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Hyperbolic Volume of Representations of Fundamental Groups of Cusped 3-Manifolds

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1 Introduction

Let W be a compact manifold and let ρ be a representation of its fundamental group into $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$. The volume of ρ is defined by taking any ρ-equivariant map from the universal cover $\widetilde{W}$ to $\mathbb{H}^3$ and then by integrating the pullback of the hyperbolic volume form on a fundamental domain. This volume does not depend on the choice of the equivariant map because two equivariant maps are always equivariantly homotopic and the cohomology class of the pullback of the volume form is invariant under homotopy.

In [3], this definition is extended to the case of a noncompact cusped 3-manifold M (see Definitions 4.1 and 2.5). When M is not compact, some problems of integrability arise if one tries to use the above definition of the volume of a representation. The idea of Dunfield for overcoming these difficulties is to use a particular (and natural) class of equivariant maps, called *pseudodeveloping maps* (see Definition 2.5), that have a nice behavior on the cusps of M allowing to control their volume. Concerning the well-definition of the volume, working with noncompact manifolds, two pseudodeveloping maps in general are not equivariantly homotopic and in [3] it is not proved that the volume of a representation does not depend on the chosen pseudodeveloping map.

In this paper, we show that the volume of a representation is well defined even in the noncompact case and we generalize to noncompact manifolds some results known in the compact case. We restrict to the orientable case. The paper is structured as follows.

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In Sections 2 and 3, we introduce the notion of pseudodeveloping map for a given representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and the notion of straightening of such a map.

In Section 4, we prove that for each orientable cusped 3-manifold M and for each representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the volume of ρ is well defined and depends only on ρ. The main theorems are the following theorems.

Theorem 1.1. Let D_ρ and F_ρ be two pseudodeveloping maps for ρ. Then $\mathrm{vol}(D_\rho) = \mathrm{vol}(F_\rho)$. □

Theorem 1.2. For any pseudodeveloping map D_ρ for ρ, $\mathrm{vol}(D_\rho) = \mathrm{Strvol}(D_\rho)$. □

Roughly speaking, Theorem 1.2 says that the volume of ρ can be computed by straightening any ideal triangulation of M and then summing the volume of the straight version of the tetrahedra.

In Section 5, generalizing the techniques used for the proof of Theorem 1.2, we show that the volume of a representation ρ is bounded from above by the relative simplicial volume.

Theorem 1.3. For all representations $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, $|\mathrm{vol}(\rho)| < V_3 \cdot \|(\overline{M}, \partial\overline{M})\|$, where V_3 is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$ and $\|(\overline{M}, \partial\overline{M})\|$ is the simplicial volume of $\overline{M}$ relative to the boundary. □

In Section 6, we prove the following rigidity theorem for representations of the fundamental group of a hyperbolic cusped manifold, that generalizes a result known for compact manifolds (see [3]).

Theorem 1.4. Let M be a noncompact, complete, orientable hyperbolic 3-manifold of finite volume. Let $\Gamma \cong \pi_1(M)$ be the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ such that $M = \mathbb{H}^3/\Gamma$. Let $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ be a representation. If $|\mathrm{vol}(\rho)| = \mathrm{vol}(M)$, then ρ is discrete and faithful. More precisely, there exists $\varphi \in \mathrm{PSL}(2, \mathbb{C})$ such that for any $\gamma \in \Gamma$,

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}. \tag{1.1}$$

□

In Section 7, we give some corollaries. In particular, we show how from Theorem 1.4 one can get a proof of Mostow's rigidity for noncompact manifolds (see [2, 12] for a more general statement and a different proof).

Theorem 1.5 (Mostow's rigidity for noncompact manifolds)**.** Let $f : M \to N$ be a proper map between two orientable noncompact, complete hyperbolic 3-manifolds of finite volume. Suppose that $\mathrm{vol}(M) = \deg(f)\,\mathrm{vol}(N)$. Then f is properly homotopic to a locally isometric covering with the same degree as f. □

Other corollaries that can be useful for checking the hyperbolicity of a 3-manifold are also shown.

Finally, we notice that the author was informed by B. Klaff that results similar to those proved in this paper have also been established in his Ph.D. thesis.

2 General definitions

We fix here the class of manifolds we consider, namely the class of ideally triangulated cusped manifolds. Since we work with cusped manifolds, we want to fix a structure on the cusps.

Definition 2.1 (cusped manifold). An orientable manifold M is called *cusped manifold* if it is diffeomorphic to the interior of a compact manifold with boundary $\overline{M}$. A *cusp* of M is a closed regular neighborhood of a component of $\partial\overline{M}$. In the following, M is required to have dimension 3 and $\partial\overline{M}$ to be a union of tori, so each cusp is homeomorphic to $T^2 \times [0, \infty)$.

We define $\widehat{M}$ as the compactification of M obtained by adding one point for each cusp of M. We denote by $\widetilde{M}$ the universal cover of M, and we call $\widehat{\widetilde{M}}$ the space obtained by adding to $\widetilde{M}$ one point for each lift of each cusp of M. We call the points added to M (or $\widetilde{M}$) *ideal points* of M (or $\widetilde{M}$). For each ideal point p of M, we fix a smooth product structure $T_p \times [0, \infty)$ on the cusp relative to p. Such a structure induces a cone structure, obtained from $T_p \times [0, \infty]$ by collapsing $T_p \times \{\infty\}$ to p, on a neighborhood C_p of p in $\widehat{M}$.

We lift such structures to the universal cover. Let $\widetilde{p}$ be an ideal point of $\widetilde{M}$ that projects to the ideal point p of M. We denote by $N_{\widetilde{p}}$ the cone at $\widetilde{p}$. The cone $N_{\widetilde{p}}$ is homeomorphic to $P_{\widetilde{p}} \times [0, \infty]$, where $P_{\widetilde{p}}$ covers the torus T_p and $P_{\widetilde{p}} \times \{\infty\}$ is collapsed to $\widetilde{p}$.

Remark 2.2. In the definition of cusped manifold, we have included a fixed product structure on the cusps. This is for technical reasons; however, we will show that the results about the volume of representations do not depend on the chosen structure.

Remark 2.3. Let $\widetilde{M}$ be the universal cover of M. In the following, when we speak about $\pi_1(M)$, we tacitly assume that a basepoint and one of its lifts have been fixed. If p is an ideal point of M, then $\pi_1(T_p)$ is well defined only up to conjugation. If we denote by $\{\widetilde{p}_i\}$ the set of the lifts of p, then there is a one-to-one correspondence between the stabilizers $\mathrm{Stab}(\widetilde{p}_i)$ of $\widetilde{p}_i$ in the group of deck transformations of $\widetilde{M} \to M$ and the conjugates of $\pi_1(T_p)$ in $\pi_1(M)$. Such a correspondence is uniquely determined once the basepoints have been fixed.

To avoid pathologies, since we are working with cusped manifolds, we need that the maps we use have a nice behavior "at infinity." Namely, we will often require that a map from a cusp to $\mathbb{H}^3$ is a cone map in the following sense.

Definition 2.4 (cone map). Let A be a set, $c \in \mathbb{R}$, and C be the cone obtained from $A \times [c, \infty]$ by collapsing $A \times \{\infty\}$ to a point, called ∞. A map $f : C \to \overline{\mathbb{H}}^n$ is a *cone map* if

(1) $f(C) \cap \partial\mathbb{H}^n = \{f(\infty)\}$;

(2) for all $a \in A$ the map $f_{|a \times [c,\infty]}$ is either the constant to $f(\infty)$ or the geodesic ray from $f(a, c)$ to $f(\infty)$, parametrized in such a way that the parameter $(t - c)$, $t \in [c, \infty]$, is the arc-length.

We recall here the definition of pseudodeveloping map for a representation (see [3]).

Definition 2.5 (pseudodeveloping map). Let M be a cusped manifold and let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation. A *pseudodeveloping map* for ρ is a piecewise smooth map $D_\rho : \widetilde{M} \to \mathbb{H}^3$ which is equivariant with respect to the actions of $\pi_1(M)$ on $\widetilde{M}$ via deck transformations and on $\mathbb{H}^3$ via ρ. Moreover, D_ρ is required to extend to a continuous map, which is still called D_ρ, from $\widehat{\widetilde{M}}$ to $\overline{\mathbb{H}}^3$ that maps the ideal points to $\partial\mathbb{H}^3$ (see Remark 2.6 for comments on this property). Finally, D_ρ is required to have the property that there exists $t_{D_\rho} \in \mathbb{R}^+$ such that for each cusp $N_p = P_p \times [0, \infty]$ of $\widetilde{M}$, the restriction of D_ρ to $P_p \times [t_{D_\rho}, \infty]$ is a cone map.

Let $\gamma \neq \mathrm{id}$ be an isometry of $\mathbb{H}^3$ and let $\mathrm{Fix}(\gamma)$ be the set of fixed points of γ. Then $\mathrm{Fix}(\gamma) \cap \partial\mathbb{H}^3$ consists of either one or two points. Moreover, if γ_1 and γ_2 commute, then $\mathrm{Fix}(\gamma_1)$ is γ_2-invariant. It follows that if Γ is an Abelian subgroup of orientation-preserving isometries and $\gamma \in \Gamma$, then $\mathrm{Fix}(\gamma)$ is Γ-invariant. Actually, for almost all Abelian Γ and for any $\gamma_1, \gamma_2 \in \Gamma \setminus \{\mathrm{id}\}$, we have

$$\mathrm{Fix}(\gamma_1) = \mathrm{Fix}(\gamma_2). \tag{2.1}$$

The only cases in which this is not true are when Γ is a dihedral group generated by two rotations of angle π around orthogonal axes. Such a group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and its unique fixed point is the intersection of the axes. It follows that any Abelian group Γ of orientation-preserving isometries has a fixed point in $\overline{\mathbb{H}}^3$ and, if Γ is not dihedral, then it has a fixed point in $\partial\mathbb{H}^3$.

Let now p be an ideal point of $\widehat{\widetilde{M}}$. Since $\mathrm{Stab}(p)$ is Abelian, then either it is dihedral or it has a fixed point in $\partial\mathbb{H}^3$. If ρ is a representation of $\pi_1(M)$ and D_ρ is a pseudodeveloping map for ρ, then $D_\rho(p)$ is a fixed point of $\rho(\mathrm{Stab}(p))$. It follows that, using

Definition 2.5, in order for a pseudodeveloping map to exist, $\rho(\mathrm{Stab}(p))$ must have a fixed point in $\partial\mathbb{H}^3$.

Remark 2.6. We included in Definition 2.5 the requirement that D_ρ maps ideal points to $\partial\mathbb{H}^3$ only for simplicity. No pathologies do occur if some ideal point is mapped to the interior of $\mathbb{H}^3$. Coherently with this fact, from now on, we suppose that for each boundary torus T, the group $\rho(\pi_1(T))$ is not dihedral. As above, we notice that this is only for simplicity and one can easily check that all the results of this paper remain true, *mutatis mutandis*, without this assumption.

Lemma 2.7. Let M be a cusped manifold and let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation. Then a pseudodeveloping map D_ρ exists.

Proof. The proof is the same as in [3], we recall it for completeness. We construct a pseudodeveloping map inductively on the n-skeleta. Let p be an ideal point of $\widetilde{M}$. Since $\mathrm{Stab}(p)$ is Abelian and not dihedral, then its ρ-image has at least one fixed point $q \in \partial\mathbb{H}^3$. We define $D_\rho(p) = q$ and, for all $\alpha \in \pi_1(M)$, we set $D_\rho(\alpha(p)) = \rho(\alpha)(q)$. We do the same for the other ideal points. Now, for each ideal point p, we define D_ρ on $P_p \times \{0\}$ in any $\mathrm{Stab}(p)$-equivariant way and then we make the cone over $D_\rho(p)$ in such a way that D_ρ has the cone property. Then we extend D_ρ in any equivariant way. The extension is possible because $\mathbb{H}^3$ is contractible.

Remark 2.8. Let p be an ideal point of $\widehat{\widetilde{M}}$. If $\rho(\mathrm{Stab}(p))$ is a parabolic nontrivial group, then it has a unique fixed point. It follows that $D_\rho(p)$ is uniquely determined. Thus, if all the ρ-images of the stabilizers of the ideal points are parabolic, then the D_ρ-images of all the ideal points are uniquely determined.

Definition 2.9 (ideally triangulated manifold). Let M be a cusped manifold. An ideal triangulation of M is a triangulation of $\widehat{M}$ having the set of ideal points as 0-skeleton. An *ideally triangulated manifold* is a cusped manifold equipped with a finite smooth ideal triangulation τ. We require the triangulation to be compatible with the product structure. That is, for each cusp N_p, we require $\tau \cap (T_p \times \{0\})$ to be a triangulation of T_p and the restriction to N_p of τ to be the product triangulation.

We will often consider the simplices of an ideal triangulation of a manifold M as subsets of $\widehat{M}$.

