,\mathbb{C})$. I need the following standard fact about $\mathrm{PSL}(2,\mathbb{C})$ (see for example [14] or [11]).

Lemma 4.2.2. *Let G be a non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *and suppose that G is not discrete. Then the closure of G is either* $\mathrm{PSL}(2,\mathbb{C})$ *or it is conjugate to* $\mathrm{PSL}(2,\mathbb{R})$ *or to a* $\mathbb{Z}_2$*-extension of* $\mathrm{PSL}(2,\mathbb{R})$.

Proposition 4.2.3. *The image of the holonomy relative to Solution* 2 *is dense in* $\mathrm{PSL}(2,\mathbb{C})$.

Proof. It is easy to check that the image of h_2 is a non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Suppose that its closure is conjugate to $\mathrm{PSL}(2,\mathbb{R})$ or to a $\mathbb{Z}_2$-extension of $\mathrm{PSL}(2,\mathbb{R})$. Then there exist a line in $\mathbb{C}\cup\{\infty\} = \partial\mathbb{H}^3$ which is h_2-invariant. Looking at the parabolic elements of h_2, it is easy to see that such a line does not exist.
The thesis follows by Lemma 4.2.2. □

The example discussed so far is interesting for several reasons. On one hand it shows that an algebraic solution of $\mathcal{C}+\mathcal{M}$ can be non-geometric. On the other hand it shows that there is no uniqueness of the algebraic solutions.

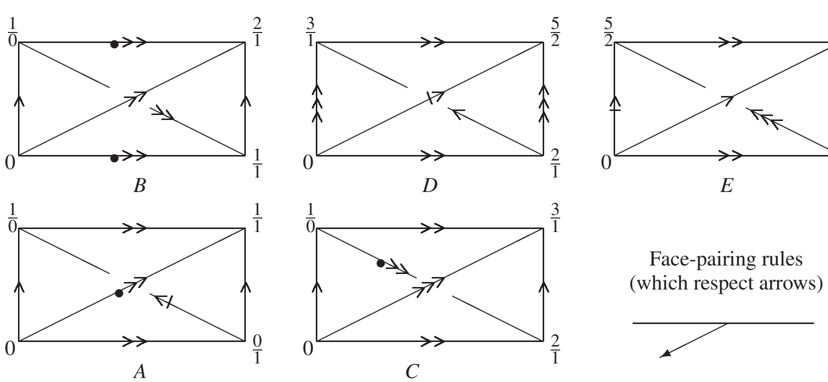
Moreover this example does not involve flat tetrahedra, so it is quite "regular." Finally, the bad solution of $\mathcal{C} + \mathcal{M}$ of LR^3 has the property that "everything works OK at the boundary," namely, the triangulation with moduli induced on the boundary torus defines on it a Euclidean structure (up to scaling). Roughly speaking, this means that the cusp of LR^3 would like to have a complete hyperbolic structure of finite volume according to the bad solution of $\mathcal{C} + \mathcal{M}$, but the rest of the manifold does not agree.

4.2.2. The manifold L^2R^3

Here I do calculations for the manifold L^2R^3.

$$\mathrm{L}^2\mathrm{R}^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}.$$

Using the algorithm described in [8], one gets the ideal triangulation τ of M with five tetrahedra, labeled A, B, C, D, E and pictured in Figure 4.8.



$$A^{\frac{0}{1}} \leftrightarrow B^{\frac{2}{1}} \quad B^{\frac{1}{1}} \leftrightarrow C^{\frac{3}{1}} \quad C^{\frac{1}{0}} \leftrightarrow D^{\frac{5}{2}} \quad D^{\frac{3}{1}} \leftrightarrow E^{\frac{7}{3}} \quad E^{\frac{5}{2}} \leftrightarrow A^{\frac{1}{1}}$$
$$A^{\frac{1}{0}} \leftrightarrow B^{0} \quad B^{\frac{1}{0}} \leftrightarrow C^{0} \quad C^{\frac{2}{1}} \leftrightarrow D^{0} \quad D^{\frac{2}{1}} \leftrightarrow E^{0} \quad E^{\frac{2}{1}} \leftrightarrow A^{0}$$

Figure 4.8. Ideal triangulation of M.

I label the vertices of the tetrahedra as in Figure 4.8. The moduli z_A and z_B are referred to the edge $\overline{0\,\frac{1}{0}}$ while z_C, z_D, z_E to the edge $\overline{0\,\frac{2}{1}}$. The induced triangulation on the boundary torus is that of Figure 4.9.

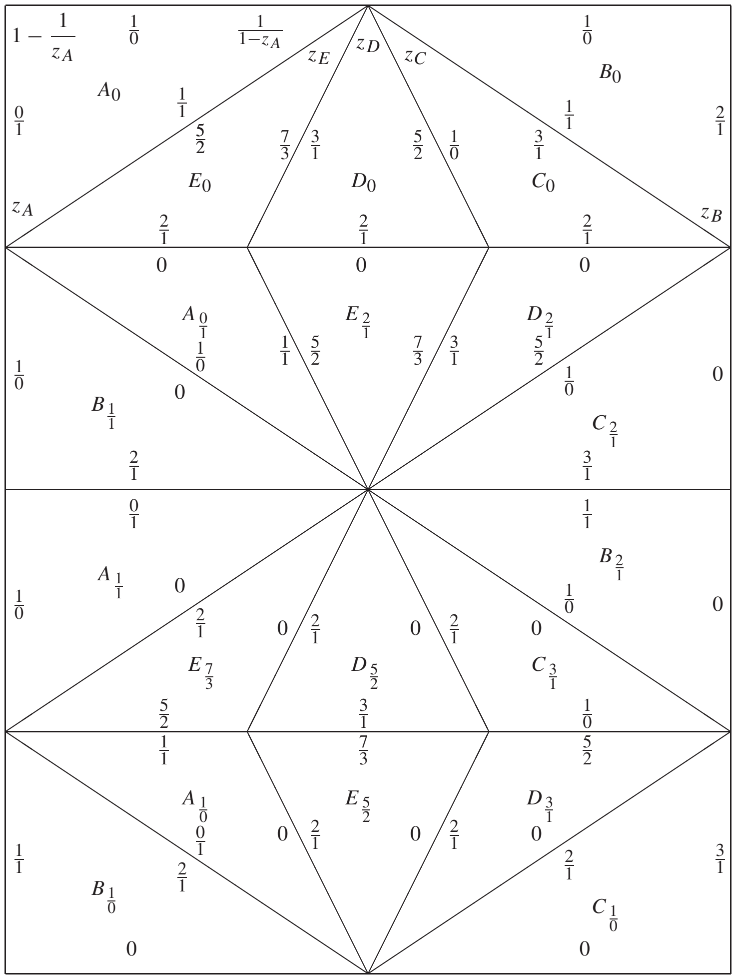


Figure 4.9. Triangulation of the boundary torus.

It is easy to see that the system of compatibility and completeness equations $\mathcal{C}+\mathcal{M}$ is equivalent to the following one:

$$\begin{cases} z_A z_B = z_C z_D z_E \\ z_C(1-z_A) = 1 \\ (1-z_D)^2 z_E^2 = (1-z_E)^2 z_D^2 \\ (z_A-1)^2 = z_A^2(1-z_B)^2 \\ \left(1-\dfrac{1}{z_E}\right)^2 \dfrac{1}{1-z_D}\left(1-\dfrac{1}{z_A}\right) = 1 \end{cases}$$

Solving this system, one founds eight solutions. The following tables contain numerical approximations of the solutions. Note that even if the modulus z_A is different from the modulus z_C, the second equation implies that the geometric versions of A and C are isometric.

	Solution 1	volume	Solution 2	volume
z_A	$0.75 + i0.6614378$	0.9626730	$0.75 - i0.6614378$	−0.9626730
z_B	$1.25 + i0.6614378$	0.7413987	$1.25 - i0.6614378$	−0.7413987
z_C	$0.5 + i1.3228756$	0.9626730	$0.5 - i1.3228756$	−0.9626730
z_D	1	*	1	*
z_E	1	*	1	*

	Solution 3	volume	Solution 4	volume
z_A	1.588633261	0	1.127804076	0
z_B	1.370528159	0	1.113321168	0
z_C	−1.69885025	0	−7.824476637	0
z_D	0.3783840018	0	0.2518509745	0
z_E	−3.387066549	0	−0.6371698130	0

	Solution 5	volume	Solution 6	volume
z_A	$0.4950484 + i0.3298695$	0.7399514	$0.4950484 - i0.3298695$	−0.7399514
z_B	$0.6011109 + i0.9321327$	1.0089809	$0.6011109 - i0.9321327$	−1.0089809
z_C	$1.3880304 + i0.9067580$	0.7399514	$1.3880304 - i0.9067580$	−0.7399514
z_D	$0.5022247 + i0.2691269$	0.6433681	$0.5022247 - i0.2691269$	−0.6433681
z_E	$0.6077815 + i0.3441339$	0.7596486	$0.6077815 - i0.3441339$	−0.7596486

	Solution 7	volume	Solution 8	volume
z_A	$0.1467328 + i1.2472524$	0.9386051	$0.1467328 - i1.2472524$	−0.9386051
z_B	$1.9069644 + i0.7908171$	0.4782906	$1.9069644 - i0.7908171$	−0.4782906
z_C	$0.3736330 + i0.5461534$	0.9386051	$0.3736330 - i0.5461534$	−0.9386051
z_D	$1.1826577 - i2.5849142$	−0.7155138	$1.1826577 + i2.5849142$	0.7155138
z_E	$-0.5956636 + i1.2429350$	0.7019645	$-0.5956636 - i1.2429350$	−0.7019645

Solutions 1 and 2 contain degenerate tetrahedra. I notice that the non-degenerate moduli of such solutions are exactly those that give the hyperbolic structure on the manifold obtained by removing the tetrahedra D and E and adding the gluing rules:

$$C^{\frac{1}{0}} \leftrightarrow A^{\frac{1}{1}} \quad \text{via} \quad \left(0, \tfrac{3}{1}, \tfrac{2}{1}\right) \leftrightarrow \left(0, \tfrac{1}{0}, \tfrac{0}{1}\right)$$

$$C^{\frac{2}{1}} \leftrightarrow A^{0} \quad \text{via} \quad \left(0, \tfrac{1}{0}, \tfrac{3}{1}\right) \leftrightarrow \left(\tfrac{0}{1}, \tfrac{1}{0}, \tfrac{1}{1}\right).$$

Now I look at the algebraic expression of Solutions 3-8. Let

$$P(x) = x^6 + 4x^5 + 3x^4 + 3x^3 - 4x^2 + 2.$$

A simple calculation shows that the moduli can be expressed in terms of roots of P by the following expressions:

$$\begin{cases} z_A = \dfrac{1}{22}(5w^5 + 19w^4 + 9w^3 + 6w^2 - 8w + 17) \\[2ex] z_B = \dfrac{1}{44}(10w^5 + 49w^4 + 62w^3 + 34w^2 - 16w + 34) \\[2ex] z_C = \dfrac{1}{11}(-12w^5 - 39w^4 - 4w^3 - 10w^2 + 72w - 32) \\[2ex] z_D = \dfrac{1}{22}(-4w^5 - 13w^4 + 6w^3 + 15w^2 + 2w + 4) \\[2ex] z_E = w \\[2ex] P(w) = 0 \end{cases}$$

Solutions 3, 4, 7, 8 are not geometric because of uniqueness of geometric solutions. Moreover, as in the case of LR^3, the polynomial P is irreducible, and the argument of Proposition 4.2.1 works in the present case.

4.2.3. A manifold with non-trivial JSJ decomposition

The manifold I consider in this subsection is obtained by gluing to the boundary torus of the complement of the figure-eight knot a Seifert manifold with incompressible boundary. The resulting manifold, which I call M, clearly is not hyperbolic because it contains an incompressible tours (the old boundary torus). This example is interesting because the manifold M admits an ideal triangulation with four tetrahedra such that there exists a positive, partially flat solution of $\mathcal{C} + \mathcal{M}$. Obviously such a solution cannot be geometric, as M is not hyperbolic. I remark that in the present example the moduli do not satisfy the equations $\mathcal{C}^*$ on the angles (compare with Theorem 2.6.3). This shows that the equations $\mathcal{C}^*$ play a fundamental role in order to have hyperbolicity.

I describe now the manifold M. I use the techniques of standard spines to construct an ideal triangulation of M, referring to [18] for details on the theory of spines. Let A be the following subset of $\mathbb{C}$:

$$A = \{z \in \mathbb{C} : |z| \leq 4,\ \ |z - 2| > 1, |z + 2| > 1\}.$$

A is a disc with two holes. Let $I \subset A$ be the set of the points with zero real part. Let S be the space obtained from $A \times [0, 1]$ by gluing $(z, 0)$ to $(-z, 1)$ and let L be the Möbius strip coming from I. The manifold S is the Seifert manifold I will glue to complement of the figure-eight knot. I call C_e and C_i the external and internal components of ∂S. Note that $\partial L \subset C_e$.

I glue C_e to the boundary torus of the complement of the figure-height knot. To do this, I specify where I glue the boundary of the Möbius strip (note that this suffices). I use the classical triangulation of the complement of the figure-eight knot. If one imagines to look from the cusp inside the complement of the figure-eight knot, one gets the following picture:

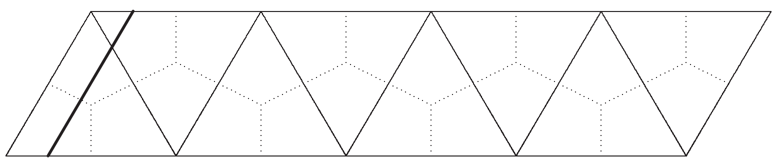


Figure 4.10. The boundary of the complement of the figure-eight knot.

The eight equilateral triangles of the boundary are pictured. The dashed lines represent the standard spine dual to the ideal triangulation, and the marked line is where I glue ∂L.

Since S retracts to $C_e \cup L$, a spine of M is obtained simply by gluing a Möbius strip to the spine of the complement of the figure-eight knot as in Figure 4.10. Such a spine has a vertex more than the old one, but it is not standard. Performing a *lune* move along the Möbius strip one obtains a standard spine of M with five vertices. As the new spine is standard, its dual is an ideal triangulation with five tetrahedra. Such a triangulation can be simplified with an MP-move, replacing the three new tetrahedra with an equivalent pair of tetrahedra. At the end, one gets the triangulation of M sketched in Figure 4.11.

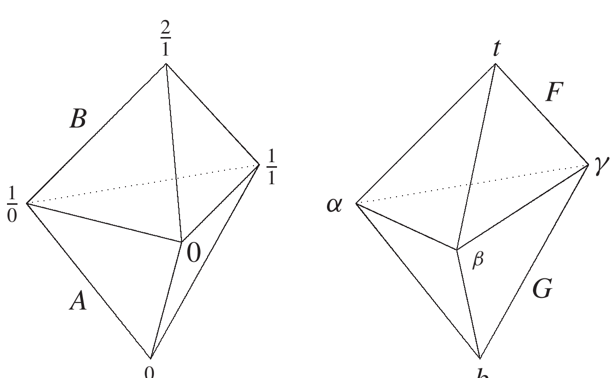


Figure 4.11. The ideal triangulation of M.