Remark 2.10. It is well known that any cusped manifold can be ideally triangulated (see, e.g., [1]).

3 The straightening

A straightening of a pseudodeveloping map D is a map that agrees with D on the ideal points and that maps each tetrahedron to a straight one. The straightening is useful to calculate the hyperbolic volume associated to a pseudodeveloping map (see Section 4). A particular case is when the manifold M is complete hyperbolic because, in this case, the straightening descends to a map from M to itself. Here, we prove that such a map is onto.

Let $\Delta \subset \mathbb{H}^3$ be an oriented geodesic ideal tetrahedron. Since Δ is the convex hull of its vertices, then the $\mathrm{Isom}^+(\mathbb{H}^3)$-class of Δ is completely determined by the $\mathrm{Isom}^+(\mathbb{H}^3)$-class of the oriented set of its vertices (the orientation of the vertices is defined up to the action of $\mathcal{A}_4$). Such a class is completely determined by a nonreal complex number called modulus, up to a three-to-one ambiguity. Such an ambiguity can be avoided by choosing a preferred pair of opposite edges of Δ (see [1, 4, 10, 11, 15]). We extend the notion of modulus to the set of flat tetrahedra, that is, to those whose vertices are distinct and lie on a hyperbolic plane of $\mathbb{H}^3$, by accepting real moduli different from 0 and 1. We want to extend this definition also to the degenerate tetrahedra, that is, to those having two or more coincident vertices. Unfortunately, for such a tetrahedron, it is not possible to encode its isometry class in a complex number. We agree that when we use a modulus in $\{0, 1, \infty\}$ for Δ, we mean that Δ is a degenerate tetrahedron and that the modulus encodes the complete information on the isometry class of Δ, that is, which vertices of Δ coincide with each other.

Definition 3.1. Let Δ^k be the standard k-simplex. Let $\varphi : \Delta^k \to \overline{\mathbb{H}}^n$ be a continuous map that maps the 0-skeleton of Δ^k to $\partial\mathbb{H}^n$. Let Q be the Euclidean convex hull of the φ-image of the vertices of Δ^k, made in a projective model of $\mathbb{H}^n$. Let $\psi : \Delta^k \to Q$ be the only simplicial map that agrees with φ on the 0-skeleton.

The map φ is *standard* if there exist two homeomorphisms $\eta : \mathrm{Im}(\varphi) \to Q$ and $\beta : \Delta^k \to \Delta^k$ such that

$$\eta \circ \varphi \circ \beta = \psi. \tag{3.1}$$

A foliation $\mathcal{F}$ of Δ^k is *standard* if there exists a standard map $\varphi : \Delta^k \to \overline{\mathbb{H}}^n$ such that $\mathcal{F} = \{\varphi^{-1}(x)\}$.

Remark 3.2. For any standard map φ, the dimension of $\mathcal{F} = \{\varphi^{-1}(x)\}$ depends only on the φ-image of the 0-skeleton.

Remark 3.3. It is not hard to show that for a map φ to be standard does not depend on the projective model we use. In other words, φ is standard if and only if $\gamma\varphi$ is standard for any isometry γ.

Remark 3.4. We introduced the notion of standard map because there is not a natural simplicial map from a simplex to an ideal simplex. For example, a straight ideal 1-simplex in the hyperbolic space is a geodesic line, and the notion of barycenter of a geodesic is not well defined.

Let M be an ideally triangulated manifold, $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ a representation, and D_ρ a pseudodeveloping map for ρ. Let Δ be a tetrahedron of τ and $\widetilde{\Delta}$ one of its lifts. The vertices of $\widetilde{\Delta}$ are ideal points, so their images under D_ρ lie in $\partial\mathbb{H}^3$. The map D_ρ determines a modulus for $\widetilde{\Delta}$ simply by considering the convex hull of the image of its vertices (the orientation is the one induced by M). Note that since D_ρ is equivariant, then it defines a modulus for Δ. For each face σ of Δ, we call $\mathrm{Str}_{D_\rho}(\widetilde{\sigma})$, or simply $\mathrm{Str}(\widetilde{\sigma})$, the straight simplex obtained as the convex hull of the D_ρ-image of the vertices of $\widetilde{\sigma}$.

Definition 3.5 (straightening). A straightening of D_ρ is a continuous, piecewise smooth, ρ-equivariant map $\mathrm{Str}(D_\rho) : \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ such that

(1) for each simplex σ of the triangulation, $\mathrm{Str}(D_\rho)$ maps $\widetilde{\sigma}$ to $\mathrm{Str}(\widetilde{\sigma})$,
(2) the restriction of $\mathrm{Str}(D_\rho)$ to any simplex σ is standard,
(3) for each cusp $N_{\widetilde{p}} = P_{\widetilde{p}} \times [0, \infty]$ there exists $c \in \mathbb{R}$ such that $\mathrm{Str}(D_\rho)$ restricted to $P_{\widetilde{p}} \times [c, \infty]$ is a cone map.

Lemma 3.6. Let M be an ideally triangulated manifold. Let ρ be a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and D_ρ a pseudodeveloping map. Then a straightening $\mathrm{Str}(D_\rho)$ of D_ρ exists. Moreover, $\mathrm{Str}(D_\rho)$ is always equivariantly homotopic to D_ρ via a homotopy that fixes the ideal points. □

Proof. A straightening of D_ρ can be constructed with the same techniques of Lemma 2.7. Regarding the homotopy, since D_ρ maps nonideal points to the interior of $\mathbb{H}^3$, then one can use a geodesic flow with the time parameter in $[0, \infty]$ (e.g., the convex combination of Definition 4.9) to construct a homotopy with the required properties. ■

Remark 3.7. A straightening in general is not a pseudodeveloping map in our setting because it can map some point of $\widetilde{M}$ to $\partial\mathbb{H}^3$. However, if there are no degenerate tetrahedra, then a straightening is also a pseudodeveloping map, and the homotopy between D_ρ and $\mathrm{Str}(D_\rho)$ can be made coherently with the cone structure of the cusps, that is, in such a way that the intermediate maps along the homotopy between D_ρ and $\mathrm{Str}(D_\rho)$ have the cone property on the cusps.

When M has a complete hyperbolic structure of finite volume, there is a natural notion of straightening of the ideal triangulation. Namely, choose the arc-length as the cone parameter on the cusps of M and consider $\mathbb{H}^3$ as the universal cover of M. Then choose ρ as the holonomy of the hyperbolic structure of M; the identity map of $\mathbb{H}^3$ is clearly a pseudodeveloping map for ρ. A natural straightening map is a straightening of the identity.

Proposition 3.8. Let M be an ideally triangulated manifold equipped with a complete, finite-volume hyperbolic structure. Then any natural straightening map projects to a map $\mathrm{Str} : \widehat{M} \to \widehat{M}$ which is onto. Moreover, $\mathrm{Str}(M) \supset M$. □

Proof. It is easy to see that $\widehat{\widetilde{M}} = \widehat{\mathbb{H}^3}$ naturally embeds into $\overline{\mathbb{H}}^3$ and that the ideal points lie on $\partial\mathbb{H}^3$. Since the straightening is equivariant, then it projects to a map $\mathrm{Str} : \widehat{M} \to \widehat{M}$. Moreover, Str fixes the ideal points. We prove that Str is onto. One can easily prove that $H_3(\widehat{M}; \mathbb{Z}) \cong H_3(\overline{M}, \partial\overline{M}; \mathbb{Z}) \cong \mathbb{Z}$. So we can define the degree of a map $f : \widehat{M} \to \widehat{M}$ by

$$f_*\big([\widehat{M}]\big) = \deg(f) \cdot [\widehat{M}], \tag{3.2}$$

where $[\widehat{M}]$ is the generator of $H_3(\widehat{M}; \mathbb{Z})$ induced by the orientation of M. Now, note that by Lemma 3.6 the natural straightening is homotopic to the identity via an equivariant homotopy. Because of equivariance, the homotopy projects to a homotopy between Str and the identity. It follows that Str_* and id_* coincide on $H_*(\widehat{M}; \mathbb{Z})$, so $\deg(\mathrm{Str}) = \deg(\mathrm{id}) = 1$. Now, suppose that Str is not onto and let x be a point in $\widehat{M}$ outside its image. If we consider Str as a map from $\widehat{M}$ to $\widehat{M} \setminus \{x\}$, we get $\mathrm{Str}_*([\widehat{M}]) = 0 \in H_3(\widehat{M} \setminus \{x\}; \mathbb{Z})$ simply because $H_3(\widehat{M} \setminus \{x\}; \mathbb{Z}) = 0$. Then $\mathrm{Str}_*([\widehat{M}])$ is a boundary in $\widehat{M} \setminus \{x\}$, consequently it is a boundary also in $\widehat{M}$. It follows that $\mathrm{Str}_*([\widehat{M}]) = 0$. This implies $\deg(\mathrm{Str}) = 0$, which is a contradiction.

The last assertion follows because Str is onto and fixes the ideal points. ■

4 Volume of representations

For this section, we fix an ideally triangulated manifold M and a representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$.

In this section, we recall the notion of the volume of an equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$. We prove that if we restrict to the class of pseudodeveloping maps, then the volume of ρ is well defined. Namely, the volume depends neither on the pseudodeveloping map nor on the product structure of the cusps. Such a volume can be calculated using a straightening of any pseudodeveloping map and it is exactly the algebraic sum of the volumes of the straightened tetrahedra.

Definition 4.1 (volume of pseudodeveloping map). Let D_ρ be a pseudodeveloping map for ρ. Let ω be the volume form of $\mathbb{H}^3$ and let $D_\rho^*\omega$ be the pullback of ω. Since D_ρ is equivariant, then $D_\rho^*\omega$ projects to a 3-form, that we still call $D_\rho^*\omega$, on M. The volume $\mathrm{vol}(D_\rho)$ of D_ρ is defined by

$$\mathrm{vol}\left(D_\rho\right) = \int_M D_\rho^*\omega. \tag{4.1}$$

Remark 4.2. We will see below that for pseudodeveloping maps, the volume is always finite. The same definition of the volume does not work for any equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$ because if the pullback of the volume form is not in L^1, then the volume is not well defined.

Definition 4.3 (straight volume). Let D_ρ be a pseudodeveloping map for ρ. Let $\{\Delta_i\}$ be the set of the tetrahedra of the ideal triangulation of M and let $\{\widetilde{\Delta}_i\}$ be a set of lifts of the Δ_i's. Let $\nu_i = 0$ if $\mathrm{Str}(\widetilde{\Delta}_i)$ is a degenerate tetrahedron, and let ν_i be the algebraic volume of $\mathrm{Str}(\widetilde{\Delta}_i)$ otherwise.

The straight volume of D_ρ is defined as $\mathrm{Strvol}(D_\rho) = \sum_i \nu_i$.

Remark 4.4. Let $\Delta_1, \dots, \Delta_n$ be the tetrahedra of τ. Let $z = (z_1, \dots, z_n) \in \{\mathbb{C} \setminus \{0,1\}\}^n$ be a solution of Thurston's hyperbolicity equations (see [1, 4, 9, 10, 11, 15]). Then there exists a developing map $D_z : \widetilde{M} \to \mathbb{H}^3$ for z that is a pseudodeveloping map for some holonomy $\rho(z)$. Such a map is already straight and we have $\mathrm{Strvol}(D_z) = \mathrm{vol}(D_z) = \sum \nu_i = \mathrm{vol}(z)$, where ν_i is the volume of the geodesic ideal tetrahedron of modulus z_i.

Let $C_p = T_p \times [0,\infty]/\sim$ be a cusp of M and let $N_{\tilde{p}} = P_{\tilde{p}} \times [0,\infty]/\sim$ be one of its lifts in $\widetilde{M}$. Then we can identify $\pi_1(C_p)$ with $\mathrm{Stab}(\tilde{p})$. Let $f : P_{\tilde{p}} \times \{0\} \to \mathbb{H}^3$ be a $\mathrm{Stab}(\tilde{p})$-equivariant map, let $\xi \in \partial\mathbb{H}^3$ be a fixed point of $\rho(\mathrm{Stab}(\tilde{p}))$, and let $F : N_{\tilde{p}} :\to \overline{\mathbb{H}}^3$ be the cone map obtained by coning f to ξ. As above, let $F^*\omega$ be the pullback of the volume form on C_p. Similarly, we can pull back the metric. We call A_t^p the area of the torus $T_p \times \{t\}$.

Lemma 4.5. In the previous setting, for $t > r$,

$$A_t^p \le A_r^p e^{-(t-r)}, \qquad \int_{T_p \times [t,\infty)} \left|F^*\omega\right| \le A_t^p. \tag{4.2}$$

□

Proof. Let (x, y) be local coordinates on $P_{\tilde{p}}$. Choose the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and assume that $\xi = \infty$. In such a model, the hyperbolic metric at the point (z, s) is the

Euclidean one rescaled by the factor $1/s$. It follows that, we call $\alpha+i\beta$ and h the complex and real components of F, we have

$$\alpha(x,y,t)+i\beta(x,y,t)=\alpha(x,y,r)+i\beta(x,y,r), \qquad h(x,y,t)=h(x,y,r)e^{(t-r)}. \tag{4.3}$$

The element of area at level t is $d\sigma_t(x,y) = \sqrt{\det({}^T JF_t \cdot H \cdot JF_t)}$, where F_t is the restriction of F to $P_{\tilde{p}} \times \{t\}$ and $H(x,y,t)=(1/h^2)\,\mathrm{Id}$ is the matrix of the hyperbolic metric. From direct calculations it follows that $d\sigma_t(x,y) \leq d\sigma_r(x,y)e^{-t+r}$ and the first inequality follows.

Now, note that the volume element $|F^*\omega|$ at the point $(x,y,t) \in C_p$ is bounded by the area element of the torus $T_p \times \{t\}$ multiplied by the length element of the ray $\{(x,y)\} \times [0,\infty]$. Since the parameter t is exactly the arc-length, then the length element is exactly dt. It follows that

$$\int_{T_p \times [t,\infty)} \left|F^*\omega\right| \leq \int_t^\infty A_s^p\,ds \leq \int_t^\infty A_t^p e^{-(s-t)}ds = A_t^p. \tag{4.4}$$

This completes the proof. ■

Remark 4.6. From Lemma 4.5 it follows, in particular, that $\int_{T_p \times [t,\infty)} |F^*\omega| \leq A_0^p e^{-t}$. This means that we have an estimate of $\int_{T_p \times [0,\infty)} |F^*\omega|$ not depending on the point $\xi = F(p)$ but only on the area of $T_p \times \{0\}$.

Remark 4.7. From Lemma 4.5, it follows that $\mathrm{vol}(D_\rho)$ is finite for any pseudodeveloping map D_ρ.

The following lemma is proved in [3].

Lemma 4.8. If D_ρ and F_ρ are two pseudodeveloping maps for ρ that agree on the ideal points, then $\mathrm{vol}(D_\rho) = \mathrm{vol}(F_\rho)$. □

This is because any two pseudodeveloping maps are equivariantly homotopic. The fact that they coincide on the ideal points allows one to construct a homotopy h that respects the cone structures of the cusps. Namely, for each ideal point $\tilde{p}$ of $\widetilde{M}$, we choose any equivariant homotopy between the restrictions of D_ρ and F_ρ to $P_{\tilde{p}} \times \{\bar{t}\}$, where $\bar{t} = \max\{t_{D_\rho}, t_{F_\rho}\}$, we cone such a homotopy to $D_\rho(\tilde{p})$ along geodesic rays, and we extend the homotopy outside the cusps in any equivariant way. For such a homotopy h, we can use the Stokes theorem on $M \times [0,1]$ for $h^*\omega$ to obtain the thesis. More precisely, let K_t be

$M \setminus \cup_p(T_p \times (t,\infty))$, where p varies on the set of the ideal points; then we have

$$0 = \int_{K_t \times [0,1]} d\big(h^*\omega\big) = \int_{\partial(K_t \times [0,1])} h^*\omega = \int_{K_t} \big(D_\rho^*\omega - F_\rho^*\omega\big) + \int_{\partial K_t \times [0,1]} h^*\omega \qquad (4.5)$$

and, as in Lemma 4.5, we can prove that the last integral goes to zero as $t \to \infty$.

We prove now Theorem 1.1, which says that the claim of Lemma 4.8 is true in general.

Proof of Theorem 1.1. For $t \in [0,\infty)$, let D_ρ^t be the map constructed as follows: D_ρ^t coincides with D_ρ until the level t of each cusp. Then for each cusp N_p, we complete D_ρ^t by coning $D|_{P_p \times \{t\}}$ to $F_\rho(p)$ along geodesic rays in such a way that the arc-length is the parameter $s-t$, where $s \in [t,\infty)$. Now, D_ρ^t is a pseudodeveloping map that agrees with F_ρ on the ideal points. Thus, by Lemma 4.8, $\mathrm{vol}(D_\rho^t) = \mathrm{vol}(F_\rho)$. Since D_ρ^t and D_ρ agree outside the cusps and where they differ they are cones on the same basis (and different vertices), from Lemma 4.5, we get

$$\big|\,\mathrm{vol}\,(D_\rho) - \mathrm{vol}\,(D_\rho^t)\big| \le 2\sum_p A_t^p \le 2\left(\sum_p A_0^p\right)e^{-t}, \qquad (4.6)$$

where p varies on the set of ideal points and A_t^p is the area of the torus $T_p \times \{t\}$. As $t \to \infty$, we get the thesis. ∎

Similar techniques actually allow to prove Theorem 1.2.

Before proving Theorem 1.2, we give the following definition.

Definition 4.9. Let f and g be two maps from a set X, respectively, to $\mathbb{H}^n$ and $\overline{\mathbb{H}}^n$. For $t \in [0,\infty]$, the convex combination Φ_t from f to g is defined by

$$\Phi_t(x) = \begin{cases} \gamma_x(t), & t \le \mathrm{dist}\,\big(f(x), g(x)\big), \\ g(x), & t \ge \mathrm{dist}\,\big(f(x), g(x)\big), \end{cases} \qquad (4.7)$$

where γ_x is the geodesic from $f(x)$ and $g(x)$, parametrized by arc-length.

Remark 4.10. In Definition 4.9, if X is a topological space and f and g are continuous, then the convex combination from f to g is continuous on $X \times [0,\infty]$ because the function $\mathrm{dist}(f(x), g(x))$ is well defined and continuous from X to $[0,\infty]$.

Proof of Theorem 1.2. For the proof assume that $t_{D_\rho} = 0$. We start by fixing a suitable homotopy h between D_ρ and $\mathrm{Str}(D_\rho)$. Define $h : \widetilde{M} \times [0,\infty] \to \mathbb{H}^3$ outside the cusps to be the convex combination from D_ρ to $\mathrm{Str}(D_\rho)$ and then for each cusp $N_{\tilde{p}}$ extend h by coning

$h((x,0),s)$ to $D_\rho(\widetilde{p})$ along geodesic rays in such a way that the parameter $t \in [0,\infty)$ of the cusp is the arc-length. Let $D_s(x) = h(x,s)$. By Lemma 4.8, we have that

$$\int_M D_\rho^*\omega = \int_M D_s^*\omega \quad \text{for } s \in (0,\infty). \tag{4.8}$$

So we only have to prove that $\int_M D_s^*\omega \to \mathrm{Strvol}(D_\rho)$ as $s \to \infty$. Clearly, it suffices to prove that for any tetrahedron Δ, we have $\int_\Delta D_s^*\omega \to \nu$, where ν is the volume of $\mathrm{Str}(\Delta)$. If Δ does not collapse in the straightening, then the distance from D_ρ to $\mathrm{Str}(D_\rho)$ is bounded outside the cusps and so $D_s = \mathrm{Str}(D_\rho)$ for $s \gg 0$; since $\mathrm{Str}(D_\rho)$ is a homeomorphism on Δ, then $\int_\Delta D_s^*\omega$ is exactly the volume of the straight version of Δ.

If Δ collapses in the straightening, then we have to show that $\int_\Delta D_s^*\omega \to 0$. This follows from direct calculations. We give only the lead line of them because they are involved but use elementary techniques. Moreover, in Section 5, we give an alternative proof of this theorem (see Theorem 1.3 and Remark 5.8).

Given the convex combination Φ_t from a map f to a map g, it is possible to calculate the Jacobian of Φ_t as a function of the derivatives of f and g, the time t, and the distance between f and g. This is not completely trivial; for example think of a tetrahedron as a convex combination of two segments: the segments have zero area but in the middle we have quadrilaterals with nonzero area. Using these calculations, we can estimate $|D_s^*\omega|$ outside the cusps, showing that its integral goes to zero as s goes to infinity. Looking inside the cusps, by Lemma 4.5, we reduce the estimate of the volume to the estimate of the area of the boundary tori, and we proceed as above, estimating the Jacobian of the restriction to the boundary tori of the convex combination Φ_t. ■

Remark 4.11. Since $\mathrm{vol}(D_\rho) = \mathrm{Strvol}(D_\rho)$, it follows that such a volume does not depend on the chosen cone structure of the cusps. Moreover, by Theorem 1.1, $\mathrm{vol}(D_\rho)$ does not depend on the developing map, but only on ρ. This allows us to give the following definition.

Definition 4.12. The volume $\mathrm{vol}(\rho)$ of ρ is the volume of any pseudodeveloping map for ρ.

As the following corollary shows, for hyperbolic manifolds, the volume of the holonomy is exactly the hyperbolic volume.

Corollary 4.13. Let M be a complete hyperbolic manifold of finite volume. If ρ is the holonomy of the hyperbolic structure, then $\mathrm{vol}(\rho) = \mathrm{vol}(M)$. □

Proof. Consider $\mathbb{H}^3$ as the universal cover of M and choose the arc-length as the cone parameter of the cusps. Clearly, the identity of $\mathbb{H}^3$ is a pseudodeveloping map for ρ. Obviously, we have $\int_M \mathrm{Id}^*(\omega) = \mathrm{vol}(M)$. ■

Corollary 4.14. Let z_i be the modulus induced by a pseudodeveloping map D_ρ on the ith tetrahedron and let v_i be the volume of a hyperbolic ideal geodesic tetrahedron of modulus z_i. Then $\mathrm{vol}(\rho) = \sum v_i$. □

Remark 4.15. Even if $\sum v_i$ depends only on ρ, the moduli z_i, induced by a pseudodeveloping map D_ρ, can actually depend on D_ρ. Namely, any ideal point p is mapped to a fixed point of $\rho(\mathrm{Stab}(p))$ and, if this is not a parabolic group, we have more than one possibility for $D_\rho(p)$. Conversely, if each $\rho(\mathrm{Stab}(p))$ is a nontrivial parabolic group, then by Remark 2.8 it follows that the moduli z_i are uniquely determined by ρ.

Proposition 4.16. Let g be a reflection of $\mathbb{H}^3$ and let $\overline{\rho}$ be the representation $g \circ \rho \circ g^{-1}$. Then $\mathrm{vol}(\overline{\rho}) = -\mathrm{vol}(\rho)$.

Proof. If D_ρ is a pseudodeveloping map for ρ, then $g \circ D_\rho$ is a pseudodeveloping map for $\overline{\rho}$ and it is easily checked that $\mathrm{vol}(g \circ D_\rho) = -\mathrm{vol}(D_\rho)$.

The following fact is proved in [3].

Proposition 4.17. Suppose that ρ factors through the fundamental group of a Dehn filling N of M. Then the volume of ρ with respect to N coincides with the volume of ρ with respect to M.

Theorem 1.2 extends from ideal to "classical" triangulations, namely, to genuine triangulations $\mathcal{T}$ of $\overline{M}$. Consider such a $\mathcal{T}$ as a triangulation of M with some simplices at infinity (those in $\partial\overline{M}$). Given a pseudodeveloping map D_ρ for ρ, define a straightening of D_ρ relative to $\mathcal{T}$, exactly as in Section 3, by considering the convex hulls of the images of the vertices of $\mathcal{T}$. Then one can give the definition of the straight volume relative to $\mathcal{T}$ of a developing map D_ρ exactly as in Definition 4.3, with the unique difference that one has to use the tetrahedra of $\mathcal{T}$ instead of the ideal tetrahedra of an ideal triangulation of M. Call such a volume $\mathrm{Strvol}^{\mathcal{T}}(D_\rho)$.

Finally, exactly as in Theorem 1.2, one can prove the following fact.

Proposition 4.18. Let $\mathcal{T}$ be a triangulation of $\overline{M}$ and let D_ρ be a pseudodeveloping map for ρ. Then $\mathrm{vol}(\rho) = \mathrm{Strvol}^{\mathcal{T}}(D_\rho)$.

5 Comparison with simplicial volume

Here, we generalize the argument used to prove Theorem 1.2 to compare $\mathrm{vol}(\rho)$ with the simplicial volume of M, obtaining exactly the expected inequality that the volume is bounded by the simplicial volume. That inequality is suggested to be true by Proposition

4.18. Indeed, if one considers a triangulation of $\overline{M}$ as a singular cycle $\sum \lambda_i \sigma_i$ representing $[\overline{M}]$ in $H_3(\overline{M}, \partial\overline{M})$, from Proposition 4.18, one gets

$$\big|\operatorname{vol}(\rho)\big| \leq \Big(\sum_i |\lambda_i|\Big) \cdot \operatorname{Max}\big\{\operatorname{vol}\big(\operatorname{Str}_{D_\rho}\sigma_i\big)\big\}. \tag{5.1}$$

So, the same calculations made for all the cycles representing $[\overline{M}]$ will lead to Theorem 1.3.