The tetrahedra labeled A and B are the old ones (those of the complement of the figure-eight knot). The pairing rules are the following:

$$A^{\frac{0}{1}} \leftrightarrow B^{\frac{2}{1}} : \left(0, \tfrac{1}{0}, \tfrac{1}{1}\right) \leftrightarrow \left(0, \tfrac{1}{0}, \tfrac{1}{1}\right) \qquad A^{\frac{1}{0}} \leftrightarrow B^{0} : \left(0, \tfrac{0}{1}, \tfrac{1}{1}\right) \leftrightarrow \left(\tfrac{1}{0}, \tfrac{1}{1}, \tfrac{2}{1}\right)$$

$$A^{\frac{1}{1}} \leftrightarrow B^{\frac{1}{0}} : \left(0, \tfrac{0}{1}, \tfrac{1}{0}\right) \leftrightarrow \left(0, \tfrac{1}{1}, \tfrac{2}{1}\right) \qquad A^{0} \leftrightarrow F^{\gamma} : \left(\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{1}{0}\right) \leftrightarrow (t, \alpha, \beta)$$

$$B^{\frac{1}{1}} \leftrightarrow G^{\gamma} : \left(0, \tfrac{1}{0}, \tfrac{2}{1}\right) \leftrightarrow (b, \beta, \alpha) \qquad F^{t} \leftrightarrow G^{b} : (\alpha, \beta, \gamma) \leftrightarrow (\alpha, \beta, \gamma)$$

$$F^{\alpha} \leftrightarrow G^{\beta} : (\beta, \gamma, t) \leftrightarrow (\gamma, \alpha, b) \qquad F^{\beta} \leftrightarrow G^{\alpha} : (\alpha, t, \gamma) \leftrightarrow (\gamma, b, \beta)$$

The moduli z_A and z_B are referred to the edge $\overline{0\frac{1}{1}}$ and z_F, z_G to $\overline{\alpha\beta}$. The triangulation of the boundary torus is that of Figure 4.12. It is readily

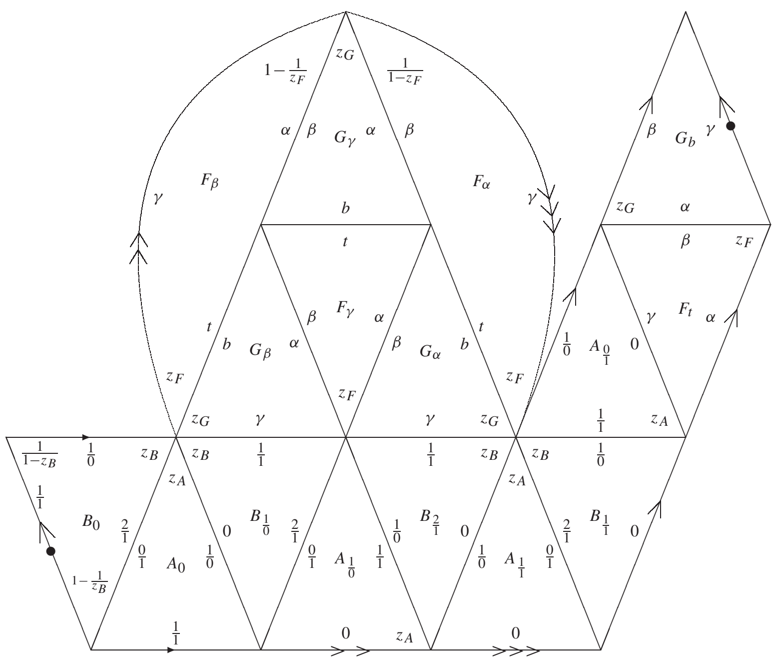


Figure 4.12. Triangulation with moduli of the boundary torus.

checked that the system of compatibility and completeness equations is equivalent to the following one:

$$\begin{cases} \dfrac{1}{1-z_A} \cdot \dfrac{1}{z_B} \cdot \dfrac{z_F}{z_G} = 1 \\ z_G z_F = 1 \end{cases} \qquad \begin{cases} \dfrac{(1-z_A)^2}{z_A} \cdot \dfrac{z_B^2}{1-z_B} = 1 \\ z_B(1-z_A) = 1 \end{cases}$$

From this one easily gets $z_G = z_F$ and $z_F^2 = 1$. Since I am looking for non-degenerate solutions, I chose $z_F = z_G = -1$. Using this one gets $z_A = z_B$ and

$$z_A^2 - z_A + 1 = 0.$$

Thus $z_A = z_B = \frac{1 \pm i\sqrt{3}}{2}$. That is, the ideal tetrahedra F and G are flat but not degenerate, while A and B are regular, exactly as in the complement of the figure-eight knot. I notice that the space obtained by gluing together the geometric versions of the tetrahedra A, B, F, G is not a manifold.

Chapter 5
Hyperbolic volume of representations and rigidity theorems

In Chapter 3, dealing with the two-dimensional case, for a solution $\mathbf{z}$ of $\mathcal{C}+\mathcal{M}$ a number A was defined as the algebraic sum of the areas of the geometric versions of the triangles of τ, and it was shown that such an area plays a central role in order for $\mathbf{z}$ to be geometric (see Corollary 3.1.41 and Proposition 3.2.11). A similar definition can be given in the three-dimensional case. Namely, if $\mathbf{z}$ is a solution of $\mathcal{C}$, define $\mathrm{vol}(\mathbf{z})$ as the algebraic sum of the volumes of the geometric versions of the tetrahedra of τ. As seen in the previous chapters, for a triangulated cusped manifold M, the set of solutions of $\mathcal{C}$ and $\mathrm{Hom}(\pi_1(M), \mathrm{Isom}^+(\mathbb{H}^3))$ are strictly related via the correspondence moduli/holonomy (see Proposition 2.4.13 and Theorem 4.1.6).

In this chapter I describe how to extend the notion of volume of a solution of $\mathcal{C}$ to the world of representations (see also [6]). Namely, if M is a cusped manifold and $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a representation, then a number $\mathrm{vol}(\rho)$ is well-defined in such a way that for any solution $\mathbf{z}$ of $\mathcal{C}$ one has $\mathrm{vol}(\mathbf{z}) = \mathrm{vol}(h(\mathbf{z}))$. The volume of representations is already well-known in the compact case, and deep results about hyperbolic manifolds have been established using it (see for example [6] and [26]). The main property of the volume is that it satisfies all the expected inequalities (very good property!). For example, it is bounded by a multiple of the Gromov norm. Moreover, for hyperbolic manifolds one has a volume-rigidity theorem for representations: the only representation of maximal volume is the holonomy of the complete hyperbolic structure. Actually, such a rigidity is proved generalizing the Gromov's proof of Mostow's theorem, and easily implies the strong version of Mostow's rigidity (Theorem 4.1.1). The non-compact situation is quite different from the compact one, and I show here how to extend the known results for the compact case to the non-compact setting.

Let W be a compact manifold and let ρ be a representation of its fundamental group into $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$. The volume of ρ is defined by taking any ρ-equivariant map from the universal cover $\widetilde{W}$ to $\mathbb{H}^3$ and then by integrating the pull-back of the hyperbolic volume form on a fundamental domain. This volume does not depend on the choice of the equivariant map because two equivariant maps are always equivariantly homotopic and the cohomology class of the pull-back of the volume form is invariant under homotopy.

In [6], Dunfield tries to extend this definition to the case of a non compact cusped 3-manifold M (see Definitions 5.2.1 and 5.1.5). When M is not compact, some problems of integrability arise if one tries to use the above definition of the volume of a representation. The idea of Dunfield for overcoming these difficulties is to use a particular (and natural) class of equivariant maps, called *pseudo-developing maps* (see Definition 5.1.5), that have a nice behavior on the cusps of M allowing to control their volume. Concerning the well-definition of the volume, working with non-compact manifolds, two pseudo-developing maps in general are not equivariantly homotopic and in [6] it is not proved that the volume of a representation does not depend on the chosen pseudo-developing map.

In this chapter I show that the volume of a representation is well-defined even in the non-compact case, and I generalize to non-compact manifolds some results know in the compact case. I restrict to the orientable case.

The chapter is structured as follows. In Section 5.1 I introduce the notion of pseudo-developing map for a given representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and the notion of straightening of such a map. In Section 5.2 I prove that for each orientable cusped 3-manifold M and for each representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the volume of ρ is well-defined and depends only on ρ. The main theorems are:

Theorem 5.2.9 *Let D_ρ and F_ρ be two pseudo-developing maps for ρ. Then* $\mathrm{vol}(D_\rho) = \mathrm{vol}(F_\rho)$.

Theorem 5.2.10 *For any pseudo-developing map D_ρ for ρ one has* $\mathrm{vol}(D_\rho) = \mathrm{Strvol}(D_\rho)$.

Roughly speaking, Theorem 5.2.10 says that the volume of ρ can be computed by straightening any ideal triangulation of M and then summing the volume of the straight versions of the tetrahedra. In Section 5.3, generalizing the techniques used for the proof of Theorem 5.2.10, I show that the volume of a representation ρ is bounded from above by a multiple of the relative simplicial volume:

Theorem 5.3.1 *For all representations* $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ *one has* $|\mathrm{vol}(\rho)| \le V_3 \cdot ||(\overline{M}, \partial\overline{M})||$*, where V_3 is the volume of a regular ideal tetrahedron in* $\mathbb{H}^3$.

In Section 5.4 I prove the volume-rigidity theorem for representations of the fundamental group of a hyperbolic manifold:

Theorem 5.4.1 *Let M be a non-compact, complete, orientable hyperbolic 3-manifold of finite volume. Let $\Gamma \cong \pi_1(M)$ be the subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *such that $M = \mathbb{H}^3/\Gamma$. Let $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. If $|\mathrm{vol}(\rho)| = \mathrm{vol}(M)$ then ρ is discrete and faithful. More precisely there exists $\varphi \in \mathrm{PSL}(2,\mathbb{C})$ such that for any $\gamma \in \Gamma$*

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}.$$

In Section 5.5 I give some corollaries. In particular I show how from Theorem 5.4.1 one can get a proof of Theorem 4.1.1.

5.1. Cone maps, pseudo-developing maps and straightening

Definition 5.1.1. (Product structure on the cusps) Let M be a cusped manifold. For any ideal point $p \in \widehat{M}$, I fix a smooth product structure $T_p \times [0,\infty)$ on the cusp relative to p. Such a structure induces a cone structure, obtained from $T_p \times [0,\infty]$ by collapsing $T_p \times \{\infty\}$ to p, on a neighborhood C_p of p in $\widehat{M}$. I lift such structures to the universal cover.

Let $\widetilde{p}$ be an ideal point of $\widehat{\widetilde{M}}$ that projects to the ideal point p of $\widehat{M}$. I denote by $N_{\widetilde{p}}$ the cone at $\widetilde{p}$. The cone $N_{\widetilde{p}}$ is homeomorphic to $P_{\widetilde{p}} \times [0,\infty]$ where $P_{\widetilde{p}}$ covers the torus T_p and $P_{\widetilde{p}} \times \{\infty\}$ is collapsed to $\widetilde{p}$.

Remark 5.1.2. The choice of a product structure on the cusps is only for technical reasons, and I will show that the results about the volume of representations do not depend on the chosen structure.

Remark 5.1.3. Let $\widetilde{M}$ be the universal cover of a cusped manifold M. In the following, when I speak about $\pi_1(M)$, I tacitly assume that a base-point and one of its lifts have been fixed. If p is an ideal point of M, then $\pi_1(T_p)$ is well-defined only up to conjugation. Called $\{\widetilde{p}_i\}$ the set of the lifts of p, there is a one-to-one correspondence between the stabilizers $\mathrm{Stab}(\widetilde{p}_i)$ of $\widetilde{p}_i$ in the group of deck transformations of $\widetilde{M} \to M$ and the conjugates of $\pi_1(T_p)$ in $\pi_1(M)$. Such a correspondence is uniquely determined once the base-points have been fixed.

To avoid pathologies, since I am working with cusped manifolds, I need that the maps I use have a nice behavior "at infinity." Namely, I will often require that a map from a cusp to $\mathbb{H}^3$ is a cone-map in the following sense.

Definition 5.1.4. (Cone-map) Let A be a set, $c \in \mathbb{R}$ and C be the cone obtained from $A \times [c,\infty]$ by collapsing $A \times \{\infty\}$ to a point, that I call

∞. I say that a map $f : C \to \overline{\mathbb{H}}^3$ is a *cone-map* (or that f has the cone property) if:

- $f(C) \cap \partial\mathbb{H}^3 = \{f(\infty)\}$;
- $\forall a \in A$ the map $f|_{a\times[c,\infty]}$ is either the constant to $f(\infty)$ or the geodesic ray from $f(a, c)$ to $f(\infty)$, parametrized in such a way that the parameter $(t - c)$, $t \in [c, \infty]$, is the arc-length.

I recall here the definition of pseudo-developing map for a representation given in [6].

Definition 5.1.5. (Pseudo-developing map) Let M be a cusped manifold and let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation. A *pseudo-developing map* for ρ is a piecewise smooth map $D_\rho : \widetilde{M} \to \mathbb{H}^3$ which is equivariant w.r.t. the actions of $\pi_1(M)$ on $\widetilde{M}$ via deck transformations and on $\mathbb{H}^3$ via ρ. Moreover I require D_ρ to extend to a continuous map, which I still call D_ρ, from $\widehat{\widetilde{M}}$ to $\overline{\mathbb{H}}^3$ that maps the ideal points to $\partial\mathbb{H}^3$ (see Remark 5.1.6 for comments on this property). Finally I require that there exists $t_{D_\rho} \in \mathbb{R}^+$ such that for each cusp $N_{\widetilde{p}} = P_{\widetilde{p}} \times [0, \infty]$ of $\widetilde{M}$, the restriction of D_ρ to $P_{\widetilde{p}} \times [t_{D_\rho}, \infty]$ is a cone-map.

Let p be an ideal point of $\widehat{\widetilde{M}}$ and let ρ be a representation of $\pi_1(M)$ in to $\mathrm{Isom}^+(\mathbb{H}^3)$. Since $\mathrm{Stab}(p)$ is Abelian, by Lemma 4.1.2 either $\rho(\mathrm{Stab}(p))$ it is dihedral or it has a fixed point in $\partial\mathbb{H}^3$. If D_ρ is a pseudo-developing map for ρ, then $D_\rho(p)$ is a fixed point of $\rho(\mathrm{Stab}(p))$. It follows that, using Definition 5.1.5, in order for a pseudo-developing map to exist, $\rho(\mathrm{Stab}(p))$ must have a fixed point in $\partial\mathbb{H}^3$.

Remark 5.1.6. I included in Definition 5.1.5 the requirement that D_ρ maps ideal points to $\partial\mathbb{H}^3$ only for simplicity. No pathologies actually occur if some ideal point is mapped to the interior of $\mathbb{H}^3$. Coherently with this fact, from now on I suppose that if $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a representation then:

For each boundary torus T, the group $\rho(\pi_1(T))$ is not dihedral.

One can easily check that all the results of this chapter remain true, *mutatis mutandis*, without this assumption.