Proof of Theorem 1.3. For the proof, we fix a representation $\rho : \pi_1(M) \to \operatorname{Isom}^+(\mathbb{H}^3)$ and a pseudodeveloping map D_ρ for ρ.

Let $c = \sum_i \lambda_i \sigma_i$ be a smooth singular chain in $\overline{M}$; here each simplex σ_i is a piecewise smooth map from the standard tetrahedron Δ^3 to $\overline{M}$. The simplicial volume of c is defined as $\|c\| = \sum |\lambda_i|$. The relative simplicial volume of $(\overline{M}, \partial\overline{M})$ is defined as

$$\big\|(\overline{M}, \partial\overline{M})\big\| = \inf\big\{\|c\| : [c] = [\overline{M}] \in H_3(\overline{M}, \partial\overline{M})\big\}. \tag{5.2}$$

We refer the reader to [1, 6, 8, 15] for more details about the simplicial volume.

The proof has two main steps.

(1) Given a smooth cycle $c = \sum_i \lambda_i \sigma_i$ representing $[\overline{M}]$, we show that $\operatorname{vol}(\rho) = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^*\omega)$, where ω is the volume form of $\mathbb{H}^3$.

(2) By replacing c with its straightening, we show that $\operatorname{vol}(\rho) = \sum_i \lambda_i \nu_i$, where ν_i is the volume of a straight version of σ_i.

From Step 2 it follows that

$$\big|\operatorname{vol}(\rho)\big| \leq \sum_i \big|\lambda_i\big| \cdot \big|\nu_i\big| \leq V_3 \cdot \|c\|. \tag{5.3}$$

Theorem 1.3 follows taking to the infimum over all c's representing $[\overline{M}]$.

Step 1. Since a pseudodeveloping map has the cone property on the cusps, the 3-form $D_\rho^*\omega$ defined on M extends to a 3-form on $\overline{M}$ that vanishes at the boundary. So we can consider the class $[D_\rho^*\omega] \in H^3(\overline{M}, \partial\overline{M})$. Since $[c] = [\overline{M}]$, then

$$\operatorname{vol}(\rho) = \int_M D_\rho^*\omega = \big\langle [D_\rho^*\omega], [\overline{M}]\big\rangle = \big\langle [D_\rho^*\omega], [c]\big\rangle = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^*\omega). \tag{5.4}$$

Step 2. The idea is the following. Consider a lift $\widetilde{c}$ of c to $\widehat{\widetilde{M}}$. Let $\overline{c} = (D_\rho)_*\widetilde{c}$ be the push-forward of $\widetilde{c}$ to $\mathbb{H}^3$ via D_ρ and let $\operatorname{Str}(\overline{c})$ be a straightening of $\overline{c}$. Since the straightening is homotopic to the identity, then there exists a chain homotopy of degree one, that is, a map H from k-chains to $(k+1)$-chains such that $\mathrm{Str} - \mathrm{Id} = H \circ \partial - \partial \circ H$. Then we have

$$\begin{aligned} \mathrm{vol}(\rho) &= \left\langle D_\rho^* \omega, \widetilde{c} \right\rangle = \left\langle \omega, \left(D_\rho\right)_* \widetilde{c} \right\rangle = \langle \omega, \overline{c} \rangle \\ &= \left\langle \omega, \mathrm{Str}(\overline{c}) \right\rangle + \langle \omega, \partial H \overline{c} \rangle - \langle \omega, H \partial \overline{c} \rangle \\ &= \sum_i \lambda_i \nu_i + \langle d\omega, H\overline{c} \rangle - \langle \omega, H \partial \overline{c} \rangle. \end{aligned} \tag{5.5}$$

The last two summands are zero because $d\omega = 0$ and, even if $\partial \overline{c} \neq 0$, everything can be made ρ-equivariantly so that the action of ρ cancels out in pairs the contributions of $\langle \omega, H \partial \overline{c} \rangle$.

We now formalize this argument. Let $C_k(X)$ denote the real vector space of finite singular, piecewise smooth k-chains in a space X. Consider the projection $\overline{M} \to \widehat{M}$ obtained by collapsing each boundary torus to a point. Given a relative cycle $c = \sum_i \lambda_i \sigma_i$ in $C_k(\overline{M})$, that is, a chain c such that $\partial c \in C_{k-1}(\partial \overline{M})$, we also call c the chain induced on $C_k(\widehat{M})$ with $\partial c \in C_k$ (ideal points of M). We call $\widetilde{c}$ a lift of c to $\widetilde{\widehat{M}}$, that is, $\widetilde{c} = \sum_i \lambda_i \widetilde{\sigma}_i \in C_k(\widetilde{\widehat{M}})$, where each $\widetilde{\sigma}_i$ is a lift of σ_i.

Remark 5.1. The chain $\widetilde{c}$ in general is not a relative cycle. Nevertheless, since c is a relative cycle, assuming $\partial \widetilde{c} = \sum_j l_j \eta_j$, there exists a family $\{\alpha_j\}$ of elements of $\pi_1(M)$ such that

$$\sum_j l_j \cdot \alpha_{j*}(\eta_j) \in C_k \text{ (ideal points of } M), \tag{5.6}$$

where $\pi_1(M)$ acts on $\widetilde{M}$ via deck transformations and $\alpha_{j*}(\eta_j)$ is the composition of α_j with η_j.

We set $\overline{\sigma}_i = (D_\rho)_*(\widetilde{\sigma}_i)$ and $\overline{c} = \sum_i \lambda_i \overline{\sigma}_i = (D_\rho)_*(\widetilde{c}) \in C_k(\overline{\mathbb{H}}^3)$. We restrict now the class of simplices we want to use.

Definition 5.2. A k-simplex $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$ is called *admissible* if, for any subsimplex η of σ, if the interior of η touches $\partial \mathbb{H}^3$, then η is constant. A chain is admissible if its simplices are admissible.

Lemma 5.3. Let $c' = \sum_i \lambda_i \sigma'_i$ be a relative cycle in $C_k(\overline{M})$. Then there exists a cycle $c = \sum_i \lambda_i \sigma_i$ (with the same λ_i's) such that $[c'] = [c] \in H_k(\overline{M}, \partial \overline{M})$ and such that $\overline{c}$ is admissible. □

Proof. For any chain $\beta \in C_k(\overline{M})$, define $\mathrm{span}(\beta)$ as the set of all the subsimplices of β (of any dimension). Given the chain c', construct c as follows: near $\partial \overline{M}$ push c' a little inside M, keeping fixed only the simplices of $\mathrm{span}(\partial c')$. This operation can be made via a homotopy, so $[c] = [c']$. Moreover, the only simplices of c that touch $\partial\overline{M}$ are the ones of $\mathrm{span}(\partial c)$. Finally, c is admissible because if $\overline{\sigma}_i(x) \in \partial\mathbb{H}^3$, then from the definition of pseudodeveloping map it follows that $\sigma_i(x)$ is an ideal point. Thus, x lies on a face F of σ_i such that the simplex $\eta = (\sigma_i)|_F$ belongs to $\mathrm{span}(\partial c)$. It follows that $\widetilde{\eta}$ is a constant map and then also $\overline{\eta}$ is constant. ■

We call $\overline{C}_k(\overline{\mathbb{H}}^3)$ the vector space of admissible chains. Note that the boundary operator is well defined on $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ (The boundary of an admissible cycle is admissible).

Definition 5.4. For any admissible simplex $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$, a straightening $\mathrm{Str}(\sigma) : \Delta^k \to \overline{\mathbb{H}}^3$ is a simplex that agrees with σ on the 0-skeleton; moreover, $\mathrm{Str}(\sigma)$ is required to be a standard map whose image is the convex hull of its vertices. For any chain $c = \sum_i \lambda_i\sigma_i$, a straightening of c is a chain $\mathrm{Str}(c) = \sum_i \lambda_i \mathrm{Str}(\sigma_i)$.

A straightening of a simplex is admissible because any straight simplex is admissible. The straightening of a simplex is not unique in general because a standard map from a simplex to $\overline{\mathbb{H}}^3$ is not uniquely determined by its restriction to the 0-skeleton. Nevertheless, as the following lemma shows, it is possible to choose a straightening for any simplex compatibly with the boundary operator of $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$.

Lemma 5.5. There exists a chain map $\mathrm{Str} : \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3) \to \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ that maps each simplex to one of its straightenings and such that for any isometry γ of $\mathbb{H}^3$, $\gamma_* \circ \mathrm{Str} = \mathrm{Str} \circ \gamma_*$. □

Proof. Let K be the set of pairs $\{(B, f)\}$, where B is a subspace of $\oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ and $f : B \to \oplus_k\overline{C}_k(\overline{\mathbb{H}}^3)$ is a linear map, such that

(1) $\partial(B) \subset B$,
(2) for all $\gamma \in \mathrm{Isom}(\mathbb{H}^3)$, $\gamma_*(B) \subset B$,
(3) for all $\sigma \in B$, $f(\sigma)$ is a straightening of σ,
(4) for all $\gamma \in \mathrm{Isom}(\mathbb{H}^3)$, $f \circ \gamma_* = \gamma_* \circ f$,
(5) $f \circ \partial = \partial \circ f$.

Note that K is not empty because each 0-simplex is admissible and it is itself its unique straightening, so that $(\overline{C}_0(\overline{\mathbb{H}}^3), \mathrm{Id}) \in K$. We order K by inclusion (i.e., $(B, f) \prec (C, g)$ if and only if $B \subset C$ and $g|_B = f$) and use Zorn's lemma. Let $\{(B_\xi, f_\xi)\}$ be an ordered sequence in K. Clearly,

$$\left(B_\infty = \cup_\xi B_\xi,\ f_\infty = \cup_\xi f_\xi\right) \tag{5.7}$$

is an upper bound for $\{(B_\xi, f_\xi)\}$. It follows that there exists a maximal element $(\overline{B}, \overline{f}) \in K$.

We claim that $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$. Suppose the contrary. Let $k = \min\{n \in \mathbb{N} : \overline{C}_n(\overline{\mathbb{H}}^3) \not\subset \overline{B}\}$ and let σ be a simplex of $\overline{C}_k(\overline{\mathbb{H}}^3) \setminus \overline{B}$. If $k = 0$, set B_1 the space spanned by $\overline{B}$ and $\bigcup_{\gamma \in \mathrm{Isom}(\mathbb{H}^3)} \gamma_*(\sigma)$, define $\overline{f}(\sigma) = \sigma$, $\overline{f}(\gamma_*(\sigma)) = \gamma_*(\overline{f}(\sigma))$, and extend $\overline{f}$ on B_1 by linearity. Then

$$(\overline{B}, \overline{f}) \prec (B_1, \overline{f}), \tag{5.8}$$

contradicting the maximality of $(\overline{B}, \overline{f})$. If $k > 0$, then $\overline{f}$ is defined on $\partial\sigma$ and, as $\overline{f}(\partial\sigma)$ is standard, it is not hard to show that it extends to a standard map $\overline{f}(\sigma)$ defined on the whole Δ^k. Then define B_1 and extend $\overline{f}$ to B_1 as above. Again we have $(\overline{B}, \overline{f}) \prec (B_1, \overline{f})$, that contradicts the maximality of $(\overline{B}, \overline{f})$.

Thus, $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ and $\overline{f}$ is the requested chain map Str. ∎

Lemma 5.6. There exists a homotopy operator $H : \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3) \to \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ between Str and the identity such that $H \circ \gamma_* = \gamma_* \circ H$ for any isometry γ of $\mathbb{H}^3$. □

Proof. A homotopy operator between Str and Id is a chain map of degree 1, that is, a map $H : \overline{C}_k(\overline{\mathbb{H}}^3) \to \overline{C}_{k+1}(\overline{\mathbb{H}}^3)$, such that

$$\mathrm{Str} - \mathrm{Id} = \partial \circ H - H \circ \partial. \tag{5.9}$$

For any admissible $\sigma : \Delta^k \to \overline{\mathbb{H}}^3$, let $h_\sigma(t, x)$ be the homotopy constructed as follows: $h_\sigma(t, x)$ is the convex combination from $\sigma(x)$ to $\mathrm{Str}(\sigma)(x)$ if $\sigma(x) \notin \partial\mathbb{H}^3$ and $h_\sigma(t, x) = \sigma(x)$ otherwise. Note that from the admissibility of σ it follows that $h_\sigma(\infty, x) = \mathrm{Str}(\sigma)(x)$ for any x. So h_σ actually is a homotopy between σ and $\mathrm{Str}(\sigma)$.

As h_σ is a map $h_\sigma : \Delta^k \times [0, \infty] \to \overline{\mathbb{H}}^3$, up to triangulating $\Delta^k \times [0, \infty]$, it is a chain in $C_{k+1}(\overline{\mathbb{H}}^3)$. Fix a canonical triangulation of $\Delta^k \times [0, \infty]$ and define $H(\sigma)$ as h_σ. Since

$$\partial\big(\Delta^k \times [0, \infty]\big) = \Delta^k \times \{\infty\} - \Delta^k \times \{0\} + \partial\Delta^k \times [0, \infty], \tag{5.10}$$

then $\partial \circ H = \mathrm{Str} - \mathrm{Id} + H \circ \partial$.

Since h_σ is constructed using geodesic rays, then for every isometry γ, we have $h_{\gamma \circ \sigma} = \gamma \circ h_\sigma$. It follows that $H \circ \gamma_* = \gamma_* \circ H$.

Finally, admissibility of h_σ follows from admissibility of σ. ∎

Lemma 5.7. Let $c = \sum_i \lambda_i \sigma_i$ be a chain in $C_k(\overline{M})$. Let $\{\gamma_j\}$ be a finite set of isometries and let A be the hyperbolic convex hull in $\mathbb{H}^3$ of $\bigcup_{i,j} \gamma_j(\mathrm{Im}(\overline{\sigma}_i))$. Then A has finite volume. □

Proof. Since D_ρ has the cone property on the cusps and since c is a finite sum of simplices, then A is contained in a geodesic polyhedron with a finite number of vertices, and such a polyhedron has finite volume. ∎

We are now ready to complete the proof of Theorem 1.3. Let $c = \sum_i \lambda_i \sigma_i$ be a relative cycle in $C_3(\overline{M})$ such that $[c] = [\overline{M}]$ in $H_3(\overline{M}, \partial\overline{M})$. By Lemma 5.3, we can suppose that c is admissible. Assume that $\partial\widetilde{c} = \sum_j l_j \eta_j$. By Remark 5.1, there exists a finite set $\{\alpha_j\} \subset \pi_1(M)$ such that $\sum_j l_j \cdot \alpha_{j_*}\eta_j \in C_2$ (ideal points of M).

Let A be as in Lemma 5.7, where we use $\{\rho(\alpha_j)\} \cup \{\mathrm{Id}\}$ as the set of isometries. Since A has finite volume, then the volume form ω of $\mathbb{H}^3$ is an element of $H^3(A)$. Moreover, the straightening of any admissible simplex in $\overline{C}_k(A)$ is contained in $\overline{C}_k(A)$ and, since the homotopy operator H is constructed using convex combinations, then H is well defined on $\oplus_k \overline{C}_k(A)$. If ν_i denotes the volume of the straight version of σ_i, then

$$
\begin{aligned}
\mathrm{vol}(\rho) &= \langle D_\rho^* \omega, \widetilde{c} \rangle = \langle \omega, (D_\rho)_*(\widetilde{c}) \rangle = \langle \omega, \overline{c} \rangle \\
&= \langle \omega, \mathrm{Str}\,\overline{c} \rangle + \langle \omega, H\partial\overline{c} \rangle - \langle \omega, \partial H\overline{c} \rangle \\
&= \sum_i \lambda_i \nu_i + \langle \omega, H\partial\overline{c} \rangle - \langle d\omega, H\overline{c} \rangle \\
&= \sum_i \lambda_i \nu_i + \langle \omega, H\partial\overline{c} \rangle.
\end{aligned} \tag{5.11}
$$

By Lemma 5.6, we have $\rho(\alpha_j)_* H = H\rho(\alpha_j)_*$. Moreover, the volume form is invariant by isometries. It follows that

$$
\begin{aligned}
\langle \omega, H\partial\overline{c} \rangle &= \left\langle \omega, H \sum_j l_j (D_\rho)_* \eta_j \right\rangle = \sum_j l_j \langle \omega, H(D_\rho)_* \eta_j \rangle \\
&= \sum_j l_j \langle \rho(\alpha_j)^* \omega, H(D_\rho)_* \eta_j \rangle = \sum_j l_j \langle \omega, \rho(\alpha_j)_* H(D_\rho)_* \eta_j \rangle \\
&= \sum_j l_j \langle \omega, H\rho(\alpha_j)_* (D_\rho)_* \eta_j \rangle = \sum_j l_j \langle \omega, H(D_\rho)_* \alpha_{j*} \eta_j \rangle \\
&= \left\langle \omega, H(D_\rho)_* \sum_j l_j \alpha_{j*} \eta_j \right\rangle.
\end{aligned} \tag{5.12}
$$

The last product is zero because $D_{\rho *} \sum_j l_j \alpha_{j*} \eta_j$ lies on the ideal points of A, where H is fixed and ω vanishes.

This completes the proof of Theorem 1.3. ■

Remark 5.8. Theorem 1.3 applies when the cycle c is an ideal triangulation. So it implies Theorem 1.2.

Corollary 5.9. Let M be a graph 3-manifold. Then for all representations $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, $\mathrm{vol}(\rho) = 0$. □

Proof. This is because for each graph manifold M, we have $\|(\overline{M}, \partial\overline{M})\| = 0$ (see [6, 8]). ■

Corollary 5.10. Let M be a complete hyperbolic 3-manifold of finite volume. Then for all representations $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, $|\operatorname{vol}(\rho)| \le \operatorname{vol}(M)$. □

Proof. This follows from the fact that for complete hyperbolic 3-manifold, $\operatorname{vol}(M) = V_3 \|(\overline{M}, \partial \overline{M})\|$ (see [6, 8]). ■

In [3] it is proved that, for compact manifolds, equality holds if and only if ρ is discrete and faithful. In Section 6, we show that this is true in general for manifolds of finite volume.

6 Rigidity of representations

This section is completely devoted to proving Theorem 1.4. More precisely, there exists $\varphi \in \mathrm{PSL}(2, \mathbb{C})$ such that for any $\gamma \in \Gamma$,

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}. \tag{6.1}$$

Remark 6.1. It is well known that in the hypotheses of Theorem 1.4, the manifold M is the interior of a compact manifold $\overline{M}$ whose boundary consists of tori. Thus, M is a cusped manifold and, by Remark 2.10, all the definitions and results we gave for ideally triangulated manifold apply.

As product structure on the cusps, we fix the horospherical one, having the arc-length as cone parameter. For this section, D_ρ will denote a fixed pseudodeveloping map for ρ.

Remark 6.2. By Proposition 4.16, we can suppose that $\operatorname{vol}(\rho) \ge 0$.

Remark 6.3. A subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is said to be *elementary* if it has an invariant set of at most two points in $\partial \mathbb{H}^3$. If the image of ρ is elementary, then one can construct a pseudodeveloping map as in Lemma 2.7 in such a way that all the tetrahedra of any ideal triangulation of M collapse in the straightening. Thus, by Theorem 1.2, $\operatorname{vol}(\rho) = 0$.

This remark implies that, in the present case, since $\operatorname{vol}(\rho) = \operatorname{vol}(M) \neq 0$, the image of ρ is nonelementary.

The idea for proving Theorem 1.4 is to rewrite the Gromov-Thurston-Goldman-Dunfield proof of Mostow's rigidity, valid in the compact case.

We will follow the lead line of [3] with the difference that we will use classical chains instead of measure chains. An alternative approach using measure chains, as employed in [3], could possibly be feasible, but we think that generalizing from the closed to the open case, the necessary preliminary results (described for instance in [13]) may require a considerable amount of work. The technique for constructing classical chains representing smear cycles is that used in [1] for the proof of Mostow's rigidity for compact manifolds. As an effect of noncompactness, we will work with infinite chains. Therefore, we have to prove that some usual homological arguments actually work for our chains.

The core of the proof is to deduce from the equality $\mathrm{vol}(\rho) = \mathrm{vol}(M)$ that D_ρ "does not shrink the volume." This allows us to construct a measurable extension of D_ρ to the whole $\overline{\mathbb{H}}^3$, whose restriction to $\partial\mathbb{H}^3$ is almost everywhere a Möbius transformation. Such a Möbius transformation will be the φ of Theorem 1.4.

The key fact is the following proposition whose proof can be found in [3, Claim 3 of Theorem 6.1].

Proposition 6.4. Let $f : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ be a measurable map that maps the vertices of almost all regular ideal tetrahedra to vertices of regular ideal tetrahedra. Then f coincides almost everywhere with the trace of an isometry φ. □

We want to apply Proposition 6.4 to D_ρ and we do it in two steps. Let M_0 be M minus the cusps and let $\pi : \mathbb{H}^3 \to M$ be the universal cover.

Proposition 6.5. The map D_ρ extends to $\overline{\mathbb{H}}^3$. More precisely, there exists a measurable map $\overline{D}_\rho : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ such that for almost all $x \in \partial\mathbb{H}^3$, for any geodesic γ^x ending at x, and for any sequence $t_n \to \infty$ such that $\pi(\gamma^x(t_n)) \in M_0$,

$$\lim_{n\to\infty} D_\rho(\gamma^x(t_n)) = \overline{D}_\rho(x). \tag{6.2}$$

□

Proposition 6.6. The map $\overline{D}_\rho$ satisfies the hypothesis of Proposition 6.4. □

Before proving Propositions 6.5 and 6.6, we show how they imply Theorem 1.4.

Proof of Theorem 1.4. By Proposition 6.6, Proposition 6.4 applies. By Proposition 6.5, the equivariance of D_ρ implies the equivariance of $\overline{D}_\rho$, getting for any $\gamma \in \Gamma$,

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}. \tag{6.3}$$

■

Remark 6.7. Both Propositions 6.5 and 6.6 will follow from Lemmas 6.21 and 6.22. We notice that Lemma 6.21 is a restatement of [3, Lemma 6.2], while Proposition 6.6 corresponds to [3, Claim 2]. Proposition 6.5 follows from Lemmas 6.21 and 6.22 exactly as in [3]. We will give a complete proof of Proposition 6.6 because the proof of [3, Claim 2] seems to be incomplete.

From now until Lemma 6.10, we describe how to construct a simplicial version of the *smearing* process of measure homology (see [13, 15]). Then we will prove Lemma 6.22. Finally, we will complete the proof of Theorem 1.4 by proving Propositions 6.5 and 6.6.

Let μ be the Haar measure on $\mathrm{Isom}(\mathbb{H}^3)$ such that for each $x \in \mathbb{H}^3$ and $A \subset \mathbb{H}^3$, we have

$$\mu\{g \in \mathrm{Isom}\left(\mathbb{H}^3\right) : g(x) \in A\} = \mathrm{vol}(A), \tag{6.4}$$

where $\mathrm{vol}(A)$ is the hyperbolic volume of A.

In the following, by a tetrahedron of $\overline{\mathbb{H}}^3$ we mean an ordered 4-tuple of points (the vertices). The volume of a tetrahedron is the hyperbolic volume with sign of the convex hull of its vertices.

Let S be the set of all genuine (nonideal, nondegenerate) tetrahedra:

$$S = \left\{\left(y_0, \dots, y_3\right) \in \left(\mathbb{H}^3\right)^4 : \mathrm{vol}\left(y_0, \dots, y_3\right) \neq 0\right\}. \tag{6.5}$$

For any $Y \in S$, let $S(Y)$ be the set of all isometric copies of Y:

$$S(Y) = \{X \in S : \exists g \in \mathrm{Isom}\left(\mathbb{H}^3\right),\ X = g(Y)\}. \tag{6.6}$$

Then a natural bijection $f_Y : \mathrm{Isom}(\mathbb{H}^3) \to S(Y)$ is well defined by

$$f_Y(g) = g(Y). \tag{6.7}$$

Thus, μ induces a measure, which we still call μ, on $S(Y)$ defined by

$$\mu(A) = \mu\left(f_Y^{-1}(A)\right). \tag{6.8}$$

We consider the sets $S_\pm(Y) = f_Y^{-1}(\mathrm{Isom}^\pm(\mathbb{H}^3))$ of tetrahedra, respectively, positively and negatively isometric to Y. Note that $S_+(Y)$ and $S_-(Y)$ are both measurable.

Set $\mathfrak{S} = \Gamma^4/\Gamma$, where Γ acts on Γ^4 by left multiplication. Each element $\sigma = [(\gamma_0, \dots, \gamma_3)] \in \mathfrak{S}$ has a unique representative with $\gamma_0 = \mathrm{Id}$. When we write $\sigma \in \mathfrak{S}$, we tacitly assume that the representative of the form $(\gamma_0, \dots, \gamma_3)$ with $\gamma_0 = \mathrm{Id}$ has been chosen. So γ_0 is always the identity.