Lemma 5.1.7. *Let M be a cusped manifold and let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation. Then a pseudo-developing map D_ρ exists.*

Proof. The proof is the same as in [6], I recall it by completeness. I construct a pseudo-developing map as follows. Let $\widetilde{p}$ be an ideal point of $\widetilde{M}$. Since $\mathrm{Stab}(\widetilde{p})$ is Abelian and not dihedral, then its ρ-image has at least one fixed point $q \in \partial\mathbb{H}^3$. I define $D_\rho(\widetilde{p}) = q$ and, for all $\alpha \in \pi_1(M)$ I set $D_\rho(\alpha(\widetilde{p})) = \rho(\alpha)(q)$. I do the same for the other ideal points. Now, for each ideal point $\widetilde{p}$, I define D_ρ on $P_{\widetilde{p}} \times \{0\}$ in any Stab(p)-equivariant way and then I make the cone over $D_\rho(\widetilde{p})$ in such a way that D_ρ has the cone property. Then I extend D_ρ in any equivariant way. The extension is possible because $\mathbb{H}^3$ is contractible. $\square$

Remark 5.1.8. Let p be an ideal point of $\widehat{\widetilde{M}}$. If $\rho(\mathrm{Stab}(p))$ is a parabolic non-trivial group, then it has a unique fixed point. It follows that $D_\rho(p)$ is uniquely determined. Thus, if all the ρ-images of the stabilizers of the ideal points are parabolic, then the D_ρ-images of all the ideal points are uniquely determined (compare with Proposition 4.1.5).

By Proposition 2.1.10 any cusped manifold can be ideally triangulated. Since for a cusped manifold I have fixed a product structure on the cusps, in the following I use the following definition.

Definition 5.1.9. (Ideally triangulated manifold) An *ideally triangulated manifold* is a cusped manifold M equipped with a finite smooth ideal triangulation τ which is compatible with the product structure. That is, for each cusp C_p of M, I require $\tau \cap (T_p \times \{0\})$ to be a triangulation of T_p, and the restriction of τ to C_p to be the product triangulation.

I will often consider the simplices of an ideal triangulation of a manifold M as subsets of $\widehat{M}$.

I introduce now the notion of straightening. Let M be an ideally triangulated manifold, let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation, and let D_ρ be a pseudo-developing map for ρ. Roughly speaking, a straightening of D_ρ is a ρ-equivariant map that agrees with D_ρ on the ideal points and that maps each tetrahedron to a straight one. The straightening is useful to calculate the hyperbolic volume associated to a pseudo-developing map (see Section 5.2). A particular case is when the manifold M is complete hyperbolic, because in this case the straightening descends to a map from M to itself. Here I prove that such a map is onto.

Let Δ be a tetrahedron of τ. By Theorem 4.1.6, the map D_ρ determines a modulus for Δ. For each face σ of Δ, call $\widetilde{\sigma}$ a lift of σ and $\mathrm{Str}_{D_\rho}(\widetilde{\sigma})$, or simply $\mathrm{Str}(\widetilde{\sigma})$, the straight simplex obtained as the convex hull of the D_ρ-image of the vertices of $\widetilde{\sigma}$ (note that $\mathrm{Str}(\widetilde{\sigma})$ can be a degenerate simplex).

Definition 5.1.10. (Straightening) A straightening of D_ρ is a continuous, piecewise smooth, ρ-equivariant map $\mathrm{Str}(D_\rho) : \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ such that:

1. For each simplex σ of the triangulation, $\mathrm{Str}(D_\rho)$ maps $\widetilde{\sigma}$ to $\mathrm{Str}(\widetilde{\sigma})$.
2. The restriction of $\mathrm{Str}(D_\rho)$ to any simplex σ is straight (see Definition 2.4.2).

3. For each cusp $N_{\widetilde{p}} = P_{\widetilde{p}} \times [0, \infty]$ there exists $c \in \mathbb{R}$ such that $\mathrm{Str}(D_\rho)$ restricted to $P_{\widetilde{p}} \times [c, \infty]$ is a cone-map.

Lemma 5.1.11. *Let M be an ideally triangulated manifold. Let ρ be a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and D_ρ be a pseudo-developing map. Then a straightening $\mathrm{Str}(D_\rho)$ of D_ρ exists. Moreover $\mathrm{Str}(D_\rho)$ is always equivariantly homotopic to D_ρ via a homotopy that fixes the ideal points.*

Proof. A straightening of D_ρ can be constructed with the same techniques of Lemma 5.1.7. Regarding the homotopy, since D_ρ maps non-ideal points to the interior of $\mathbb{H}^3$, then one can use a geodesic flow with the time-parameter in $[0, \infty]$ (for example the convex combination of Definition 5.2.11) to construct a homotopy with the required properties. □

Remark 5.1.12. A straightening in general is not a pseudo-developing map in the present setting, because it can map some point of $\widetilde{M}$ to $\partial\mathbb{H}^3$. However, if there are no degenerate tetrahedra, then a straightening is also a pseudo-developing map, and the homotopy between D_ρ and $\mathrm{Str}(D_\rho)$ can be made coherently with the cone structure of the cusps, *i.e.* in such a way that the intermediate maps along the homotopy between D_ρ and $\mathrm{Str}(D_\rho)$ have the cone property on the cusps.

When M has a complete hyperbolic structure of finite volume, there is a natural notion of straightening of an ideal triangulation. Namely, choose the arc-length as the cone parameter on the cusps of M and consider $\mathbb{H}^3$ as the universal cover of M. Then choose ρ as the holonomy of the hyperbolic structure of M; the identity map of $\mathbb{H}^3$ clearly is a pseudo-developing map for ρ. A natural straightening map is a straightening of the identity.

Proposition 5.1.13. *Let M be an ideally triangulated manifold equipped with a complete, finite-volume hyperbolic structure. Then any natural straightening map projects to a map $\mathrm{Str} : \widehat{M} \to \widehat{M}$ which is onto. Moreover $\mathrm{Str}(M) \supset M$.*

Proof. It is easy to see that $\widehat{\widetilde{M}}$ naturally embeds into $\overline{\mathbb{H}}^3$ and that the ideal points lie on $\partial\mathbb{H}^3$. Since the straightening is equivariant, then it projects to a map $\mathrm{Str} : \widehat{M} \to \widehat{M}$. Moreover, Str fixes the ideal points. I prove that Str is onto. One can easily prove that $H_3(\widehat{M}; \mathbb{Z}) \cong H_3(\overline{M}, \partial\overline{M}; \mathbb{Z}) \cong \mathbb{Z}$. So I can define the degree of a map $f : \widehat{M} \to \widehat{M}$ by

$$f_*([\widehat{M}]) = \deg(f) \cdot [\widehat{M}]$$

where $[\widehat{M}]$ is the generator of $H_3(\widehat{M};\mathbb{Z})$ induced by the orientation of M. Now note that by Lemma 5.1.11 the natural straightening is homotopic to the identity via an equivariant homotopy. Because of equivariance, the homotopy projects to a homotopy between Str and the identity. It follows that Str_* and id_* coincide on $H_*(\widehat{M};\mathbb{Z})$, so $\deg(\mathrm{Str}) = \deg(\mathrm{id}) = 1$. Now suppose that Str is not onto and let x be a point in $\widehat{M}$ outside its image. If one considers Str as a map from $\widehat{M}$ to $\widehat{M} \setminus \{x\}$, one gets

$$\mathrm{Str}_*([\widehat{M}]) = 0 \in H_3(\widehat{M} \setminus \{x\};\mathbb{Z})$$

simply because $H_3(\widehat{M} \setminus \{x\};\mathbb{Z}) = 0$. Then $\mathrm{Str}_*([\widehat{M}])$ is a boundary in $\widehat{M} \setminus \{x\}$, and consequently it is a boundary also in $\widehat{M}$. It follows that $\mathrm{Str}_*([\widehat{M}]) = 0$. This implies that $\deg(\mathrm{Str}) = 0$, that is a contradiction. The last assertion follows because Str is onto and fixes the ideal points. □

5.2. Volume of representations

Notation. For this section M will denote a fixed ideally triangulated manifold and ρ will denote a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that for any boundary torus T, the group $\rho(\pi_1(T))$ is not dihedral.

In this section I recall the notion of volume of an equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$. I prove that if one restricts to the class of pseudo-developing maps, then the volume of ρ is well-defined. Namely the volume does not depend neither on the pseudo-developing map nor on the product structure of the cusps. I show that such a volume can be calculated using a straightening of any pseudo-developing map and that it is exactly the algebraic sum of the volumes of the straightened tetrahedra.

Definition 5.2.1. (Volume of pseudo-developing map) Let D_ρ be a pseudo-developing map for ρ. Let ω be the volume form of $\mathbb{H}^3$ and let $D_\rho^*\omega$ be the pull-back of ω. Since D_ρ is equivariant, then $D_\rho^*\omega$ projects to a 3-form, that I still call $D_\rho^*\omega$, on M. The volume $\mathrm{vol}(D_\rho)$ of D_ρ is defined by:

$$\mathrm{vol}(D_\rho) = \int_M D_\rho^*\omega.$$

Remark 5.2.2. I will show below that for pseudo-developing maps the volume is always finite. The same definition of volume does not work for any equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$ because if the pull-back of the volume form is not in L^1, then its integral is not well-defined.

Definition 5.2.3. [Straight volume] Let D_ρ be a pseudo developing map for ρ. Let $\{\Delta_i\}$ be the set of the tetrahedra of the ideal triangulation of M

and $\{\widetilde{\Delta}_i\}$ be a set of lifts of the Δ_i's. Let $v_i = 0$ if $\mathrm{Str}(\widetilde{\Delta}_i)$ is a degenerate tetrahedron, and let v_i be the algebraic volume of $\mathrm{Str}(\widetilde{\Delta}_i)$ otherwise. The straight volume of (D_ρ) is defined by

$$\mathrm{Strvol}(D_\rho) = \sum_i v_i .$$

Remark 5.2.4. If $\mathbf{z}$ is a solution of $\mathcal{C}$ and $D_{\mathbf{z}}$ is a developing map for $\mathbf{z}$ having the cone property, then $D_{\mathbf{z}}$ is also a pseudo-developing map for the holonomy $h(\mathbf{z})$. Such a map is already straight and one has

$$\mathrm{Strvol}(D_{\mathbf{z}}) = \mathrm{vol}(D_{\mathbf{z}}) = \sum v_i = \mathrm{vol}(\mathbf{z})$$

where v_i is the volume of the geodesic ideal tetrahedron of modulus z_i.

Let $C_p = T_p \times [0,\infty]/_\sim$ be a cusp of M and let $N_{\widetilde{p}} = P_{\widetilde{p}} \times [0,\infty]/_\sim$ be one of its lifts in $\widetilde{M}$. Let $f : P_{\widetilde{p}} \times \{0\} \to \mathbb{H}^3$ be a $\mathrm{Stab}(\widetilde{p})$-equivariant map, let $\xi \in \partial\mathbb{H}^3$ be a fixed point of $\rho(\mathrm{Stab}(\widetilde{p}))$ and let $F : N_{\widetilde{p}} \to \overline{\mathbb{H}}^3$ be the cone-map obtained by coning f to ξ. As above, let $F^*\omega$ be the pull-back of the volume-form on C_p. Similarly one can pull-back the metric. I call A_t^p the area of the torus $T_p \times \{t\}$.

Lemma 5.2.5. *In the previous setting, for $t > r$:*

$$A_t^p \le A_r^p e^{-(t-r)} \qquad \textit{and} \qquad \int_{T_p \times [t,\infty)} |F^*\omega| \le A_t^p .$$

Proof. Let (x, y) be local coordinates on $P_{\widetilde{p}}$. Choose the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and assume that $\xi = \infty$. In such a model the hyperbolic metric at the point (z, s) is the Euclidean one rescaled by the factor $1/s$. It follows that, if $\alpha + i\beta$ and h are the complex and real components of F, then

$$\alpha(x,y,t)+i\beta(x,y,t)=\alpha(x,y,r)+i\beta(x,y,r) \quad h(x,y,t)=h(x,y,r)e^{(t-r)}.$$

The element of area at level t is

$$d\sigma_t(x, y) = \sqrt{\det({}^T JF_t \cdot H \cdot JF_t)}$$

where F_t is the restriction of F to $P_{\widetilde{p}} \times \{t\}$ and $H(x, y, t) = \frac{1}{h^2}\mathrm{Id}$ is the matrix of the hyperbolic metric. From direct calculations it follows that $d\sigma_t(x, y) \le d\sigma_r(x, y)e^{-t+r}$ and the first inequality follows.

Now note that the volume element $|F^*\omega|$ at the point $(x, y, t) \in C_p$ is bounded by the area element of the torus $T_p \times \{t\}$ multiplied by the length element of the ray $\{(x, y)\} \times [0, \infty]$. Since the parameter t is exactly the arc-length, then the length element is exactly dt. It follows that

$$\int_{T_p \times [t,\infty)} |F^*\omega| \leq \int_t^\infty A_s^p ds \leq \int_t^\infty A_t^p e^{-(s-t)} ds = A_t^p.$$

This completes the proof. $\square$

Remark 5.2.6. From Lemma 5.2.5 it follows in particular that

$$\int_{T_p \times [t,\infty)} |F^*\omega| \leq A_0^p e^{-t}.$$

This implies that one has an estimate of $\int_{T_p \times [0,\infty)} |F^*\omega|$ not depending on the point $\xi = F(p)$ but only on the area of $T_p \times \{0\}$.

Remark 5.2.7. From Lemma 5.2.5 it follows that $\mathrm{vol}(D_\rho)$ is finite for any pseudo-developing map D_ρ.

The following lemma is proved in [6].

Lemma 5.2.8. *If D_ρ and F_ρ are two pseudo-developing maps for ρ that agree on the ideal points, then* $\mathrm{vol}(D_\rho) = \mathrm{vol}(F_\rho)$.

This is because any two pseudo-developing maps are equivariantly homotopic. The fact that they coincide on the ideal points allows one to construct a homotopy h that respects the cone structures of the cusps. Namely, for each ideal point $\widetilde{p}$ of $\widetilde{M}$ one chooses any equivariant homotopy between the restrictions of D_ρ and F_ρ to $P_{\widetilde{p}} \times \{\bar{t}\}$, where $\bar{t} = \max\{t_{D_\rho}, t_{F_\rho}\}$, then one cones such a homotopy to $D_\rho(\widetilde{p})$ along geodesic rays, and extends the homotopy outside the cusps in any equivariant way. For such a homotopy h one can use the Stokes theorem on $M \times [0, 1]$ for $h^*\omega$ to obtain the thesis. More precisely, let K_t be $M \setminus \cup_p (T_p \times (t, \infty))$, where p varies in the set of the ideal points; then

$$0 = \int_{K_t \times [0,1]} d(h^*\omega) = \int_{\partial(K_t \times [0,1])} h^*\omega = \int_{K_t} (D_\rho^*\omega - F_\rho^*\omega) + \int_{\partial K_t \times [0,1]} h^*\omega$$

and, as in Lemma 5.2.5, one can prove that the last integral goes to zero as $t \to \infty$.

I now prove that the claim of Lemma 5.2.8 is true in general.