For the rest of the section, we fix a fundamental polyhedron $F \subset \mathbb{H}^3$ for M. For all $\varepsilon > 0$ let $\mathcal{F}^\varepsilon$ be a locally finite ε-net in F. For any $\xi \in \mathcal{F}^\varepsilon$, let

$$F_\xi = \left\{x \in F : d(x, \xi) = d\left(x, \mathcal{F}^\varepsilon\right)\right\}. \tag{6.9}$$

Each F_ξ is a geodesic polyhedron of diameter less than ε. From the cone property of D_ρ, it follows that the diameters of $D_\rho(F_\xi)$ are bounded by a constant δ that depends on ε. Moreover, by removing some boundary face from some F_ξ, we get that F is the disjoint union of the F_ξ's. We set

$$S^0(Y) = \{X \in S(Y) \text{ with the first vertex in } F\}. \tag{6.10}$$

We define now a family of special simplices. Let

$$\mathfrak{N} = \{(\gamma_0, \ldots, \gamma_3, \xi_0, \ldots, \xi_3) : (\gamma_0, \ldots, \gamma_3) \in \mathfrak{S},\ \xi_i \in \mathcal{F}^\varepsilon\ \forall i\}. \tag{6.11}$$

For each $\eta \in \mathfrak{N}$, define Δ_η as the straight geodesic singular 3-simplex whose vertices are the points ξ_0, $\gamma_1(\xi_1)$, $\gamma_2(\xi_2)$, and $\gamma_3(\xi_3)$; more precisely,

$$\Delta_\eta : \Delta^3 \ni t \longmapsto \pi\left(\sum_{i=0}^{3} t_i \gamma_i(\xi_i)\right). \tag{6.12}$$

For each tetrahedron $X = (x_0, \ldots, x_3) \in S^0(Y)$, there exists a unique $\eta = (\gamma_0, \ldots, \gamma_3, \xi_0, \ldots, \xi_3) \in \mathfrak{N}$ such that $x_i \in \gamma_i(F_{\xi_i})$ for $i = 0, \ldots, 3$. This defines a function

$$s_Y : S^0(Y) \longrightarrow \mathfrak{N}. \tag{6.13}$$

Roughly speaking, $\mathfrak{N}$ is a locally finite ε-net in the space of 3-simplices of M and s_Y is the "closest point" projection.

For any $\eta \in \mathfrak{N}$, define

$$\begin{aligned} a_Y^\pm(\eta) &= \mu\{s_Y^{-1}(\eta) \cap S_\pm(Y)\} = \mu\{X \in S_\pm(Y) : x_i \in \gamma_i(F_{\xi_i})\}, \\ a_Y(\eta) &= a_Y^+(\eta) - a_Y^-(\eta). \end{aligned} \tag{6.14}$$

In the language of measures, one can think of $a_Y^\pm$ as the pushforward of the measure μ under the map $s_Y : S^0(Y) \cap S_\pm(Y) \to \mathfrak{N}$. This is the key for the passage from measure chains to classical ones.

The smearing of the tetrahedron Y is the cycle

$$Z_Y = \sum_{\eta \in \mathfrak{N}} a_Y(\eta)\Delta_\eta. \tag{6.15}$$

We notice that, as $\mathfrak{N}$ depends on the family $\mathcal{F}^\varepsilon$, the cycle Z_Y actually depends on ε.

Remark 6.8. The smearing of a tetrahedron in general is not a finite sum. Nevertheless, as the following lemma shows, it has bounded l^1-norm.

Lemma 6.9. For any $Y \in S$, $\sum_\eta |a_Y(\eta)| < \mathrm{vol}(M)$. □

Proof. If $Y = (y_0, \dots, y_3)$, then

$$\begin{aligned}\sum_\eta |a_Y(\eta)| &\le \sum_\eta \left(a_Y^+(\eta) + a_Y^-(\eta)\right) = \sum_\eta \mu\{s_Y^{-1}(\eta)\} = \mu\left\{\bigcup_\eta s_Y^{-1}(\eta)\right\} \\ &= \mu\{s_Y^{-1}(\mathfrak{N})\} = \mu\{f_Y^{-1}s_Y^{-1}(\mathfrak{N})\} = \mu\{g : g(y_0) \in F\} \\ &= \mathrm{vol}(F) = \mathrm{vol}(M).\end{aligned} \tag{6.16}$$

■

Lemma 6.10. The infinite chain Z_Y is a cycle, that is, $\partial Z_Y = 0$. □

Proof. First note that the l^1-norm of ∂Z_Y is bounded by 4 times the l^1-norm of Z_Y. Thus, all the sums we will consider make sense.

Let υ be a simplex of ∂Z_Y. By construction, υ is obtained as the projection of an $(n-1)$-simplex having vertices in F_{ξ_0}, $\gamma_1(F_{\xi_1})$, and $\gamma_2(F_{\xi_2})$, for some $\gamma_1, \gamma_2 \in \Gamma$ and $\xi_0, \xi_1, \xi_2 \in \mathcal{F}^\varepsilon$. Let A_υ be the set of the elements of $\mathfrak{N}$ of the form $\eta = (\gamma_0, \gamma_1, \gamma_2, \gamma, \xi_0, \xi_1, \xi_2, \xi)$ with $\gamma \in \Gamma$ and $\xi \in \mathcal{F}^\varepsilon$. The simplices Δ_η of Z_Y having υ as the last face contribute to the coefficient of υ in ∂Z_Y by

$$\begin{aligned}\sum_{\eta \in A_\upsilon} a_Y(\eta) &= \sum_{\eta \in A_\upsilon} \mu\left(s_Y^{-1}(\eta) \cap S_+(Y)\right) - \sum_{\eta \in A_\upsilon} \mu\left(s_Y^{-1}(\eta) \cap S_-(Y)\right) \\ &= \mu\left(s_Y^{-1}(A_\upsilon) \cap S_+(Y)\right) - \mu\left(s_Y^{-1}(A_\upsilon) \cap S_-(Y)\right) = 0.\end{aligned} \tag{6.17}$$

The same calculation, made with the simplices having υ as the ith face, shows that the coefficient of υ in ∂Z_Y is zero. ■

For any ideal, nonflat tetrahedron $Y = (y_0, \dots, y_3)$, let $t \mapsto y_i(t)$ be the geodesic ray from the center of mass of Y to y_i, $i = 0, \dots, 3$. For any $R > 0$, let Y_R be the following element of S:

$$Y_R = \left(y_0(R), \dots, y_3(R)\right). \tag{6.18}$$

Remark 6.11. From now on, we fix a positively oriented regular ideal tetrahedron Y, and we write $S_\pm(R)$, f_R, s_R, $a_R(\eta)$, and Z_R for $S_\pm(Y_R)$, f_{Y_R}, s_{Y_R}, $a_{Y_R}(\eta)$, and Z_{Y_R}.

We say that a 3-simplex Δ is ε-close to a tetrahedron X if the vertices of Δ are ε-close to X. We define

$$\epsilon(R, \varepsilon) = \sup\left\{V_3 - \mathrm{vol}(\Delta) : \Delta \text{ is } \varepsilon\text{-close to an element of } S(R)\right\}. \tag{6.19}$$

Lemma 6.12. For any fixed ε, for large R, the function $\epsilon(R, \varepsilon)$ goes to zero exponentially in R. □

This is because $V_3 - \mathrm{vol}(Y_R)$ goes to zero like e^{-R} and the volume of any Δ which is ε-close to Y_R is close to the volume of Y_R. See [1, 3, 15] for details.

Remark 6.13. What we actually need to prove our claims is a restatement for Z_R of Step 2 of the proof of Theorem 1.3. From now until Proposition 6.19, we prove facts that are standard for finite chains, but need a proof for Z_R.

For $\eta \in \mathfrak{N}$, we set $\nu_\eta = \mathrm{vol}(\Delta_\eta)$. Using the fact that all the F_ξ's have diameters less than ε, one can prove the following lemma (see [1] for details). Recall that $\mathfrak{N}$ depends on $\mathcal{F}^\varepsilon$ and so it depends on ε.

Lemma 6.14. For any $\varepsilon > 0$, for large enough R, for any $\eta \in \mathfrak{N}$,

(1) $a_R^+(\eta) \cdot a_R^-(\eta) = 0$,

(2) $a_R(\eta) \neq 0 \implies a_R(\eta) \cdot \nu_\eta \geq 0$. □

Lemma 6.15. There exists a constant c such that $|D_\rho^* \omega| < c|\omega|$, where ω is the volume form of $\mathbb{H}^3$. □

Proof. Let M_0 be M minus the cusps. The function $|D_\rho^*\omega|/|\omega|$ is continuous and hence bounded on M_0. In the cusps, by direct calculation and using the cone property of D_ρ, one can show that the same bound holds. ■

Lemma 6.16. The integrals $\langle \omega, Z_R\rangle$ and $\langle D_\rho^*\omega, Z_R\rangle$ are well defined. □

Proof. As $\sum |a_R(\eta)| < +\infty$, since $|\langle \omega, \Delta_\eta\rangle|$ is bounded by V_3, then $\langle \omega, Z_R\rangle$ is well defined. Consider now $D_\rho^*\omega$. From Lemma 6.15, it follows that the integral of $|D_\rho^*|$ over straight geodesic simplices is bounded by cV_3. Hence, also $\langle D_\rho^*\omega, Z_R\rangle$ is well defined. ■

As above, let M_0 denote M minus the cusps and, for $k \in \mathbb{N}^*$, let

$$M_k = \bigcup_{T \subset \partial M_0} T \times [k-1, k). \tag{6.20}$$

Let $\mathcal{F}_k^\varepsilon = \mathcal{F}^\varepsilon \cap \pi^{-1}(M_k)$ and $\mathfrak{N}_k = \{\eta \in \mathfrak{N} : \xi_0 \in \mathcal{F}_k^\varepsilon\}$. We have

$$Z_R = \sum_{k \in \mathbb{N}} \sum_{\eta \in \mathfrak{N}_k} a_R(\eta)\Delta_\eta. \tag{6.21}$$

Lemma 6.17. For any k, the chain $\sum_{\eta \in \mathfrak{N}_k} a_R(\eta)\Delta_\eta$ is a finite sum. □

Proof. If $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_k$, then Δ_η is ε-close to an element $X \in S(R)$ having first vertex in F_{ξ_0} with $\xi_0 \in \mathcal{F}^\varepsilon_k$. Since $\mathcal{F}^\varepsilon$ is locally finite and $\overline{M}_k$ is compact, $\mathcal{F}^\varepsilon_k$ is finite, so there is only a finite number of possibilities for ξ_0. Since $\overline{F}_{\xi_0}$ is compact, any $X \in S(R)$ with first vertex in F_{ξ_0} lies on a compact ball B of $\mathbb{H}^3$. Since F is a fundamental domain, then there exists only a finite number of elements $\gamma \in \Gamma$ so that $\gamma(F)$ intersects B. Then for any ξ_0, there is only a finite number of possibilities for ξ_1, ξ_2, and ξ_3. It follows that there exists only a finite number of $\eta \in \mathfrak{N}_k$ such that $a_R(\eta) \neq 0$. ■

Lemma 6.18. For any R, if k is large enough, then for any $\eta \in \mathfrak{N}_k$ with $a_R(\eta) \neq 0$, the simplex Δ_η is completely contained in a cusp of M. □

Proof. If $X = (x_0, \dots, x_3) \in S(R)$, then X lies in the ball $B(x_0, 2R)$. Since M has a finite number of cusps, for any R there exists $m \in \mathbb{N}$ such that for $k \geq m$ if $x_0 \in M_k$, then the whole ball $B(x_0, 2R + \varepsilon)$ is contained in the cusp containing x_0. If $\eta \in \mathfrak{N}_k$ and $a_R(\eta) \neq 0$, then there exists $X \in S(R)$ with $x_0 \in \pi^{-1}(M_k) \cap F$, hence Δ_η is ε-close to X. Thus, $\Delta_\eta \subset B(x_0, 2R + \varepsilon)$ is contained in the cusp that contains x_0. ■

Now, for $k \in \mathbb{N}$, define

$$Z_{R,k} = \sum_{j<k} \sum_{\eta \in \mathfrak{N}_j} a_R(\eta)\Delta_\eta \tag{6.22}$$

which is a finite chain by Lemma 6.17. Moreover, since $\partial Z_R = 0$, then each simplex υ of $\partial Z_{R,k}$ appears as a face of a simplex Δ_η with $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_j$ for some $j \geq k$. Therefore, by Lemma 6.18, for k large enough, each simplex υ of $\partial Z_{R,k}$ is contained in a cusp of M. Thus, to each υ there corresponds an ideal point of $\widehat{M}$. For each $\upsilon \in \partial Z_{R,k}$, let $\lambda_{R,k}(\upsilon)$ be the coefficient of υ in $\partial Z_{R,k}$ and let C_υ be the cone from υ to the corresponding ideal point.

Let $\widehat{Z}_{R,k}$ be the chain obtained by adding to $Z_{R,k}$ the cones C_υ:

$$\widehat{Z}_{R,k} = Z_{R,k} + \sum_{\upsilon \in \partial Z_{R,k}} \lambda_{R,k}(\upsilon) C_\upsilon. \tag{6.23}$$

The chain $\widehat{Z}_{R,k}$ is a finite sum and it is easily checked that it is a cycle.

For any 3-simplex Δ, let $\mathrm{Strvol}(\Delta)$ denote the volume of the convex hull of the vertices of $D_\rho(\Delta)$. For any $\eta \in \mathfrak{N}$, set $w_\eta = \mathrm{Strvol}(\Delta_\eta)$.

Proposition 6.19. For any $R > 0$,

$$\sum_\eta a_R(\eta) v_\eta = \langle \omega, Z_R \rangle = \langle D^*_\rho \omega, Z_R \rangle = \sum_\eta a_R(\eta) w_\eta. \tag{6.24}$$

□

Proof. The first equality is tautological. We use now the cycles $\widehat{Z}_{R,k}$ to approximate Z_R. Since $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, then $[\omega] = [D_\rho^*\omega]$ as elements of $H^3(\widehat{M})$. Thus, for any $k \in \mathbb{N}$, we have $\langle \omega, \widehat{Z}_{R,k}\rangle = \langle D_\rho^*\omega, \widehat{Z}_{R,k}\rangle$. As in Step 2 of the proof of Theorem 1.3, we can straighten the finite cycle $\widehat{Z}_{R,k}$ getting

$$\left\langle \omega, \widehat{Z}_{R,k}\right\rangle = \left\langle D_\rho^*\omega, \widehat{Z}_{R,k}\right\rangle = \sum_{j<k}\sum_{\eta\in\mathfrak{N}_k} a_R(\eta) w_\eta + \sum_{v\in\partial Z_{R,k}} \lambda_{R,k}(v)\,\mathrm{Strvol}\left(C_v\right). \tag{6.25}$$

For each simplex α of $\widehat{Z}_{R,k}$, we have $|\mathrm{vol}(\alpha)| \le V_3$, $|\mathrm{Strvol}(\alpha)| \le V_3$, and, by Lemma 6.15, $|\langle D_\rho^*\omega, \alpha\rangle| \le cV_3$. It follows that to get the remaining inequalities, it suffices to show that

$$\lim_{k\to\infty} \sum_{v\in\partial Z_{R,k}} \left|\lambda_{R,k}(v)\right| = 0. \tag{6.26}$$

Since $\partial Z_R = 0$, if $v \in \partial Z_{R,k}$, then $v \in \partial\Delta_\eta$ with $a_R(\eta) \neq 0$ and $\eta \in \mathfrak{N}_j$ for some $j \ge k$. So we have

$$\begin{aligned}\sum_{v\in\partial Z_{R,k}} \left|\lambda_{R,k}(v)\right| &\le 4\sum_{j\ge k}\sum_{\eta\in\mathfrak{N}_j} a_R^+(\eta) + a_R^-(\eta) = 4\sum_{j\ge k}\sum_{\eta\in\mathfrak{N}_j} \mu\left\{s_R^{-1}(\eta)\right\}\\ &= 4\sum_{j\ge k}\mu\left\{s_R^{-1}\left(\mathfrak{N}_j\right)\right\} = 4\sum_{j\ge k}\mu\left\{Y\in S(R) : \exists\xi\in\mathcal{F}_j^\varepsilon,\ y_0\in F_\xi\right\}\\ &= 4\sum_{j\ge k}\sum_{\xi\in\mathcal{F}_j^\varepsilon} \mathrm{vol}\left(F_\xi\right) \le 4\,\mathrm{vol}\left(\bigcup_{j\ge k-\varepsilon} M_j\right).\end{aligned} \tag{6.27}$$

The last term goes to zero as $k\to\infty$ because M has finite volume and the desired equality follows. ■

Now that we have Proposition 6.19, forget about the cycles $\widehat{Z}_{R,k}$.

From triangular inequality, Proposition 6.19, and Lemma 6.14, we have

$$\begin{aligned}\sum_\eta \left|a_R(\eta)\right|\cdot\left|w_\eta\right| &\ge \left|\sum_\eta a_R(\eta) w_\eta\right| = \left|\sum_\eta a_R(\eta) v_\eta\right|\\ &= \sum_\eta \left|a_R(\eta)\right|\cdot\left|v_\eta\right| \ge \sum_\eta \left|a_R(\eta)\right|\left(V_3 - \epsilon(R,\varepsilon)\right)\end{aligned} \tag{6.28}$$

from which and from Lemma 6.9, we get the following proposition.

Proposition 6.20. For R large enough,

$$\sum_{\eta\in\mathfrak{N}} \left|a_R(\eta)\right|\left(V_3 - \left|w_\eta\right|\right) \le \sum_{\eta\in\mathfrak{N}} \left|a_R(\eta)\right|\epsilon(R,\varepsilon) \le \mathrm{vol}(M)\epsilon(R,\varepsilon). \tag{6.29}$$

□

For any $R > 0$, let $A_R \subset \mathfrak{N}$ be the set of tetrahedra with "small" straight volume:

$$A_R = \left\{\eta \in \mathfrak{N} : V_3 - \left|w_\eta\right| > R^2 \cdot \mathrm{vol}(M) \cdot \epsilon(R, \varepsilon)\right\}. \tag{6.30}$$

Lemma 6.21. For R large enough,

$$\sum_{\eta \in A_R} \left|a_R(\eta)\right| \leq \frac{1}{R^2}. \tag{6.31}$$

□

Proof. From Proposition 6.20, we get

$$\begin{aligned} R^2 \,\mathrm{vol}(M)\epsilon(R, \varepsilon) \cdot \sum_{\eta \in A_R} \left|a_R(\eta)\right| &\leq \sum_{\eta \in A_R} \left|a_R(\eta)\right|\left(V_3 - \left|w_\eta\right|\right) \\ &\leq \sum_{\eta \in \mathfrak{N}} \left|a_R(\eta)\right|\left(V_3 - \left|w_\eta\right|\right) \\ &\leq \mathrm{vol}(M)\epsilon(R, \varepsilon). \end{aligned} \tag{6.32}$$

The claimed inequality follows. ■

Lemma 6.22. For almost all isometries g,

$$\lim_{n \to \infty} \mathrm{Strvol}\left(g(Y_n)\right) = V_3. \tag{6.33}$$

□

Proof. Since $a_R^+ \cdot a_R^- = 0$, then $\sum_{\eta \in A_R} |a_R(\eta)| = \mu(s_R^{-1}(A_R))$. Thus, for any fixed $R > 0$, we have

$$\mu\left(\bigcup_{\mathbb{N} \ni n > R} s_R^{-1}(A_n)\right) \leq \sum_{n > R} \frac{1}{n^2} < \frac{1}{R}. \tag{6.34}$$

Recalling that for any set $A \subset \mathfrak{N}$ we have $\mu(s_R^{-1}(A)) = \mu(f_R^{-1} s_R^{-1}(A))$, we get

$$\mu\left\{g \in \mathrm{Isom}\left(\mathbb{H}^3\right) : \exists n > R,\ w_{s_n(g(Y_n))} < V_3 - n^2 \cdot \mathrm{vol}(M) \cdot \epsilon(n, \varepsilon)\right\} < \frac{1}{R}. \tag{6.35}$$

From Lemma 6.12, it follows that $\lim_{n \to \infty} n^2 \epsilon(n, \varepsilon) = 0$. As $R \to \infty$, this implies that for any $\varepsilon > 0$, for almost any isometry g, we have

$$\lim_{n \to \infty} w_{s_n(g(Y_n))} = V_3. \tag{6.36}$$

Let g be one of such maps. Since the diameters of the $D_\rho(F_\xi)$ are bounded by δ, then $D_\rho(\Delta_{s_R(g(Y_R))})$ is δ-close to $D_\rho(g(Y_R))$. Recalling that $w_{s_R(g(Y_R))} = \mathrm{Strvol}(\Delta_{s_R(g(Y_R))})$, we have that

$$\lim_{n\to\infty} \operatorname{Strvol}\left(\Delta_{s_n(g(Y_n))}\right) = V_3 \tag{6.37}$$

and, since $D_\rho(g(Y_R))$ is δ-close to $D_\rho(\Delta_{s_R(g(Y_R))})$, then also

$$\lim_{n\to\infty} \operatorname{Strvol}\left(g(Y_n)\right) = V_3. \tag{6.38}$$

■

We sketch here the proof of Proposition 6.5, refer to [3] for details.

Proof of Proposition 6.5. In the disc model, let γ be a geodesic from 0 to a point in $\partial\mathbb{H}^3$. Let X_R be a family of regular tetrahedra of edge R with first vertex in 0 and second in $\gamma(R)$. All the claims from Lemma 6.9 to Lemma 6.22 hold for $\{X_R\}$. It follows that for almost all isometries g, we have

$$\lim_{n\to\infty} \operatorname{Strvol}\left(g(X_n)\right) = V_3. \tag{6.39}$$

Then $D_\rho(g(\gamma(n)))$ must reach the boundary of $\mathbb{H}^3$. Using again the above property of the limit, one can estimate the angle $\alpha(n)$ between the geodesic from $D_\rho(g(0))$ to $D_\rho(g(\gamma(n)))$ and the geodesic from $D_\rho(g(0))$ to $D_\rho(g(\gamma(n+1)))$. Such estimate shows that $\sum \alpha(n) < \infty$, which implies that $D_\rho(g(\gamma(n)))$ converges. The claim follows because D_ρ is locally Lipschitz outside the cusps. Measurability follows because the extension can be viewed as a pointwise limit of measurable functions. ■

Remark 6.23. In general, D_ρ is not uniformly continuous in the cusps. So it cannot be locally Lipschitz on the whole $\mathbb{H}^3$.

We come now to the proof of Proposition 6.6.

Lemma 6.24. Let $X = (x_0, x_1, x_2, x_3)$ be an ideal tetrahedron in $\overline{\mathbb{H}}^3$. Suppose that no three vertices of X coincide. Then for any $\varepsilon > 0$, there exist neighborhoods U_i of x_i in $\overline{\mathbb{H}}^3$ such that for any tetrahedron $Y = (y_0, \dots, y_3)$ with $y_i \in U_i$, $|\operatorname{vol}(Y) - \operatorname{vol}(X)| < \varepsilon$. □

This follows from the formula of the volume for ideal tetrahedra, see [1] for details.

Remark 6.25. Lemma 6.24 does not hold if three vertices of X coincide. To see this, let Y be a regular ideal tetrahedron and let γ be a parabolic or hyperbolic isometry. Then $\gamma^n(Y)$ is a family of tetrahedra with maximal volume, but at least three of the vertices of $\gamma^n(Y)$ converge to the same point.

Lemma 6.26. For almost all regular ideal tetrahedra Y, the ideal tetrahedron $\overline{D}_\rho(Y)$ is defined. Moreover, for almost all Y, either $\overline{D}_\rho(Y)$ is regular (whence $\mathrm{vol}(\overline{D}_\rho(Y)) = V_3$) or at least three of its vertices coincide (whence $\mathrm{vol}(\overline{D}_\rho(Y)) = 0$). □

Proof. Without loss of generality, we can restrict the first claim to the space of positive regular ideal tetrahedra. We parametrize such a space with

$$\left\{(a, b, c) \in S^2_\infty \times S^2_\infty \times S^2_\infty : a \neq b \neq c\right\}, \tag{6.40}$$

where $S^2_\infty = \partial\mathbb{H}^3$, by mapping (a, b, c) to the unique positive regular ideal tetrahedron with (a, b, c) as the first three vertices. We denote by $Q(a, b, c)$ the fourth vertex of such tetrahedron. Since $\overline{D}_\rho$ is defined almost everywhere, the first claim follows from Fubini's theorem. The second claim follows from Lemmas 6.22 and 6.24. ■

From Lemma 6.26, we can restate Proposition 6.6 as follows.

Proposition 6.27. The set $\{Y \in S^2_\infty \times S^2_\infty \times S^2_\infty : \mathrm{vol}(\overline{D}_\rho(Y)) = 0\}$ has zero measure. □

The proof of this result will follow from the next lemma.

Lemma 6.28. If the set

$$\left\{Y \in S^2_\infty \times S^2_\infty \times S^2_\infty : \mathrm{vol}\left(\overline{D}_\rho(Y)\right) = 0\right\} \tag{6.41}$$

has positive measure, then the map $\overline{D}_\rho$ is constant almost everywhere. □

Before proving Lemma 6.28, we show how it implies Proposition 6.27.

Proof of Proposition 6.27. By contradiction, we apply Lemma 6.28 deducing that $\overline{D}_\rho$ is almost everywhere a constant p. From the equivariance of $\overline{D}_\rho$, it follows that for any $\gamma \in \Gamma$ and $x \in \partial\mathbb{H}^3$, we have

$$p = \overline{D}_\rho \gamma(x) = \rho(\gamma)\left(\overline{D}_\rho(x)\right) = \rho(\gamma)(p). \tag{6.42}$$

Thus, p is a fixed point of any element of Γ. This implies that the image of ρ is elementary, but this cannot happen because of Remark 6.3. ■

We now prove Lemma 6.28.