Theorem 5.2.9. *Let D_ρ and F_ρ be two pseudo-developing maps for ρ. Then* $\mathrm{vol}(D_\rho) = \mathrm{vol}(F_\rho)$.

Proof. For $t \in [0, \infty)$, let D^t_ρ be the map constructed as follows: D^t_ρ coincides with D_ρ up to the level t of each cusp. Then for each cusp $N_{\widetilde{p}}$ complete D^t_ρ by coning $D|_{P_{\widetilde{p}}\times\{t\}}$ to $F_\rho(\widetilde{p})$ along geodesic rays in such a way that the arc-length is the parameter $s-t$, where $s \in [t, \infty)$. Now, D^t_ρ is a pseudo-developing map that agrees with F_ρ on the ideal points. Thus by Lemma 5.2.8 $\mathrm{vol}(D^t_\rho) = \mathrm{vol}(F_\rho)$. Since D^t_ρ and D_ρ agree outside the cusps and where they differ they are cones on the same basis (and different vertices), from Lemma 5.2.5 it follows that

$$|\mathrm{vol}(D_\rho) - \mathrm{vol}(D^t_\rho)| \leq 2\sum_p A^p_t \leq 2(\sum_p A^p_0)e^{-t}$$

where p varies in the set of ideal points and A^p_t is the area of the torus $T_p \times \{t\}$. As $t \to \infty$ one gets the thesis. □

Similar techniques actually allow to prove the following theorem.

Theorem 5.2.10. *For any pseudo-developing map D_ρ for ρ*

$$\mathrm{vol}(D_\rho) = \mathrm{Strvol}(D_\rho).$$

Before proving Theorem 5.2.10, I give the following definition.

Definition 5.2.11. (Convex combination) Let f, g be two maps from a set X respectively to $\mathbb{H}^n$ and $\overline{\mathbb{H}}^n$. For $t \in [0, \infty]$ the convex combination Φ_t from f to g is defined by:

$$\Phi_t(x) = \begin{cases} \gamma_x(t) & t \leq \mathrm{dist}(f(x), g(x)) \\ g(x) & t \geq \mathrm{dist}(f(x), g(x)) \end{cases}$$

where γ_x is the geodesic from $f(x)$ and $g(x)$, parametrized by arc-length.

Remark 5.2.12. In Definition 5.2.11, if X is a topological space and f and g are continuous, then the convex combination from f to g is continuous on $X \times [0, \infty]$ because the function $\mathrm{dist}(f(x), g(x))$ is well-defined and continuous from X to $[0, \infty]$.

Proof of 5.2.10. For the proof assume that $t_{D_\rho} = 0$. I start by fixing a suitable homotopy h between D_ρ and $\mathrm{Str}(D_\rho)$. Define $h : \widetilde{M} \times [0, \infty] \to \mathbb{H}^3$ outside the cusps to be the convex combination from D_ρ to $\mathrm{Str}(D_\rho)$ and then for each cusp $N_{\widetilde{p}}$ extend h by coning $h((x, 0), s)$ to $D_\rho(\widetilde{p})$ along geodesic rays in such a way that the parameter $t \in [0, \infty)$ of the cusp is the arc-length. Let $D_s(x) = h(x, s)$. By Lemma 5.2.8

$$\int_M D^*_\rho \omega = \int_M D^*_s \omega \qquad \text{for } s \in (0, \infty).$$

So one only has to prove that $\int_M D_s^*\omega \to \mathrm{Strvol}(D_\rho)$ as $s \to \infty$. Clearly, it suffices to prove that for any tetrahedron Δ we have

$$\int_\Delta D_s^*\omega \to v$$

where v is the volume of $\mathrm{Str}(\Delta)$. If Δ does not collapse in the straightening, then the distance from D_ρ to $\mathrm{Str}(D_\rho)$ is bounded outside the cusps so $D_s = \mathrm{Str}(D_\rho)$ for $s \gg 0$; since $\mathrm{Str}(D_\rho)$ is a homeomorphism on Δ, then $\int_\Delta D_s^*\omega$ is exactly the volume of the straight version of Δ.

If Δ collapses in the straightening, then one has to show that $\int_\Delta D_s^*\omega \to 0$. This follows from direct calculations, which I only sketch because they are involved but use elementary techniques. Moreover, in the next section, I will give an alternative proof of this theorem (see Theorem 5.3.1 and Remark 5.3.10).

Given the convex combination Φ_t from a map f to a map g, it is possible to calculate the Jacobian of Φ_t as a function of the derivatives of f and g, the time t and the distance between f and g. This is not completely trivial, for example think of a tetrahedron as a convex combination of two segments: the segments have zero area but in the middle one has quadrilaterals with non-zero area. Using these calculations, one can estimate $|D_s^*\omega|$ outside the cusps, showing that its integral goes to zero as s goes to infinity. Looking inside the cusps, by Lemma 5.2.5 one reduces the estimate to the same estimate as above, made with 2-dimensional objects (the bases of the cusps). □

Remark 5.2.13. Since $\mathrm{vol}(D_\rho) = \mathrm{Strvol}(D_\rho)$ it follows that such a volume does not depend on the chosen cone structure of the cusps. Moreover, by Theorem 5.2.9, $\mathrm{vol}(D_\rho)$ does not depend on the pseudo-developing map, but only on ρ. This allows one to give the following definition.

Definition 5.2.14. The volume $\mathrm{vol}(\rho)$ of ρ is the volume of any pseudo-developing map for ρ.

As the following corollary shows, for hyperbolic manifolds the volume of the holonomy is exactly the hyperbolic volume.

Corollary 5.2.15. *Let M be a complete hyperbolic manifold of finite volume. If ρ is the holonomy of the hyperbolic structure then* $\mathrm{vol}(\rho) = \mathrm{vol}(M)$.

Proof. Consider $\mathbb{H}^3$ as the universal cover of M and choose the arc length as the cone parameter of the cusps. Clearly the identity of $\mathbb{H}^3$ is a pseudo-developing map for ρ, and $\int_M \mathrm{Id}^*(\omega) = \mathrm{vol}(M)$. □

Corollary 5.2.16. *Let z_i be the modulus induced by a pseudo-developing map D_ρ on Δ_i and let v_i be the volume of a hyperbolic ideal geodesic tetrahedron of modulus z_i. Then* $\mathrm{vol}(\rho) = \sum v_i$.

Remark 5.2.17. Even if $\sum v_i$ depends only on ρ, the moduli z_i induced by a pseudo-developing map D_ρ actually can depend on D_ρ (see Section 4.1).

Proposition 5.2.18. *Let g be a reflection of $\mathbb{H}^3$ and let $\overline{\rho}$ be the representation $g \circ \rho \circ g^{-1}$. Then* $\mathrm{vol}(\overline{\rho}) = -\mathrm{vol}(\rho)$.

Proof. If D_ρ is a pseudo-developing map for ρ, then $g \circ D_\rho$ is a pseudo-developing map for $\overline{\rho}$ and it is easily checked that $\mathrm{vol}(g \circ D_\rho) = -\mathrm{vol}(D_\rho)$. $\square$

The following fact is proved in [6].

Proposition 5.2.19. *Suppose that ρ factors through the fundamental group of a Dehn filling N of M. Then the volume of ρ w.r.t. N coincides with the volume of ρ w.r.t. M.*

Theorem 5.2.10 extends from ideal to "classical" triangulations, namely to genuine triangulations $\mathcal{T}$ of $\overline{M}$. Consider such a $\mathcal{T}$ as a triangulation of M with some simplices at infinity (those in $\partial\overline{M}$). Given a pseudo-developing map D_ρ for ρ, define a straightening of D_ρ relative to $\mathcal{T}$, exactly as in Section 5.1, by considering the convex hulls of the images of the vertices of $\mathcal{T}$. Then one can give the definition of the straight volume relative to $\mathcal{T}$ of a developing map D_ρ exactly as in Definition 5.2.3, with the unique difference that one has to use the tetrahedra of $\mathcal{T}$ instead of the ideal tetrahedra of an ideal triangulation of M. Call such a volume $\mathrm{Strvol}^{\mathcal{T}}(D_\rho)$.

Finally, exactly as in Theorem 5.2.10, one can prove the following fact:

Proposition 5.2.20. *Let $\mathcal{T}$ be a triangulation of $\overline{M}$ and D_ρ be a pseudo-developing map for ρ. Then* $\mathrm{vol}(\rho) = \mathrm{Strvol}^{\mathcal{T}}(D_\rho)$.

5.3. Comparison with simplicial volume

In this section I generalize the argument used to prove Theorem 5.2.10 to compare $\mathrm{vol}(\rho)$ with the simplicial volume of M, obtaining exactly the expected inequality. I keep here the notation fixed at the beginning of Section 5.2.

Let $||(\overline{M}, \partial\overline{M})||$ be the simplicial volume of $\overline{M}$ relative to the boundary (see [1], [12], [16], [26] for more details), and let V_3 be the volume of a regular straight ideal tetrahedron of $\mathbb{H}^3$.

Theorem 5.3.1. *For any representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$*

$$|\mathrm{vol}(\rho)| \leq V_3 \cdot ||(\overline{M}, \partial\overline{M})||.$$

Proof. For the proof I fix a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and a pseudo-developing map D_ρ for ρ.

Let $c = \sum_i \lambda_i \sigma_i$ be a smooth singular chain in $\overline{M}$; here each simplex σ_i is a piecewise smooth map from the standard tetrahedron Δ^3 to $\overline{M}$. The simplicial volume of c is defined as $||c|| = \sum |\lambda_i|$. The relative simplicial volume of $(\overline{M}, \partial\overline{M})$ is defined as

$$||(\overline{M}, \partial\overline{M})|| = \inf\{||c|| : [c] = [\overline{M}] \in H_3(\overline{M}, \partial\overline{M})\}.$$

The proof has two main steps:

1. Given a smooth cycle $c = \sum_i \lambda_i \sigma_i$ representing $[\overline{M}]$, show that
$$\mathrm{vol}(\rho) = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^* \omega)$$
where ω is the volume form of $\mathbb{H}^3$.
2. By replacing c with its straightening, show that $\mathrm{vol}(\rho) = \sum_i \lambda_i v_i$, where v_i is the volume of a straight version of σ_i.

From Step 2 it follows that

$$|\mathrm{vol}(\rho)| \leq \sum_i |\lambda_i| \cdot |v_i| \leq V_3 \cdot ||c||.$$

Theorem 5.3.1 follows taking to the infimum over all relative cycles c representing $[\overline{M}]$.

Step 1. Since a pseudo-developing map has the cone property on the cusps, the 3-form $D_\rho^* \omega$ defined on M extends to a 3-form on $\overline{M}$ that vanishes at the boundary. So one can consider the class $[D_\rho^* \omega] \in H^3(\overline{M}, \partial\overline{M})$. Since $[c] = [\overline{M}]$,

$$\mathrm{vol}(\rho) = \int_M D_\rho^* \omega = \langle [D_\rho^* \omega], [\overline{M}] \rangle = \langle [D_\rho^* \omega], [c] \rangle = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^* \omega).$$

Step 2. I first give an outline of the proof. Consider a lift $\widetilde{c}$ of c to $\widehat{\widetilde{M}}$. Let $\overline{c} = (D_\rho)_* \widetilde{c}$ be the push-forward of $\widetilde{c}$ to $\overline{\mathbb{H}}^3$ via D_ρ and let $\mathrm{Str}(\overline{c})$ be a straightening of $\overline{c}$. Since the straightening is homotopic to the identity, then there exists a degree-one chain-homotopy, that is a map H from k-chains to $(k+1)$-chains such that

$$\mathrm{Str} - \mathrm{Id} = H \circ \partial - \partial \circ H.$$

Then

$$\begin{aligned} \mathrm{vol}(\rho) &= \langle D_\rho^* \omega, \widetilde{c} \rangle = \langle \omega, (D_\rho)_* \widetilde{c} \rangle = \langle \omega, \overline{c} \rangle \\ &= \langle \omega, \mathrm{Str}(\overline{c}) \rangle + \langle \omega, \partial H \overline{c} \rangle - \langle \omega, H \partial \overline{c} \rangle \\ &= \sum_i \lambda_i v_i + \langle d\omega, H\overline{c} \rangle - \langle \omega, H \partial \overline{c} \rangle. \end{aligned}$$

The last two summands are zero because $d\omega = 0$ and, even if $\partial \overline{c} \neq 0$, everything can be made ρ-equivariantly so that the action of ρ cancels out in pairs the contributions of $\langle \omega, H \partial \overline{c} \rangle$.

I formalize now this argument. Let $C_k(X)$ denote the real vector space of finite singular, piecewise smooth k-chains in a space X. Consider the projection $\overline{M} \to \widehat{M}$ obtained by collapsing each boundary torus to a point. Let $c = \sum_i \lambda_i \sigma_i$ be a relative cycle in $C_k(\overline{M})$, *i.e.* a chain c such that $\partial c \in C_{k-1}(\partial \overline{M})$. I also call c the chain induced on $C_k(\widehat{M})$ with $\partial c \in C_{k-1}(\text{ideal points})$, and I call $\widetilde{c}$ a lift of c to $\widetilde{\widehat{M}}$, that is

$$\widetilde{c} = \sum_i \lambda_i \widetilde{\sigma}_i \in C_k(\widetilde{\widehat{M}})$$

where each $\widetilde{\sigma}_i$ is a lift of σ_i.

Remark 5.3.2. The chain $\widetilde{c}$ in general is not a relative cycle. Nevertheless, since c is a relative cycle, assuming $\partial \widetilde{c} = \sum_j l_j \eta_j$, there exists a family $\{\alpha_j\}$ of elements of $\pi_1(M)$ such that

$$\sum_j l_j \cdot \alpha_{j*}(\eta_j) \in C_{k-1}(\text{ideal points})$$

where $\pi_1(M)$ acts on $\widetilde{M}$ via deck transformations and $\alpha_{j*}(\eta_j)$ is the composition of α_j with η_j.

I set

$$\overline{\sigma}_i = (D_\rho)_*(\widetilde{\sigma}_i) \qquad \text{and} \qquad \overline{c} = \sum_i \lambda_i \overline{\sigma}_i = (D_\rho)_*(\widetilde{c}) \in C_k(\overline{\mathbb{H}}^3).$$

I restrict now the class of simplices I want to use.

Definition 5.3.3. I call a k-simplex $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$ *admissible* if for any sub-simplex η of σ, if the interior of η touches $\partial \mathbb{H}^3$ then η is constant. A chain is admissible if its simplices are admissible.

Definition 5.3.4. For any chain $\beta \in C_k(\overline{M})$, define span$(\beta)$ as the set of all the subsimplices of β (of any dimension).

Lemma 5.3.5. *Let $c' = \sum_i \lambda_i \sigma_i'$ be a relative cycle in $C_k(\overline{M}, \partial\overline{M})$. Then there exists a cycle $c = \sum_i \lambda_i \sigma_i$ (with the same λ_i's) such that*

- *c is a relative k-cycle in $(\overline{M}, \partial\overline{M})$ with $\partial c' = \partial c$.*
- *For each simplex η (of any dimension) of c, if the interior of η touches $\partial\overline{M}$ then η is completely contained in $\partial\overline{M}$. This implies that $\overline{c}$ is admissible.*
- *The only simplices that touch $\partial\overline{M}$ are those of span$(\partial c')$.*
- *$[c] = [c']$ in $H_k(\overline{M}, \partial\overline{M})$.*

Proof. Here I do not distinguish between a simplex and its support, when speaking of a sub-simplex, I consider the support as a subset. The idea is the following. Given the chain c', construct c as follows: near $\partial\overline{M}$ push c' a little inside M, keeping fixed only the simplices of span$(\partial c')$. This operation can be made via an homotopy, so $[c] = [c']$. Moreover, the only simplices of c that touch $\partial\overline{M}$ are those of span(∂c). Finally, c is admissible because, if $\overline{\sigma}_i(x) \in \partial\mathbb{H}^3$, then from the definition of pseudo-developing map it follows that $\sigma_i(x)$ is an ideal point. Thus x lies on a face F of σ_i such that the simplex $\eta = (\sigma_i)|_F$ belongs to span(∂c). It follows that $\widetilde{\eta}$ is a constant map and then also $\overline{\eta}$ is constant.
I now work out the details. First of all I construct suitable neighborhoods for sub-simplices of the standard k-simplex Δ^k. For each sub-simplex η of Δ^k define $\theta_\eta : \Delta^k \to \mathbb{R}^+$ as

$$\theta_\eta(x) = \sup\left\{\mathrm{dist}_{\Delta^k}(x,\xi) : \xi \text{ is a } (k\text{–}1)\text{-subsimplex of } \Delta^k \text{ such that } \eta \subset \xi\right\}$$

where $\theta_\eta(x) = 0$ if $\eta = \Delta^k$. Note that if $\eta \subset \xi$ then $\theta_\eta(x) \geq \theta_\xi(x)$. For any set A of subsimplices of Δ^k (closed by passage to subsimplices) define the following function on Δ^k:

$$\delta^A(x) = \inf\{\theta_\eta(x) : \eta \in A\}.$$

Thus, given the chain c', any simplex σ' of c' induces a function on Δ^k by:

$$\delta_{\sigma'}(x) = \delta^{\mathrm{span}(\partial c')}(x)$$

Note that if η is a subsimplex of σ_1' and σ_2', then

$$\delta_{\sigma_1'}|_\eta = \delta_{\sigma_2'}|_\eta. \tag{5.1}$$

Now fix a product structure, different from the one used for defining the cone-maps, of a neighborhood U of $\partial\overline{M}$ as

$$U = \partial\overline{M} \times [0, 1)$$

where $\partial\overline{M} = \partial\overline{M} \times \{0\}$. Call P_M and P_t the projections from U respectively to $\partial\overline{M}$ and $[0, 1)$. Now fix a small enough ε and modify each $\delta_{\sigma'}$ by setting:

$$\delta_{\sigma'}(x) = \min\left(\delta_{\sigma'}(x), \varepsilon\right).$$

For each simplex σ_i' of c' define a homotopy $h_i : \Delta^k \times [0, 1] \to \overline{M}$ as follows:

$$h_i(x, s) = \begin{cases} \sigma_i'(x) & \sigma_i'(x) \notin U \\ \left(P_M(\sigma_i'(x), \inf\left(1, P_t(\sigma_i'(x)) + s\delta_{\sigma_i'}(x)\right)\right) & \text{otherwise} \end{cases}$$

Finally, setting $\sigma_i = h_i(x, s)$, the chain $\sum \lambda_i\sigma_i$ has the required properties for any $s > 0$. Note the the last property of c follows because by condition (5.1) the chains c' and c hare homotopic. $\square$

I call $\overline{C}_k(\overline{\mathbb{H}}^3)$ the vector space of admissible chains. Note that the boundary operator is well-defined on $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ (The boundary of an admissible cycle is admissible).

Definition 5.3.6. For any admissible simplex $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$, a straightening $\mathrm{Str}(\sigma) : \Delta^k \to \overline{\mathbb{H}}^3$ is a simplex that agrees with σ on the 0-skeleton, moreover I require $\mathrm{Str}(\sigma)$ to be a straight map whose image is the convex hull of its vertices. For any chain $c = \sum_i \lambda_i\sigma_i$ a straightening of c is a chain $\mathrm{Str}(c) = \sum_i \lambda_i \mathrm{Str}(\sigma_i)$.

A straightening of a simplex is admissible because any straight simplex is admissible. The straightening of a simplex is not unique in general. Nevertheless, as the following lemma shows, it is possible to choose a straightening for any simplex compatibly with the boundary operator of $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$.

Lemma 5.3.7. *There exists a chain-map* $\mathrm{Str} : \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3) \to \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ *that maps each simplex to one of its straightenings and such that for any isometry γ of $\mathbb{H}^3$,* $\gamma_* \circ \mathrm{Str} = \mathrm{Str} \circ \gamma_*$.

Proof. Let K be the set of pairs $\{(B, f)\}$ where B is a subspace of $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ and $f : B \to \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ is a linear map, such that:

- $\partial(B) \subset B$.
- $\forall\gamma \in \mathrm{Isom}(\mathbb{H}^3)$, $\gamma_*(B) \subset B$.
- $\forall\sigma \in B$, $f(\sigma)$ is a straightening of σ.
- $\forall\gamma \in \mathrm{Isom}(\mathbb{H}^3)$, $f \circ \gamma_* = \gamma_* \circ f$.
- $f \circ \partial = \partial \circ f$.

Note that K is not empty because each 0-simplex is admissible and it is itself its unique straightening, so that $(\overline{C}_0(\overline{\mathbb{H}}^3), Id) \in K$. I order K by inclusion (*i.e.* $(B, f) \prec (C, g)$ iff $B \subset C$ and $g|_B = f$) and use Zorn's lemma. Let $\{(B_\xi, f_\xi)\}$ be an ordered sequence in K. Clearly

$$(B_\infty = \cup_\xi B_\xi, f_\infty = \cup_\xi f_\xi)$$

is an upper bound for $\{(B_\xi, f_\xi)\}$. It follows that there exists a maximal element $(\overline{B}, \overline{f}) \in K$. I claim that $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$. Suppose the contrary. Let

$$k = \min\{n \in \mathbb{N} : \overline{C}_n(\overline{\mathbb{H}}^3) \not\subset \overline{B}\}$$

and let σ be a simplex of $\overline{C}_k(\overline{\mathbb{H}}^3) \setminus \overline{B}$. If $k = 0$, set B_1 to be the space spanned by $\overline{B}$ and $\bigcup_{\gamma \in \mathrm{Isom}(\mathbb{H}^3)} \gamma_*(\sigma)$, define $\overline{f}(\sigma) = \sigma$, $\overline{f}(\gamma_*(\sigma)) = \gamma_*(\overline{f}(\sigma))$ and extend $\overline{f}$ on B_1 by linearity. Then

$$(\overline{B}, \overline{f}) \prec (B_1, \overline{f})$$

contradicting the maximality of $(\overline{B}, \overline{f})$. If $k > 0$, then $\overline{f}$ is defined on $\partial\sigma$ and, as $\overline{f}(\partial\sigma)$ is straight, it is not hard to show that it extends to a straight map $\overline{f}(\sigma)$ defined on the whole Δ^k. Then define B_1 and extend $\overline{f}$ to B_1 as above. Again one has $(\overline{B}, \overline{f}) \prec (B_1, \overline{f})$, that contradicts the maximality of $(\overline{B}, \overline{f})$.

Thus $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ and $\overline{f}$ is the requested chain map Str. □

Lemma 5.3.8. *There exists a homotopy operator $H : \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3) \to \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ between Str and the identity such that $H \circ \gamma_* = \gamma_* \circ H$ for any isometry γ of $\mathbb{H}^3$.*

Proof. A homotopy operator between Str and Id is a chain-map of degree one, *i.e.* a map $H : \overline{C}_k(\overline{\mathbb{H}}^3) \to \overline{C}_{k+1}(\overline{\mathbb{H}}^3)$, such that

$$\mathrm{Str} - Id = \partial \circ H - H \circ \partial.$$

For any admissible $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$, let $h_\sigma : [0, \infty] \times \Delta^k \to \mathbb{H}^3$ be the homotopy constructed as follows: $h_\sigma(t, x)$ is the convex combination from $\sigma(x)$ to $\mathrm{Str}(\sigma)(x)$ if $\sigma(x) \notin \partial\mathbb{H}^3$ and $h_\sigma(t, x) = \sigma(x)$ otherwise. Note that from the admissibility of σ it follows that $h_\sigma(\infty, x) = \mathrm{Str}(\sigma)(x)$ for any x. So h_σ actually is a homotopy between σ and $\mathrm{Str}(\sigma)$.

As h_σ is a map $h_\sigma : [0,\infty]\times\Delta^k \to \overline{\mathbb{H}}^3$, up to triangulating $[0,\infty]\times\Delta^k$, it is a chain in $C_{k+1}(\overline{\mathbb{H}}^3)$. Fix a canonical triangulation of $[0,\infty]\times\Delta^k$ and define $H(\sigma)$ as h_σ. Since

$$\partial([0,\infty]\times\Delta^k) = \{\infty\}\times\Delta^k - \{0\}\times\Delta^k + [0,\infty]\times\partial\Delta^k$$

one gets $\partial\circ H = \mathrm{Str} - Id + H\circ\partial$.

Since h_σ is constructed using geodesic rays, then for every isometry γ one has $h_{\gamma\circ\sigma} = \gamma\circ h_\sigma$. It follows that $H\circ\gamma_* = \gamma_*\circ H$. Finally, admissibility of h_σ follows from admissibility of σ. □

Lemma 5.3.9. *Let $c = \sum_i \lambda_i\sigma_i$ be a chain in $C_k(\overline{M})$. Let $\{\gamma_j\}$ be a finite set of isometries and let A be the hyperbolic convex hull in $\mathbb{H}^3$ of*

$$\bigcup_{i,j}\gamma_j(\mathrm{Im}(\overline{\sigma}_i)).$$

Then A has finite volume.

Proof. Since D_ρ has the cone property on the cusps and since c is a finite sum of simplices, then A is contained in a geodesic polyhedron with a finite number of vertices, and such a polyhedron has finite volume. □

I am now ready to complete the proof of Theorem 5.3.1. Let $c = \sum_i \lambda_i\sigma_i$ be a relative cycle in $C_3(\overline{M})$ such that $[c] = [\overline{M}]$ in $H_3(\overline{M}, \partial\overline{M})$. By Lemma 5.3.5 I can suppose that c is admissible. Assume $\partial\widetilde{c} = \sum_j l_j\eta_j$. By Remark 5.3.2, there exists a finite set $\{\alpha_j\}\subset\pi_1(M)$ such that $\sum_j l_j\cdot\alpha_{j_*}\eta_j \in C_2(\text{ideal points})$.

Let A be as in Lemma 5.3.9, where I use $\{\rho(\alpha_j)\}\cup\{\mathrm{Id}\}$ as the set of isometries. Since A has finite volume, then the volume form ω of $\mathbb{H}^3$ is a cocycle in A. Moreover, the straightening of any admissible simplex in $\overline{C}_k(A)$ is contained in $\overline{C}_k(A)$ and, since the homotopy operator H between Str and Id is constructed using convex combinations, H is well-defined on $\oplus_k\overline{C}_k(A)$. Called v_i the volume of the straight version of σ_i, one has

$$\begin{aligned}\mathrm{vol}(\rho) &= \langle D_\rho^*\omega, \widetilde{c}\rangle = \langle\omega, (D_\rho)_*(\widetilde{c})\rangle = \langle\omega, \overline{c}\rangle\\ &= \langle\omega, \mathrm{Str}\overline{c}\rangle + \langle\omega, H\partial\overline{c}\rangle - \langle\omega, \partial H\overline{c}\rangle\\ &= \sum_i\lambda_i v_i + \langle\omega, H\partial\overline{c}\rangle - \langle d\omega, H\overline{c}\rangle = \sum_i\lambda_i v_i + \langle\omega, H\partial\overline{c}\rangle\end{aligned}$$

By Lemma 5.3.8

$$\rho(\alpha_j)_* H = H\rho(\alpha_j)_*.$$

Moreover, the volume form is invariant by isometries. It follows that

$$\begin{aligned}
\langle \omega, H\partial\overline{c}\rangle = \ & \langle \omega, H\sum_j l_j (D_\rho)_* \eta_j\rangle = \sum_j l_j \langle \omega, H(D_\rho)_* \eta_j\rangle \\
& = \sum_j l_j \langle \rho(\alpha_j)^* \omega, H(D_\rho)_* \eta_j\rangle = \sum_j l_j \langle \omega, \rho(\alpha_j)_* H(D_\rho)_* \eta_j\rangle \\
& = \sum_j l_j \langle \omega, H\rho(\alpha_j)_* (D_\rho)_* \eta_j\rangle = \sum_j l_j \langle \omega, H(D_\rho)_* \alpha_{j*} \eta_j\rangle \\
& = \langle \omega, H(D_\rho)_* \sum_j l_j \alpha_{j*} \eta_j\rangle.
\end{aligned}$$

The last product is zero because $D_{\rho *} \sum_j l_j \alpha_{j*} \eta_j$ lies on the ideal points of A, where H is fixed and ω vanishes.

This completes the proof of Theorem 5.3.1. □

Remark 5.3.10. The proof of Theorem 5.3.1 applies when the cycle c is an ideal triangulation. So it implies Theorem 5.2.10.

Corollary 5.3.11. *Let M be a graph 3-manifold. Then each representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ has volume zero.*

Proof. This is because for each graph manifold M one has $||(M, \partial M)|| = 0$ (see [12], [16]). □

Corollary 5.3.12. *Let M be a complete hyperbolic 3-manifold of finite volume. Then for all representations $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$*

$$|\mathrm{vol}(\rho)| \leq \mathrm{vol}(M).$$

Proof. This follows because if M is a complete hyperbolic 3-manifold, then $\mathrm{vol}(M) = V_3 ||(\overline{M}, \partial\overline{M})||$ (see [12], [16]). □

In [6] it is proved that, for compact manifolds, equality holds if and only if ρ is discrete and faithful. In the next section I show that this is true in general for manifolds of finite volume.

5.4. Rigidity of representations

This section is completely devoted to proving the following:

Theorem 5.4.1. *Let M be a non-compact, complete, orientable hyperbolic 3-manifold of finite volume. Let $\Gamma \cong \pi_1(M)$ be the sub-group of*

$\mathrm{PSL}(2,\mathbb{C})$ *such that* $M = \mathbb{H}^3/\Gamma$. *Let* $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ *be a representation. If* $|\mathrm{vol}(\rho)| = \mathrm{vol}(M)$ *then* ρ *is discrete and faithful. More precisely, there exists* $\varphi \in \mathrm{PSL}(2,\mathbb{C})$ *such that for any* $\gamma \in \Gamma$

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}.$$

Remark 5.4.2. It is well-known that, in the hypotheses of Theorem 5.4.1, the manifold M is the interior of a compact manifold $\overline{M}$ whose boundary consists of tori. Thus M is a cusped manifold and, by Proposition 2.1.10 it can be ideally triangulated. Then all the definitions and results I gave for ideally triangulated manifolds apply.