Lemma 6.29. In the hypothesis of Lemma 6.28, there exists a positive-measure set $A \subset S^2_\infty$ such that $\overline{D}_\rho$ is constant on A. □

Proof. By Lemma 6.26, it is not restrictive to suppose that the set

$$\{(a,b,c) \in S^2_\infty \times S^2_\infty \times S^2_\infty : \overline{D}_\rho(a) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\} \tag{6.43}$$

has positive measure. Then by Fubini's theorem there exists a positive-measure set $A_0 \subset S^2_\infty$ such that for all $a_0 \in A_0$, the set

$$\{(b,c) \in S^2_\infty \times S^2_\infty : \overline{D}_\rho(a_0) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\} \tag{6.44}$$

has positive measure in $S^2_\infty \times S^2_\infty$. Again by Fubini's theorem for all $a_0 \in A_0$, there exists a positive-measure set $A_1 \in S^2_\infty$ such that for any $a_1 \in A_1$, the set

$$\{c \in S^2_\infty : \overline{D}_\rho(a_0) = \overline{D}_\rho(a_1) = \overline{D}_\rho(c)\} \tag{6.45}$$

has positive measure. In particular, $\overline{D}_\rho$ is constant on A_1. ■

We set $p = \overline{D}_\rho(A_1)$ and $A = \overline{D}_\rho^{-1}(p)$.

Remark 6.30. In the sequel we use the symbol $\widetilde{\forall}$ to mean *"for almost all."*

By Lemma 6.26, the set A has the following property:

$$\widetilde{\forall}(a_0, a_1, x) \in A \times A \times A^c, \quad Q(a_0, a_1, x) \in A. \tag{6.46}$$

We work now in the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$. So, $S^2_\infty = \mathbb{C} \cup \{\infty\}$. In that model

$$Q(\infty, a, z) = \alpha(z - a) + a, \tag{6.47}$$

where $\alpha = (1 + i\sqrt{3})/2$. Again by Fubini's theorem $\widetilde{\forall}a_0 \in A$, $\widetilde{\forall}(a_1, x) \in A \times A^c$, we have $Q(a_0, a_1, x) \in A$ and we can suppose that this holds for $a_0 = \infty$.

In other words, $\widetilde{\forall}(a, x) \in A \times A^c$, the third vertex of the equilateral triangle, with the first two vertices in a and x, is in A. For any $a, x \in \mathbb{C}$, we call $E_x(a)$ the set of the vertices of the regular hexagon centered at x and with a vertex in a. Then we have

$$\widetilde{\forall}(a, x) \in A \times A^c, \quad E_x(a) \subset A \tag{6.48}$$

and in particular $\widetilde{\forall}(a, x) \in A \times A^c$, $2x - a \in A$. Note that x is the midpoint of the segment between a and $2x - a$.

Lemma 6.31. For any open set $B \subset \mathbb{C}$, $\mu(A \cap B) > 0$. □

Proof. Suppose the contrary. Then there exists an open set B such that $\mu(A \cap B) = 0$. That is, almost all the points of B are in A^c. Moreover, from (6.48) and Fubini's theorem it follows that $\widetilde{\forall} x \in A^c, \widetilde{\forall} a \in A, E_x(a) \in A$. Therefore, there exists a point $x_0 \in B$ such that a small ball $B_0 = B(x_0, r_0)$ is contained in B and

$$\widetilde{\forall} a \in A, \quad E_{x_0}(a) \in A. \tag{6.49}$$

Since $\mu(A) > 0$, then there exists a small ball $B_1 = B(x_1, r_1)$ such that $\mu(A \cap B_1) > 0$. Let $x_2 = (x_1 + x_0)/2$. If there exists $r > 0$ such that $\mu(A \cap B(x_2, r)) = 0$, then applying the same argument, we can find a point y arbitrarily close to x_2 such that (6.49) holds for y. In particular, we get that almost all the points of the set $C = \{2y - a : a \in B_1 \cap A\}$ are in A. But if y is close enough to x_2, then $C \cap B_0$ has positive measure, contradicting that $\mu(A \cap B) = 0$.

It follows that for all $r_2 > 0$, we have $\mu(A \cap B(x_2, r_2)) > 0$; in particular, we choose $r_2 < r_0/2$. By iterating this construction, we find a sequence of points $x_n \to x_0$ and radii $r_0/2 > r_n > 0$ such that $\mu(A \cap B(x_n, r_n)) > 0$. For n large enough, this contradicts the fact that $\mu(A \cap B) = 0$. ∎

Lemma 6.32. For all $z \in \mathbb{C}$,

$$\mu\big(B(z,r) \cap A\big) \geq \frac{1}{2}\mu\big(B(z,r)\big) \quad \forall r > 0. \tag{6.50}$$

□

Proof. From Fubini's theorem and condition (6.48), it follows that $\widetilde{\forall} a \in A$, we have

$$\widetilde{\forall} x \in A^c, \quad E_x(a) \subset A. \tag{6.51}$$

Note that if (6.51) holds for a, then (6.50) holds for a.

Let $z \in \mathbb{C}$. From Lemma 6.31, it follows that there exists a sequence $x_n \to z$ such that (6.51) (and hence (6.50)) holds for x_n. As the function $x \mapsto \mu(A \cap B(x, r))$ is continuous, then the claim holds for z. ∎

Lemma 6.33. Let $X \subset \mathbb{R}^2$ be a measurable set. If there exists $\alpha > 0$ such that, for any ball B,

$$\mu(B \cap X) \geq \alpha \mu(B), \tag{6.52}$$

then $\mu(\mathbb{R}^2 \setminus X) = 0$. □

This is a standard fact of integration theory and it follows from Lebesgue's differentiation theorem (see, e.g., [14]).

From this lemma and Lemma 6.32 it follows that the set A has full measure. Since $A = \overline{D}_\rho^{-1}(p)$, then $\overline{D}_\rho$ is constant almost everywhere and Lemma 6.28 is proved.

This completes the proof of Theorem 1.4.

7 Corollaries

In this section, we prove some corollaries that can be useful for studying hyperbolic 3-manifolds.

First, we show how from Theorem 1.4 one gets a proof of Theorem 1.5 (see [2, 12] for a more general statement and a different proof).

Proof of Theorem 1.5. Let ω be the volume form of N. For $X = M, N$, let $\Gamma_X \cong \pi_1(X)$ be the subgroup of $PSL(2,\mathbb{C})$ such that $X = \mathbb{H}^3/\Gamma_X$. Let f_* denote both the map induced in homology and the representation $f_* : \pi_1(M) \to PSL(2,\mathbb{C})$.

First, assume that the lift $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ has the cone property on the cusps. This implies that $\widetilde{f}$ is a pseudodeveloping map for f_*. Since $f_*[M] = \deg(f)[N]$, we have

$$\begin{aligned} \mathrm{vol}(M) &= \deg(f) \cdot \mathrm{vol}(N) = \big\langle \omega, \deg(f)[N] \big\rangle \\ &= \big\langle \omega, f_*[M] \big\rangle = \big\langle f^*\omega, [M] \big\rangle = \mathrm{vol}\left(f_*\right). \end{aligned} \tag{7.1}$$

Thus, by Theorem 1.4 there exists an isometry φ such that for any $\gamma \in \Gamma_M$,

$$f_*(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}. \tag{7.2}$$

As $\widetilde{M} \cong \mathbb{H}^3$, we consider the isometry φ as an f_*-equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$. Namely, for any $x \in \mathbb{H}^3$ and $\gamma \in \Gamma_M$,

$$\varphi\big(\gamma(x)\big) = f_*(\gamma)\big(\varphi(x)\big). \tag{7.3}$$

∎

It follows that φ projects to a locally isometric covering $\varphi : M \to N$ and the convex combination from $\widetilde{f}$ to φ projects to a proper homotopy from f to φ. Since the degree of a map is invariant under proper homotopies, then $\deg(\varphi) = \deg(f)$.

We prove now that f is always properly homotopic to a map whose lift has the cone property on the cusps. Let $\widetilde{f}$ be a lift of f. For each cusp $N_p = P_p \times [0,\infty)$, let $f_p = \widetilde{f}|_{P_p \times \{0\}}$. Since f is proper, it follows that $\widetilde{f}(N_p \times \{\infty\})$ is well defined. Let $F_p : N_p \times [0,\infty) \to \mathbb{H}^3$ be the map obtained by coning f_p to $\widetilde{f}(N_p \times \{\infty\})$ along geodesic rays. Let $\widetilde{f}'$ be the map obtained by replacing, on each cusp N_p, the map $\widetilde{f}|_{N_p}$ with the map F_p. The map $\widetilde{f}'$ obviously has the cone property on the cusps and projects to a map $f' : M \to N$. Moreover, the convex combination from $\widetilde{f}$ to $\widetilde{f}'$ projects to a proper homotopy between f and f'.

From Theorems 1.4 and 1.5, Corollary 5.10, and the corresponding statements for compact manifolds, we get the following statement.

Theorem 7.1. Let M be a complete, oriented hyperbolic 3-manifold of finite volume. Let $\Gamma \cong \pi_1(M)$ be the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ such that $M = \mathbb{H}^3/\Gamma$. Let $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. Then $|\mathrm{vol}(\rho)| \leq |\mathrm{vol}(M)|$ and equality holds if and only if ρ is discrete and faithful. □

Corollary 7.2. Let M be an atoroidal, irreducible, ideally triangulated 3-manifold. Let $\boldsymbol{z} \in \{\mathbb{C} \setminus \{0,1\}\}^n$ be a solution of the hyperbolicity equations such that $\mathrm{vol}(\boldsymbol{z}) \neq 0$. Then M is hyperbolic. □

Proof. This immediately follows from Corollary 5.9 and Thurston's hyperbolization theorem (see [16]). ■

In [4], the notion of *geometric* solution of the hyperbolicity equations is introduced. Roughly speaking, a geometric solution of the hyperbolicity equations for a given ideal triangulation τ is a choice of moduli which is compatible with a global hyperbolic structure on M. In [4], it is shown that not each solution of hyperbolicity equations is geometric (see [4] for more details on algebraic and geometric solutions of hyperbolicity equations).

Corollary 7.3. Let M be a complete hyperbolic 3-manifold of finite volume and let τ be an ideal triangulation of M. If there exists a solution $\boldsymbol{z} \in \{\mathbb{C} \setminus \{0,1\}\}^n$ of the hyperbolicity equations for τ, then there exists a solution z' of hyperbolicity equations that is geometric. Moreover, such a solution is the one of maximal volume. □

Proof. Consider a natural straightening of τ and let $\boldsymbol{z}'$ be the moduli induced on τ. By Proposition 3.8, we only have to prove that the moduli are not in $\{0,1,\infty\}$. Suppose there is a degenerate tetrahedron Δ_i. Then at least two vertices, say v and w, of Δ_i coincide.

Let $\rho(\boldsymbol{z})$ be the holonomy relative to $\boldsymbol{z}$ and let D_z be a developing map that is also a pseudodeveloping map for $\rho(\boldsymbol{z})$. Then D_z maps Δ_i into a tetrahedron of modulus z_i. But by hypothesis, $\boldsymbol{z}$ is in $\{\mathbb{C}\setminus\{0,1\}\}^n$ and so the vertices of Δ_i are four distinct points. The last assertion follows from Corollary 5.10 and Theorem 1.4. ■

Corollary 7.3 tells that, once one has a solution $\boldsymbol{z} \in \{\mathbb{C} \setminus \{0,1\}\}^n$ of the hyperbolicity equations for a triangulation τ of a cusped manifold M, in order to know if M admits a complete hyperbolic structure of finite volume, it suffices to study the solution of maximal volume. Namely, if one succeeds to prove that the solution of maximal volume is geometric, then M is hyperbolic. Conversely, if one proves that such a solution is not geometric (e.g., if its holonomy is not discrete), then M cannot be hyperbolic, and this does not depend on the chosen triangulation.

As an example of application of Corollary 7.3 we give the following corollary.

Corollary 7.4. Let M be a cusped 3-manifold equipped with an ideal triangulation τ. If there exists a solution $z \in \{\mathbb{C} \setminus \{0,1\}\}^n$ of the hyperbolicity equations for τ and all the solutions have zero volume, then M is not hyperbolic. □

We notice that the hypothesis that all the solutions have zero volume can be replaced by requiring that the volumes are *too small*. This is because the set of the volumes of the hyperbolic manifolds is bounded from below by a positive constant.

Finally, we obtain another proof of the well-known fact that no Dehn filling of a Seifert manifold is hyperbolic.

Corollary 7.5. Let M be a 3-manifold such that $\|(M, \partial M)\| = 0$ and let N be a Dehn filling of M. Then N is not hyperbolic. □

Proof. Suppose the contrary. Let ρ be the holonomy of the hyperbolic structure of N. From Theorem 1.3 it follows that $\mathrm{vol}(\rho) = 0$, but from Proposition 4.17 and Corollary 4.13 it follows that $\mathrm{vol}(\rho) = \mathrm{vol}(N) > 0$. ■

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