As product structure on the cusps I fix the horospherical one, having the arc-length as cone parameter. For this section D_ρ will denote a fixed pseudo-developing map for ρ.

Remark 5.4.3. By Proposition 5.2.18 I can suppose $\mathrm{vol}(\rho) \geq 0$.

Remark 5.4.4. A subgroup of $\mathrm{PSL}(2,\mathbb{C})$ is said to be *elementary* if it has an invariant set of at most two points in $\partial\mathbb{H}^3$. If the image of ρ is elementary, then one can construct a pseudo-developing map as in Lemma 5.1.7 in such a way that all the tetrahedra of any ideal triangulation of M collapse in the straightening. Thus, by Theorem 5.2.10, $\mathrm{vol}(\rho) = 0$.

This remark implies that, in the present case, since $\mathrm{vol}(\rho) = \mathrm{vol}(M) \neq 0$, the image of ρ is non-elementary.

The idea for proving Theorem 5.4.1 is to rewrite the Gromov-Thurston-Goldman-Dunfield proof of Mostow's rigidity, valid in the compact case.

I will follow the lead-line of [6], with the difference that I will use classical chains instead of measure-chains. The technique for constructing classical chains representing smear-cycles is that used in [1] for the proof of Mostow's rigidity for compact manifolds. As an effect of non-compactness I will work with infinite chains. Therefore, I have to prove that some of usual homological arguments actually work for these chains.

The core of the proof is to deduce from the equality $\mathrm{vol}(\rho) = \mathrm{vol}(M)$ that D_ρ "does not shrink the volume." This allows one to construct a measurable extension of D_ρ to the whole $\overline{\mathbb{H}}^3$, whose restriction to $\partial\mathbb{H}^3$ is almost everywhere a Möbius transformation. Such a Möbius transformation will be the φ of Theorem 5.4.1. To prove this, I need the following fact, whose proof can be found in [6] (claim 3 of Theorem 6.1).

Proposition 5.4.5. *Let* $f : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ *be a measurable map that maps the vertices of almost all regular ideal tetrahedra to vertices of regular ideal tetrahedra. Then* f *coincides almost everywhere with the trace of an isometry* φ.

I want to apply Proposition 5.4.5 to D_ρ, and I will do it in two steps. Let M_0 be M minus the cusps and let $\pi : \mathbb{H}^3 \to M$ be the universal cover. I state two results I will prove below:

Proposition 5.4.6. *The map D_ρ extends to $\overline{\mathbb{H}}^3$. More precisely, there exists a measurable map $\overline{D}_\rho : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ such that for almost all $x \in \partial\mathbb{H}^3$, for any geodesic γ^x ending at x, for any sequence $t_n \to \infty$ such that $\pi(\gamma^x(t_n)) \in M_0$, it holds*

$$\lim_{n\to\infty} D_\rho(\gamma^x(t_n)) = \overline{D}_\rho(x).$$

Proposition 5.4.7. *The map $\overline{D}_\rho$ satisfies the hypothesis of Proposition* 5.4.5.

Before proving Propositions 5.4.6, and 5.4.7 I show how they imply Theorem 5.4.1

Proof of 5.4.1. By Proposition 5.4.7, Proposition 5.4.5 applies. By Proposition 5.4.6 the equivariance of D_ρ implies the equivariance of $\overline{D}_\rho$, yielding for any $\gamma \in \Gamma$

$$\rho(\gamma) = \varphi\circ\gamma\circ\varphi^{-1}. \qquad \square$$

Remark 5.4.8. Both Propositions 5.4.6, and 5.4.7 will follow from Lemmas 5.4.22 and 5.4.23 below. I notice that Lemma 5.4.22 is a restatement of Lemma 6.2 of [6], while Proposition 5.4.7 corresponds to *Claim* 2 of [6]. Proposition 5.4.6 follows from Lemmas 5.4.22 and 5.4.23 exactly as in [6]. I will give a complete proof of Proposition 5.4.7 because the proof of *Claim* 2 in [6] seems to be incomplete.

From now until Lemma 5.4.11 I will describe how to construct a simplicial version of the *smearing* process of measure-homology (see [26] or [24]). Then I will prove Lemma 5.4.23. Finally I will complete the proof of Theorem 5.4.1 by proving Propositions 5.4.6 and 5.4.7.

Let μ be the Haar measure on $\mathrm{Isom}(\mathbb{H}^3)$ such that for each $x \in \mathbb{H}^3$ and $A \subset \mathbb{H}^3$ it is

$$\mu\{g \in \mathrm{Isom}(\mathbb{H}^3) : g(x) \in A\} = \mathrm{vol}(A)$$

where $\mathrm{vol}(A)$ is the hyperbolic volume of A.

In the following, by a tetrahedron of $\overline{\mathbb{H}}^3$ I mean an ordered 4-tuple of points (the vertices). The volume of a tetrahedron is the hyperbolic volume with sign of the convex hull of its vertices.

Let S be the set of all genuine (non-ideal, non-degenerate) tetrahedra:

$$S = \{(y_0, \dots, y_3) \in (\mathbb{H}^3)^4 : \operatorname{vol}(y_0, \dots, y_3) \neq 0\}.$$

For any $Y \in S$ let $S(Y)$ be the set of all isometric copies of Y:

$$S(Y) = \{X \in S : \exists g \in \operatorname{Isom}(\mathbb{H}^3),\ X = g(Y)\}.$$

Then a natural bijection $f_Y : \operatorname{Isom}(\mathbb{H}^3) \to S(Y)$ is well-defined by

$$f_Y(g) = g(Y).$$

Thus μ induces a measure, which I still call μ, on $S(Y)$ defined by

$$\mu(A) = \mu(f_Y^{-1}(A)).$$

I consider the sets $S_\pm(Y) = f_Y^{-1}(\operatorname{Isom}^\pm(\mathbb{H}^3))$ of tetrahedra respectively positively and negatively isometric to Y. Note that $S_+(Y)$ and $S(Y)_-$ are both measurable.

Set $\mathfrak{S} = \Gamma^4/\Gamma$ where Γ acts on Γ^4 by left multiplication. Each element $\sigma = [(\gamma_0, \dots, \gamma_3)] \in \mathfrak{S}$ has a unique representative with $\gamma_0 = \mathrm{Id}$. When I write $\sigma \in \mathfrak{S}$, I tacitly assume that the representative of the form $(\gamma_0, \dots, \gamma_3)$ with $\gamma_0 = \mathrm{Id}$ has been chosen. So γ_0 is always the identity.

For the rest of the section I fix a fundamental polyhedron $F \subset \mathbb{H}^3$ for M. For all $\varepsilon > 0$ let $\mathcal{F}^\varepsilon$ be a locally finite ε-net in F. For any $\xi \in \mathcal{F}^\varepsilon$ let

$$F_\xi = \{x \in F : d(x, \xi) = d(x, \mathcal{F}^\varepsilon)\}.$$

Each F_ξ is a geodesic polyhedron of diameter less than ε. From the cone-property of D_ρ it follows that the diameters of the $D_\rho(F_\xi)$'s are bounded by a constant δ that depends on ε. Moreover, by removing some boundary face from some the F_ξ's, one gets that F is the disjoint union of the F_ξ's. I set

$$S^0(Y) = \{X \in S(Y) \text{ with first vertex in } F\}.$$

I define now a family of special simplices. Let

$$\mathfrak{N} = \{(\gamma_0, \dots, \gamma_3, \xi_0, \dots, \xi_3) : (\gamma_0, \dots, \gamma_3) \in \mathfrak{S},\ \xi_i \in \mathcal{F}^\varepsilon \text{ for all } i\}.$$

For each $\eta \in \mathfrak{N}$ define Δ_η as the straight geodesic singular 3-simplex whose vertices are the points $\xi_0, \gamma_1(\xi_1), \gamma_2(\xi_2), \gamma_3(\xi_3)$, more precisely

$$\Delta_\eta : \Delta^3 \ni t \mapsto \pi\Big(\sum_{i=0}^{3} t_i \gamma_i(\xi_i)\Big).$$

For each tetrahedron $X = (x_0, \dots, x_3) \in S^0(Y)$ there exists a unique $\eta = (\gamma_0, \dots, \gamma_3, \xi_0, \dots, \xi_3) \in \mathfrak{N}$ such that $x_i \in \gamma_i(F_{\xi_i})$ for $i = 0, \dots, 3$. This defines a function

$$s_Y : S^0(Y) \to \mathfrak{N}.$$

Roughly speaking, $\mathfrak{N}$ is a locally finite ε-net in the space of 3-simplices of M and s_Y is the "closest point"-projection.

For any $\eta \in \mathfrak{N}$ define

$$a_Y^{\pm}(\eta) = \mu\{s_Y^{-1}(\eta) \cap S_{\pm}(Y)\} = \mu\{X \in S_{\pm}(Y) : \ x_i \in \gamma_i(F_{\xi_i})\}$$

and

$$a_Y(\eta) = a_Y^{+}(\eta) - a_Y^{-}(\eta).$$

In the language of measures, one can think of $a_Y^{\pm}$ as the push-forward of the measure μ under the map $s_Y : S^0(Y) \cap S_{\pm}(Y) \to \mathfrak{N}$. This is the key for the passage from measure-chains to classical ones.

The smearing of the tetrahedron Y is the cycle:

$$Z_Y = \sum_{\eta \in \mathfrak{N}} a_Y(\eta)\Delta_\eta.$$

I notice that, as $\mathfrak{N}$ depends on the family $\mathcal{F}^\varepsilon$, the cycle Z_Y actually depends on ε.

Remark 5.4.9. The smearing of a tetrahedron in general is not a finite sum. Nevertheless, as the following lemma shows, it has bounded l^1-norm.

Lemma 5.4.10. *For any $Y \in S$, it holds* $\sum_{\eta} |a_Y(\eta)| < \mathrm{vol}(M)$.

Proof. If $Y = (y_0, \dots y_3)$, then

$$\begin{aligned}
\sum_{\eta} |a_Y(\eta)| &\le \sum_{\eta} \left(a_Y^{+}(\eta) + a_Y^{-}(\eta)\right) \\
&= \sum_{\eta} \mu\{s_Y^{-1}(\eta)\} \\
&= \mu\Big\{ \bigcup_{\eta} s_Y^{-1}(\eta) \Big\} \\
&= \mu\{ s_Y^{-1}(\mathfrak{N}) \} = \mu\{ f_Y^{-1} s_Y^{-1}(\mathfrak{N}) \} \\
&= \mu\{g : \ g(y_0) \in F\} = \mathrm{vol}(F) = \mathrm{vol}(M).
\end{aligned}$$

□

Lemma 5.4.11. *The infinite chain Z_Y is a cycle,* i.e. $\partial Z_Y = 0$.

Proof. First note that the l^1-norm of ∂Z_Y is bounded by 4 times the l^1-norm of Z_Y. Thus all the sums I will consider make sense.

Let υ be a simplex of ∂Z_Y. By construction υ is obtained as the projection of an 2-simplex having vertices in $F_{\xi_0}, \gamma_1(F_{\xi_1}), \gamma_2(F_{\xi_2})$ for some $\gamma_1, \gamma_2 \in \Gamma$ and $\xi_0, \xi_1, \xi_2 \in \mathcal{F}^\varepsilon$. Let A_υ be the set of the elements of $\mathfrak{N}$ of the form $\eta = (\gamma_0, \gamma_1, \gamma_2, \gamma, \xi_0, \xi_1, \xi_2, \xi)$ with $\gamma \in \Gamma$ and $\xi \in \mathcal{F}^\varepsilon$. The simplices Δ_η of Z_Y having υ as the last face contribute to the coefficient of υ in ∂Z_Y by

$$\begin{aligned}\sum_{\eta \in A_\upsilon} a_Y(\eta) &= \sum_{\eta \in A_\upsilon} \mu(s_Y^{-1}(\eta) \cap S_+(Y)) - \sum_{\eta \in A_\upsilon} \mu(s_Y^{-1}(\eta) \cap S_-(Y)) \\ &= \mu(s_Y^{-1}(A_\upsilon) \cap S_+(Y)) - \mu(s_Y^{-1}(A_\upsilon) \cap S_-(Y)) = 0.\end{aligned}$$

The same calculation, made with the simplices having υ as the ith face, shows that the coefficient of υ in ∂Z_Y is zero. □

For any ideal, non-flat, tetrahedron $Y = (y_0, \dots, y_3)$ let $t \mapsto y_i(t)$ be the geodesic ray from the center of mass of Y to y_i, $i = 0, \dots, 3$. For any $R > 0$ let Y_R be the following element of S:

$$Y_R = (y_0(R), \dots, y_3(R)).$$

Remark 5.4.12. From now on I fix a positively oriented regular ideal tetrahedron Y, and I write $S_\pm(R)$, f_R, s_R, $a_R(\eta)$ and Z_R for $S_\pm(Y_R)$, f_{Y_R}, s_{Y_R}, $a_{Y_R}(\eta)$ and Z_{Y_R}.

I say that a 3-simplex Δ is ε-close to a tetrahedron X if the vertices of Δ are ε-close to X. I define

$$\epsilon(R, \varepsilon) = \sup\{V_3 - \mathrm{vol}(\Delta) : \ \Delta \text{ is } \varepsilon\text{-close to an element of } S(R)\}$$

Lemma 5.4.13. *For any fixed ε, for large R the function $\epsilon(R, \varepsilon)$ goes to zero exponentially in R.*

This is because $V_3 - \mathrm{vol}(Y_R)$ goes to zero like e^{-R} and the volume of any Δ which is ε-close to Y_R is close to the volume of Y_R. See [1], [6], [26] for details.

Remark 5.4.14. What I actually need to prove my claims is a restatement for Z_R of Step 2 of Theorem 5.3.1. From now until Proposition 5.4.20, I prove facts that are standard for finite chains, but need a proof for Z_R.

For $\eta \in \mathfrak{N}$, I set $v_\eta = \mathrm{vol}(\Delta_\eta)$. Using the fact that all the F_ξ's have diameter less than ε, one can prove the following lemma (see [1] for details). Recall that $\mathfrak{N}$ depends on $\mathcal{F}^\varepsilon$ and so it depends on ε.

Lemma 5.4.15. *For any $\varepsilon > 0$, for large enough R one has that for any $\eta \in \mathfrak{N}$*

- $a_R^+(\eta) \cdot a_R^-(\eta) = 0$.
- $a_R(\eta) \neq 0 \implies a_R(\eta) \cdot v_\eta \geq 0$.

Lemma 5.4.16. *There exists a constant c such that $|D_\rho^* \omega| < c|\omega|$, where ω is the volume-form of $\mathbb{H}^3$.*

Proof. Let M_0 be M minus the cusps. The function $|D_\rho^*\omega|/|\omega|$ is continuous and hence bounded on M_0. I prove by direct calculation that the same bound holds on M. Let C_p be a cusp of M and let $P_{\widetilde{p}} \times [0, \infty)$ be a lift of C_p to $\widetilde{M} = \mathbb{H}^3$. Since the product structure of the cusps is the horospherical one (see Remark 5.4.2) the metric on $P_{\widetilde{p}} \times \{t\}$ is a Euclidean metric on $P_{\widetilde{p}} \times \{0\}$ rescaled by $1/e^t$ and the parameter t is the arc length. It follows that at the point $(x, y, t) \in P_{\widetilde{p}} \times [0, \infty)$

$$\omega(x, y, t) = c_1(x, y)\frac{1}{e^{2t}}dx \wedge dy \wedge dt$$

where c_1 is a continuous function not depending on t. Moreover by the cone property of developing maps, the restriction of D_ρ to $P_{\widetilde{p}} \times [0, \infty)$ can be written as

$$D_\rho(x, y, t) = \Big(\alpha(x, y) + i\beta(x, y), h(x, y)e^t\Big)$$

where, using the half space model of $\mathbb{H}^3$, $(\alpha + i\beta, h) \in \mathbb{C} \times \mathbb{R}^+$. It follows that the metric at the point $D_\rho(x, y, t)$ is the Euclidean one rescaled by $1/(h(x, y)e^t)$, and that

$$|D_\rho^*\omega|(x, y, t) = \frac{c_2(x, y)}{h^2(x, y)e^{2t}}$$

where c_2 is a continuous function not depending on t. The thesis follows. □

Lemma 5.4.17. *The integrals $\langle \omega, Z_R \rangle$ and $\langle D_\rho^*\omega, Z_R \rangle$ are well-defined.*

Proof. As $\sum |a_R(\eta)| < +\infty$, since $|\langle \omega, \Delta_\eta \rangle|$ is bounded by V_3, then $\langle \omega, Z_R \rangle$ is well-defined. Consider now $D_\rho^*\omega$. From Lemma 5.4.16 it follows that the integral of $|D_\rho^*|$ over straight geodesic simplices is bounded by cV_3. Hence also $\langle D_\rho^*\omega, Z_R \rangle$ is well-defined. □

As above, let M_0 denote M minus the cusps and, for $k \in \mathbb{N}^*$ let

$$M_k = \bigcup_{T \subset \partial M_0} T \times [k-1, k).$$

Let $\mathcal{F}_k^\varepsilon = \mathcal{F}^\varepsilon \cap \pi^{-1}(M_k)$ and $\mathfrak{N}_k = \{\eta \in \mathfrak{N} : \xi_0 \in \mathcal{F}_k^\varepsilon\}$. Then

$$Z_R = \sum_{k \in \mathbb{N}} \sum_{\eta \in \mathfrak{N}_k} a_R(\eta) \Delta_\eta.$$

Lemma 5.4.18. *For any k the chain $\sum_{\eta \in \mathfrak{N}_k} a_R(\eta)\Delta_\eta$ is a finite sum.*

Proof. If $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_k$ then Δ_η is ε-close to an element $X \in S(R)$ having first vertex in F_{ξ_0} with $\xi_0 \in \mathcal{F}_k^\varepsilon$. Since $\mathcal{F}^\varepsilon$ is locally finite and $\overline{M}_k$ is compact, $\mathcal{F}_k^\varepsilon$ is finite, so there is only a finite number of possibilities for ξ_0. Since $\overline{F}_{\xi_0}$ is compact, any $X \in S(R)$ with first vertex in F_{ξ_0} lies in a compact ball B of $\mathbb{H}^3$. Since F is a fundamental domain, then there exists only a finite number of elements $\gamma \in \Gamma$ so that $\gamma(F)$ intersects B. Then for any ξ_0 there is only a finite number of possibilities for ξ_1, ξ_2 and ξ_3. It follows that there exists only a finite number of $\eta \in \mathfrak{N}_k$ such that $a_R(\eta) \neq 0$. □

Lemma 5.4.19. *For any R, if k is large enough, then for any $\eta \in \mathfrak{N}_k$ with $a_R(\eta) \neq 0$, the simplex Δ_η is completely contained in a cusp of M.*

Proof. If $X = (x_0, \ldots, x_3) \in S(R)$ then X lies in the ball $B(x_0, 2R)$. Since M has a finite number of cusps, for any R there exists $m \in \mathbb{N}$ such that for $k \geq m$ if $x_0 \in M_k$ then the whole ball $B(x_0, 2R+\varepsilon)$ is contained in the cusp containing x_0. If $\eta \in \mathfrak{N}_k$ and $a_R(\eta) \neq 0$, then there exists $X \in S(R)$ with $x_0 \in \pi^{-1}(M_k) \cap F$ hence Δ_η is ε-close to X. Thus $\Delta_\eta \subset B(x_0, 2R+\varepsilon)$ is contained in the cusp that contains x_0. □

Now for $k \in \mathbb{N}$ define

$$Z_{R,k} = \sum_{j<k} \sum_{\eta \in \mathfrak{N}_j} a_R(\eta)\Delta_\eta.$$

$Z_{R,k}$ is a finite chain by Lemma 5.4.18. Moreover, since $\partial Z_R = 0$, each simplex υ of $\partial Z_{R,k}$ appears as a face of a simplex Δ_η with $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_j$ for some $j \geq k$. Therefore, by Lemma 5.4.19, for k large enough each simplex υ of $\partial Z_{R,k}$ is contained in a cusp of M. Thus to each υ there corresponds an ideal point of $\widehat{M}$. For each $\upsilon \in \partial Z_{R,k}$ let $\lambda_{R,k}(\upsilon)$ be the coefficient of υ in $\partial Z_{R,k}$ and let C_υ be the cone from υ to the corresponding ideal point.

Let $\widehat{Z}_{R,k}$ be the chain obtained by adding to $Z_{R,k}$ the cones C_υ:

$$\widehat{Z}_{R,k} = Z_{R,k} + \sum_{\upsilon \in \partial Z_{R,k}} \lambda_{R,k}(\upsilon) C_\upsilon.$$

The chain $\widehat{Z}_{R,k}$ is a finite sum and it is easily checked that it is a cycle.

For any 3-simplex Δ let $\mathrm{Strvol}(\Delta)$ denote the volume of the convex hull of the vertices of $D_\rho(\Delta)$. For any $\eta \in \mathfrak{N}$ set $w_\eta = \mathrm{Strvol}(\Delta_\eta)$.

Proposition 5.4.20. *For any* $R > 0$

$$\sum_\eta a_R(\eta) v_\eta = \langle \omega, Z_R \rangle = \langle D_\rho^* \omega, Z_R \rangle = \sum_\eta a_R(\eta) w_\eta.$$

Proof. The first equality is tautological. I use now the cycles $\widehat{Z}_{R,k}$ to approximate Z_R. Since $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, then $[\omega] = [D_\rho^* \omega]$ as elements of $H^3(\widehat{M})$. Thus for any $k \in \mathbb{N}$

$$\langle \omega, \widehat{Z}_{R,k} \rangle = \langle D_\rho^* \omega, \widehat{Z}_{R,k} \rangle.$$

As in Step 2 of Theorem 5.3.1, one can straighten the finite cycle $\widehat{Z}_{R,k}$, getting:

$$\langle \omega, \widehat{Z}_{R,k} \rangle = \langle D_\rho^* \omega, \widehat{Z}_{R,k} \rangle = \sum_{j<k} \sum_{\eta \in \mathfrak{N}_k} a_R(\eta) w_\eta + \sum_{\upsilon \in \partial Z_{R,k}} \lambda_{R,k}(\upsilon) \mathrm{Strvol}(C_\upsilon).$$

For each simplex α of $\widehat{Z}_{R,k}$ it is $|\mathrm{vol}(\alpha)| \leq V_3$, $|\mathrm{Strvol}(\alpha)| \leq V_3$ and, by Lemma 5.4.16, $|\langle D_\rho^* \omega, \alpha \rangle| \leq c V_3$. It follows that to get the remaining equalities it suffices to show that

$$\lim_{k \to \infty} \sum_{\upsilon \in \partial Z_{R,k}} |\lambda_{R,k}(\upsilon)| = 0.$$

Since $\partial Z_R = 0$, if $\upsilon \in \partial Z_{R,k}$ then $\upsilon \in \partial \Delta_\eta$ with $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_j$ for some $j \geq k$. So one has

$$\begin{aligned}
\sum_{\upsilon \in \partial Z_{R,k}} |\lambda_{R,k}(\upsilon)| &\leq 4 \sum_{j \geq k} \sum_{\eta \in \mathfrak{N}_j} a_R^+(\eta) + a_R^-(\eta) \\
&= 4 \sum_{j \geq k} \sum_{\eta \in \mathfrak{N}_j} \mu\{s_R^{-1}(\eta)\} \\
&= 4 \sum_{j \geq k} \mu\{s_R^{-1}(\mathfrak{N}_j)\} \\
&= 4 \sum_{j \geq k} \mu\{Y \in S(R) : \exists \xi \in \mathcal{F}_j^\varepsilon,\ y_0 \in F_\xi\} \\
&= 4 \sum_{j \geq k} \sum_{\xi \in \mathcal{F}_j^\varepsilon} \mathrm{vol}(F_\xi) \leq 4 \mathrm{vol}\left(\bigcup_{j \geq k - \varepsilon} M_j \right)
\end{aligned}$$

The last term goes to zero as $k \to \infty$ because M has finite volume and the desired equality follows. $\square$

Now that Proposition 5.4.20 is proved, forget about the cycles $\widehat{Z}_{R,k}$. From the triangular inequality, Proposition 5.4.20 and Lemma 5.4.15

$$\begin{aligned}\sum_\eta |a_R(\eta)| \cdot |w_\eta| &\geq \Big|\sum_\eta a_R(\eta) w_\eta\Big| \\ &= \Big|\sum_\eta a_R(\eta) v_\eta\Big| \\ &= \sum_\eta |a_R(\eta)| \cdot |v_\eta| \\ &\geq \sum_\eta |a_R(\eta)|(V_3 - \epsilon(R,\varepsilon))\end{aligned}$$

from which and Lemma 5.4.10 one gets:

Proposition 5.4.21. *For R large enough*

$$\sum_{\eta\in\mathfrak{N}} |a_R(\eta)|(V_3 - |w_\eta|) \leq \sum_{\eta\in\mathfrak{N}} |a_R(\eta)|\epsilon(R,\varepsilon) \leq \mathrm{vol}(M)\epsilon(R,\varepsilon).$$

For any $R > 0$ let $A_R \subset \mathfrak{N}$ be the set of tetrahedra with "small" straight volume:

$$A_R = \{\eta \in \mathfrak{N} : \ V_3 - |w_\eta| > R^2 \cdot \mathrm{vol}(M) \cdot \epsilon(R,\varepsilon)\}.$$

Lemma 5.4.22. *For R large enough*

$$\sum_{\eta\in A_R} |a_R(\eta)| \leq \frac{1}{R^2}.$$

Proof. From Proposition 5.4.21 one gets

$$\begin{aligned}R^2\mathrm{vol}(M)\epsilon(R,\varepsilon) \cdot \sum_{\eta\in A_R} |a_R(\eta)| &\leq \sum_{\eta\in A_R} |a_R(\eta)|(V_3 - |w_\eta|) \\ &\leq \sum_{\eta\in\mathfrak{N}} |a_R(\eta)|(V_3 - |w_\eta|) \\ &\leq \mathrm{vol}(M)\epsilon(R,\varepsilon)\end{aligned}$$

The claimed inequality follows. $\square$

Lemma 5.4.23. *For almost all isometries g*

$$\lim_{n\to\infty} \operatorname{Strvol}(g(Y_n)) = V_3.$$

Proof. Since $a_R^+ \cdot a_R^- = 0$, then $\sum_{\eta\in A_R} |a_R(\eta)| = \mu(s_R^{-1}(A_R))$. Thus for any fixed $R > 0$

$$\mu\left(\bigcup_{\mathbb{N}\ni n>R} s_R^{-1}(A_n)\right) \le \sum_{n>R} \frac{1}{n^2} < \frac{1}{R}.$$

Recalling that for any set $A \subset \mathfrak{N}$

$$\mu(s_R^{-1}(A)) = \mu(f_R^{-1} s_R^{-1}(A))$$

one gets

$$\mu\{g \in \operatorname{Isom}(\mathbb{H}^3) : \exists n > R,\ w_{s_n(g(Y_n))} < V_3 - n^2 \cdot \operatorname{vol}(M) \cdot \epsilon(n,\varepsilon)\} < \frac{1}{R}.$$

From Lemma 5.4.13 it follows that $\lim_{n\to\infty} n^2 \epsilon(n,\varepsilon) = 0$. As $R \to \infty$, this implies that for any $\varepsilon > 0$, for almost any isometry g

$$\lim_{n\to\infty} w_{s_n(g(Y_n))} = V_3.$$

Let g be one of such maps. Since the diameters of the $D_\rho(F_\xi)$ are bounded by δ, then $D_\rho(\Delta_{s_R(g(Y_R))})$ is δ-close to $D_\rho(g(Y_R))$. Recalling that $w_{s_R(g(Y_R))} = \operatorname{Strvol}(\Delta_{s_R(g(Y_R))})$, one gets that

$$\lim_{n\to\infty} \operatorname{Strvol}(\Delta_{s_n(g(Y_n))}) = V_3$$

and, since $D_\rho(g(Y_R))$ is δ-close to $D_\rho(\Delta_{s_R(g(Y_R))})$, then also

$$\lim_{n\to\infty} \operatorname{Strvol}(g(Y_n)) = V_3. \qquad \square$$

I sketch here the proof of Proposition 5.4.6, referring to [6] for details.

Proof of Proposition 5.4.6. In the disc model let γ be a geodesic from 0 to a point in $\partial\mathbb{H}^3$. Let X_R be a family of regular tetrahedra of edge R with first vertex in 0 and second in $\gamma(R)$. All the claims from Lemma 5.4.10 to Lemma 5.4.23 hold for $\{X_R\}$. It follows that for almost all isometries g

$$\lim_{n\to\infty} \operatorname{Strvol}(g(X_n)) = V_3.$$

Then $D_\rho(g(\gamma(n)))$ must reach the boundary of $\mathbb{H}^3$. Using again the above property of the limit, one can estimate the angle $\alpha(n)$ between the geodesic from $D_\rho(g(0))$ to $D_\rho(g(\gamma(n)))$ and the geodesic from $D_\rho(g(0))$ and $D_\rho(g(\gamma(n+1)))$. Such estimate shows that $\sum \alpha(n) < \infty$, which implies that $D_\rho(g(\gamma(n)))$ converges. The claim follows because D_ρ is locally Lipschitz outside the cusps. Measurability follows because the extension can be viewed as a point-wise limit of measurable functions.□

Remark 5.4.24. In general D_ρ is not uniformly continuous in the cusps. So it cannot be locally Lipschitz on the whole $\mathbb{H}^3$.

I come now to the proof of Proposition 5.4.7.

Lemma 5.4.25. *Let $X = (x_0, x_1, x_2, x_3)$ be an ideal tetrahedron in $\overline{\mathbb{H}}^3$. Suppose that no three vertices of X coincide. Then for any $\varepsilon > 0$ there exist neighborhoods U_i of x_i in $\overline{\mathbb{H}}^3$ such that for any tetrahedron $Y = (y_0, \dots, y_3)$ with $y_i \in U_i$ it holds $|\mathrm{vol}(Y) - \mathrm{vol}(X)| < \varepsilon$.*

This follows from the formula of the volume for ideal tetrahedra, see [1] for details.

Remark 5.4.26. Lemma 5.4.25 does not hold if three vertices of X coincide. To see this, let Y be a regular ideal tetrahedron and let γ be a parabolic or hyperbolic isometry. Then $\gamma^n(Y)$ is a family of tetrahedra with maximal volume, but at least three of the vertices of $\gamma^n(Y)$ converge to the same point.

Lemma 5.4.27. *For almost all regular ideal tetrahedra Y, the ideal tetrahedron $\overline{D}_\rho(Y)$ is defined. Moreover, for almost all Y either $\overline{D}_\rho(Y)$ is regular (whence $\mathrm{vol}(\overline{D}_\rho(Y)) = V_3$) or at least three of its vertices coincide (whence $\mathrm{vol}(\overline{D}_\rho(Y)) = 0$).*

Proof. Without loss of generality, I can restrict the first claim to the space of positive regular ideal tetrahedra. I parametrize such a space with

$$\{(a, b, c) \in S^2_\infty \times S^2_\infty \times S^2_\infty : a \neq b \neq c\}$$

where $S^2_\infty = \partial\mathbb{H}^3$, by mapping (a, b, c) to the unique positive regular ideal tetrahedron with (a, b, c) as the first three vertices. I denote by $Q(a, b, c)$ the fourth vertex of such tetrahedron. Since $\overline{D}_\rho$ is defined almost everywhere, the first claim follows from Fubini's theorem. The second claim follows from Lemmas 5.4.23 and 5.4.25. □

With the above notation, by Lemma 5.4.27 I can restate Proposition 5.4.7 as follows.

Proposition 5.4.28. *The set $\{Y \in S^2_\infty \times S^2_\infty \times S^2_\infty : \mathrm{vol}(\overline{D}_\rho(Y)) = 0\}$ has zero measure.*

The proof of this result will follow from the next:

Lemma 5.4.29. *If the set*

$$\{Y \in S^2_\infty \times S^2_\infty \times S^2_\infty : \operatorname{vol}(\overline{D}_\rho(Y)) = 0\}$$

has positive measure, then the map $\overline{D}_\rho$ is constant almost everywhere.

Before proving Lemma 5.4.29 I show how it implies Proposition 5.4.28.

Proof of 5.4.28. By contradiction, I apply Lemma 5.4.29 deducing that $\overline{D}_\rho$ is almost everywhere a constant p. From the equivariance of $\overline{D}_\rho$ it follows that for any $\gamma \in \Gamma$ and $x \in \partial\mathbb{H}^3$ one has

$$p = \overline{D}_\rho\gamma(x) = \rho(\gamma)(\overline{D}_\rho(x)) = \rho(\gamma)(p).$$

Thus p is a fixed point of any element of Γ. This implies that the image of ρ is elementary, but this cannot happen because of Remark 5.4.4. $\square$

I now prove Lemma 5.4.29.

Lemma 5.4.30. *In the hypothesis of Lemma* 5.4.29 *there exists a positive-measure set $A \subset S^2_\infty$ such that $\overline{D}_\rho$ is constant on A.*

Proof. By Lemma 5.4.27 it is not restrictive to suppose that the set

$$\{(a, b, c) \in S^2_\infty \times S^2_\infty \times S^2_\infty : \overline{D}_\rho(a) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\}$$

has positive measure. Then by Fubini's theorem there exists a positive-measure set $A_0 \subset S^2_\infty$ such that for all $a_0 \in A_0$ the set

$$\{(b, c) \in S^2_\infty \times S^2_\infty : \overline{D}_\rho(a_0) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\}$$

has positive measure in $S^2_\infty \times S^2_\infty$. Again by Fubini's theorem for all $a_0 \in A_0$ there exists a positive-measure set $A_1 \in S^2_\infty$ such that for any $a_1 \in A_1$ the set

$$\{c \in S^2_\infty : \overline{D}_\rho(a_0) = \overline{D}_\rho(a_1) = \overline{D}_\rho(c)\}$$

has positive measure. In particular $\overline{D}_\rho$ is constant on A_1. $\square$

I set $p = \overline{D}_\rho(A_1)$ and $A = \overline{D}_\rho^{-1}(p)$.

Remark 5.4.31. In the sequel I use the symbol $\widetilde{\forall}$ to mean *"for almost all."*

By Lemma 5.4.27 the set A has the following property

$$\widetilde{\forall}(a_0, a_1, x) \in A \times A \times A^c, \quad Q(a_0, a_1, x) \in A.$$

We work now in the half space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$. So $S^2_\infty = \mathbb{C} \cup \{\infty\}$. In that model

$$Q(\infty, a, z) = \alpha(z - a) + a$$

where $\alpha = (1+i\sqrt{3})/2$. Again by Fubini's theorem $\widetilde{\forall} a_0 \in A$, $\widetilde{\forall}(a_1, x) \in A \times A^c$, we have $Q(a_0, a_1, x) \in A$ and we can suppose that this holds for $a_0 = \infty$.

In other words, for almost all $(a, x) \in A \times A^c$ the third vertex of the equilateral triangle with the first two vertices in a and x is in A. For any $a, x \in \mathbb{C}$ we call $E_x(a)$ the set of the vertices of the regular hexagon centered at x and with a vertex in a. Then we have

$$\widetilde{\forall}(a, x) \in A \times A^c, \ E_x(a) \subset A \tag{5.1}$$

and in particular $\widetilde{\forall}(a, x) \in A \times A^c$, $2x - a \in A$. Note that x is the middle-point of the segment between a and $2x - a$.

Lemma 5.4.32. *For any open set $B \subset \mathbb{C}$ one has $\mu(A \cap B) > 0$.*

Proof. Suppose the contrary. Then there exists an open set B such that $\mu(A \cap B) = 0$. That is, almost all the points of B are in A^c. Moreover, from (5.1) and Fubini's theorem it follows that $\widetilde{\forall} x \in A^c, \widetilde{\forall} a \in A$, $E_x(a) \in A$. Therefore there exists a point $x_0 \in B$ such that a small ball $B_0 = B(x_0, r_0)$ is contained in B and

$$\widetilde{\forall} a \in A, \ E_{x_0}(a) \in A. \tag{5.2}$$

Since $\mu(A) > 0$ then there exists a small ball $B_1 = B(x_1, r_1)$ such that $\mu(A \cap B_1) > 0$. Let $x_2 = (x_1 + x_0)/2$. If there exists $r > 0$ such that $\mu(A \cap B(x_2, r)) = 0$, then applying the same argument one can find a point y arbitrarily close to x_2 such that (5.2) holds for y. In particular one gets that almost all the points of the set $C = \{2y - a : \ a \in B_1 \cap A\}$ are in A. But if y is close enough to x_2 then $C \cap B_0$ has positive measure, contradicting that $\mu(A \cap B) = 0$.

It follows that for all $r_2 > 0$ one has $\mu(A \cap B(x_2, r_2)) > 0$, in particular I choose $r_2 < r_0/2$. By iterating this construction, I find a sequence of points $x_n \to x_0$ and radii $r_0/2 > r_n > 0$ such that $\mu(A \cap B(x_n, r_n)) > 0$. For n large enough this contradicts the fact that $\mu(A \cap B) = 0$. $\square$

Lemma 5.4.33. *For all $z \in \mathbb{C}$*

$$\forall r > 0 \ \ \mu(B(z,r) \cap A) \geq \frac{1}{2}\mu(B(z,r)). \tag{5.3}$$

Proof. From Fubini's theorem, and condition (5.1), it follows that for almost all $a \in A$

$$\widetilde{\forall} x \in A^c, \ \ E_x(a) \subset A. \tag{5.4}$$

Note that if (5.4) holds for a, then (5.3) holds for a.

Let $z \in \mathbb{C}$. From Lemma 5.4.32 it follows that there exists a sequence $x_n \to z$ such that (5.4) (and hence (5.3)) holds for x_n. As the function $x \mapsto \mu(A \cap B(x,r))$ is continuous, then the claim holds for z. □

Lemma 5.4.34. *Let $X \subset \mathbb{R}^2$ be a measurable set. If there exists $\alpha > 0$ such that for any ball B*

$$\mu(B \cap X) \geq \alpha\mu(B)$$

then $\mu(\mathbb{R}^2 \setminus X) = 0$.

Proof. This is a standard fact of integration theory and it follows from Lebesgue's differentiation theorem (see for example [25]). I briefly outline the proof. The inequality for the balls easily implies the same inequality for any measurable set, so

$$0 = \mu(X^c \cap X) \geq \alpha\mu(X^c). \qquad \square$$

From this lemma and Lemma 5.4.33 it follows that the set A has full measure. Since $A = \overline{D}_\rho^{-1}(p)$, the map $\overline{D}_\rho$ is constant almost everywhere and Lemma 5.4.29 is proved. □

This completes the proof of Theorem 5.4.1.

5.5. Corollaries

In this section I prove some corollaries that can be useful for studying hyperbolic 3-manifolds.

First I show how from Theorem 5.4.1 one gets a proof of Mostow's rigidity for non-compact manifolds (see [3] for a more general statement and a different proof).

Theorem 5.5.1. (Mostow's rigidity for non-compact manifolds) *Let $f : M \to N$ be a proper map between two orientable non-compact, complete hyperbolic 3-manifolds of finite volume. Suppose that*

$$\mathrm{vol}(M) = \deg(f)\mathrm{vol}(N).$$

Then f is properly homotopic to a locally isometric covering with the same degree as f.

Proof. Let ω be the volume form of N. For $X = M, N$ let $\Gamma_X \cong \pi_1(X)$ be the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ such that $X = \mathbb{H}^3/\Gamma_X$. Let f_* denote both the map induced in homology and the representation $f_* : \pi_1(M) \to \Gamma_N < \mathrm{PSL}(2,\mathbb{C})$.

First assume that the lift $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ has the cone-property on the cusps. This implies that $\widetilde{f}$ is a pseudo-developing map for f_*. Since $f_*[M] = \deg(f)[N]$,

$$\begin{aligned}\mathrm{vol}(M) &= \deg(f)\cdot\mathrm{vol}(N) = \langle\omega, \deg(f)[N]\rangle\\ &= \langle\omega, f_*[M]\rangle = \langle f^*\omega, [M]\rangle = \mathrm{vol}(f_*).\end{aligned}$$

Thus, by Theorem 5.4.1 there exists an isometry φ such that for any $\gamma \in \Gamma_M$

$$f_*(\gamma) = \varphi\circ\gamma\circ\varphi^{-1}.$$

As $\widetilde{M} \cong \mathbb{H}^3$, I consider the isometry φ as an f_*-equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$. Namely, for any $x \in \mathbb{H}^3$ and $\gamma \in \Gamma_M$

$$\varphi(\gamma(x)) = f_*(\gamma)(\varphi(x)).$$

It follows that φ projects to a locally isometric covering $\varphi : M \to N$ and the convex combination from $\widetilde{f}$ to φ projects to a proper homotopy from f to φ. Since the degree of a map is invariant under proper homotopies, then $\deg(\varphi) = \deg(f)$.

I prove now that f is always properly homotopic to a map whose lift has the cone property on the cusps. Let $\widetilde{f}$ be a lift of f. For each cusp $N_p = P_p \times [0,\infty)$ let $f_p = \widetilde{f}|_{P_p\times\{0\}}$. Since f is proper it follows that $\widetilde{f}(N_p \times \{\infty\})$ is well-defined. Let $F_p : N_p \times [0,\infty) \to \mathbb{H}^3$ be the map obtained by coning f_p to $\widetilde{f}(N_p \times \{\infty\})$ along geodesic rays. Let $\widetilde{f}'$ be the map obtained by replacing, on each cusp N_p, the map $\widetilde{f}|_{N_p}$ with the map F_p. The map $\widetilde{f}'$ obviously has the cone-property on the cusps, and projects to a map $f' : M \to N$. Moreover, the convex combination from $\widetilde{f}$ to $\widetilde{f}'$ projects to a proper homotopy between f and f'. □

From Theorem 5.4.1, Theorem 5.5.1, Corollary 5.3.12 and the corresponding statements for compact manifolds, one gets the following statement.

Theorem 5.5.2. *Let M be a complete, oriented hyperbolic 3-manifold of finite volume. Let $\Gamma \cong \pi_1(M)$ be the subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *such that $M = \mathbb{H}^3/\Gamma$. Let $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ be a representation. Then $|\mathrm{vol}(\rho)| \leq |\mathrm{vol}(M)|$ and equality holds if and only if ρ is discrete and faithful.*

Corollary 5.5.3. *Let M be a complete hyperbolic 3-manifold of finite volume and let τ be an ideal triangulation of M. If there exists a solution $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$ of the hyperbolicity equations for τ, then there exists a solution z' of the hyperbolicity equations which is geometric. Moreover such a solution is the one of maximal volume.*

Proof. Consider a natural straightening of τ, and let $\mathbf{z}'$ be the moduli induced on τ. By Proposition 5.1.13, I have only to prove that the moduli are not in $\{0, 1, \infty\}$. Suppose that there is a degenerate tetrahedron Δ_i. Then at least two vertices, say v and w, of Δ_i coincide.

Let $\rho(\mathbf{z})$ be the holonomy relative to $\mathbf{z}$ and let $D_{\mathbf{z}}$ be a developing map that is also a pseudo-developing map for $\rho(\mathbf{z})$. Then $D_{\mathbf{z}}$ maps Δ_i into a tetrahedron of modulus z_i. But by hypothesis, $\mathbf{z}$ is in $\{\mathbb{C} \setminus \{0, 1\}\}^n$ and so the vertices of Δ_i are four distinct points. The last assertion follows from Corollary 5.3.12 and Theorem 5.4.1 □

Corollary 5.5.3 tells that, once one has a solution $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$ of the hyperbolicity equations for a triangulation τ of a cusped manifold M, checking hyperbolicity of M boils down to showing that the solution of maximal volume is geometric. Namely, if one succeeds to prove that the solution of maximal volume is not geometric (for example because its holonomy is not discrete, or simply because a solution of maximal volume does not exist or it is not unique) then M cannot be hyperbolic, and this does not depend on the chosen triangulation. In any case, to try to show that a solution of non-maximal volume is geometric is a waste of time.

As an example of application of Corollary 5.5.3 I give the following:

Corollary 5.5.4. *Let M be a cusped 3-manifold equipped with an ideal triangulation τ. If there exists a solution $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$ of the hyperbolicity equations for τ, and all the solutions have zero volume, then M is not hyperbolic.*

I notice that the hypothesis that all the solutions have zero volume can be replaced by requiring that the volumes are *too small*. This is because the set of the volumes of the hyperbolic manifolds is bounded from below by a positive constant.

Finally, I obtain another proof of the well-know fact that no Dehn filling of a Seifert manifold is hyperbolic.

Corollary 5.5.5. *Let M be a 3-manifold such that $||(M, \partial M)|| = 0$ and let N be a Dehn filling of M. Then N is not hyperbolic.*

Proof. Suppose the contrary. Let ρ be the holonomy of the hyperbolic structure of N. From Theorem 5.3.1 it follows that $\mathrm{vol}(\rho) = 0$, but from Proposition 5.2.19 and Corollary 5.2.15 it follows that $\mathrm{vol}(\rho) = \mathrm{vol}(N) > 0$. □

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