SPECTRAL RIGIDITY OF AUTOMORPHIC ORBITS IN FREE GROUPS

MATHIEU CARETTE, STEFANO FRANCAVIGLIA, ILYA KAPOVICH, AND ARMANDO MARTINO

ABSTRACT. It is well-known that a point $T \in \mathrm{cv}_N$ in the (unprojectivized) Culler-Vogtmann Outer space cv_N is uniquely determined by its *translation length function* $||.||_T : F_N \to \mathbb{R}$. A subset S of a free group F_N is called *spectrally rigid* if, whenever $T, T' \in \mathrm{cv}_N$ are such that $||g||_T = ||g||_{T'}$ for every $g \in S$ then $T = T'$ in cv_N. By contrast to the similar questions for the Teichmüller space, it is known that for $N \geq 2$ there does not exist a finite spectrally rigid subset of F_N.

In this paper we prove that for $N \geq 3$ if $H \leq \mathrm{Aut}(F_N)$ is a subgroup that projects to an infinite normal subgroup in $\mathrm{Out}(F_N)$ then the H-orbit of an arbitrary nontrivial element $g \in F_N$ is spectrally rigid. We also establish a similar statement for $F_2 = F(a, b)$, provided that $g \in F_2$ is not conjugate to a power of $[a, b]$.

CONTENTS

2000 *Mathematics Subject Classification.* Primary 20F, Secondary 57M, 37D.

The first author is a Postdoctoral Researcher of the F.R.S.-FNRS (Belgium). The third author was supported by the NSF grant DMS-0904200.

1. Introduction

The phenomenon of *marked length spectrum rigidity* plays an important role in Riemannian geometry and adjacent areas. If M is a closed manifold with a Riemannian metric ρ of negative (but not necessarily constant) curvature, the associated *marked length spectrum* is the function $\ell_\rho : G \to \mathbb{R}$, where $G = \pi_1(M)$ and where for $\gamma \in G$ $\ell_\rho(\gamma)$ is the shortest length with respect to ρ among all free homotopy representatives of γ. It is easy to see that $\ell_\rho(\gamma) = \ell_\rho(\gamma_1^{-1}\gamma\gamma_1)$ for all $\gamma, \gamma_1 \in G$. Thus ℓ_ρ may be also viewed as a function from the set of conjugacy classes in G to $\mathbb{R}$. One can also think of ℓ_ρ as the "translation length" function on G. Namely, let $X = (\widetilde{M}, d_\rho)$, where d_ρ is the distance function on $\widetilde{M}$ corresponding to the lift $\widetilde{\rho}$ of ρ to $\widetilde{M}$. Then the natural action of G on $\widetilde{M}$ by covering transformation is an action of G by isometries on X (and thus can be thought of as a representation $\Gamma \to Isom(X)$) and for every $\gamma \in G$ we have

$$\ell_\rho(\gamma) = \inf_{x \in X} d_\rho(x, \gamma x) = \min_{x \in X} d_\rho(x, \gamma x)$$

is the *translation length* of γ as the isometry of X. The *Marked Length Spectrum Rigidity Conjecture* (MLSRC) states that knowing the function ℓ_ρ uniquely determines the isometry type of (M, ρ). More precisely, if ρ, ρ' are two smooth negatively curved Riemannian metrics on M such that $\ell_\rho = \ell\rho'$ then there exists an isometry from (M, ρ) to (M, ρ') which is isotopic to the identity. There are also various generalizations of MLSRC to other contexts, such as allowing more general types of metrics on M. There are also generalizations with the set-up where, given two representations $\tau G \to Isom(X_1)$ and $\tau_2 : G \to Isom(X_2)$ with the same marked length spectrum $\ell_{\tau_1} = \ell_{\tau_2} : G \to \mathbb{R}$ (where X_1 and X_2 are required to satisfy various kinds of negative or non-positve curvature conditions). Then the desired conclusion of MLSRC is that there exists an isometry $X_1 \to X_2$ conjugating ρ_1 to ρ_2. MLSRC is known to hold for surfaces, including various generalizations of the types of metrics ρ included under consideration, see [17, 51, 20]. There are also a number of known results establishing versions of MLSRC for representations $\rho_1 : G \to Isom(X_1)$, $\rho_2 : G \to Isom(X_2)$ where X_1, X_2 are allowed to be higher dimensional, but there are more significant restrictions on the geometry of X_1, X_2 that in the surface case results (see, for example, [19, 33, 47, 48, 25, 49]). However, the original version of MLSRC is still mostly open (except for rather special classes of metrics) in dimensions bigger than two. We refer the reader to the survey [18] for a more extended discussion on the topic.

In any context where MLSRC is known to hold, it is natural to ask if there are smaller subsets of G such that knowing the restriction of the marked length spectrum to such a subset uniquely determines the entire marked length spectrum. Namely, for a given class of length functions $\ell : G \to \mathbb{R}$ where MLSRC is known to hold, say that a subset $S \subseteq G$ is *spectrally rigid* if whenever ℓ, ℓ' are two length functions from the class in question such that $\ell|_S = \ell'|_S$ then $\ell = \ell'$. For closed surfaces with metrics of constant curvature -1 the situation is particularly well-behaved. Thus it is known (see, for example, [27]) that if Σ is a closed oriented surface of genus ≥ 2, then there exist elements $h_1, \dots, h_{6g-5} \in G = \pi_1(\Sigma)$ such that whenever ρ_1, ρ_2 are two points in the Teichmuller space $\mathcal{T}(\Sigma)$ (i.e. marked hyperbolic metrics on Σ) such that $\ell_{\rho_1}(h_i) = \ell_{\rho_2}(h_i)$ for $i = 1, \dots, 6g-5$ then $\ell_{\rho_1} = \ell_{\rho_2}$ and $\rho_1 = \rho_2$ in $\mathcal{T}(\Sigma)$. Thus the subset $\{h_1, \dots, h_{6g-5}\} \subseteq G$ "spectrally rigid" with respect to the class of hyperbolic metrics on Σ. We believe that looking for "small" spectrally rigid sets in other situations, where MLSRC is known to hold, is an interesting general problem representing the next level in the study of length spectrum rigidity.

If G is a finitely generated group acting by isometries on an $\mathbb{R}$-tree X, there is also a naturally associated *translation length function* $||.||_X = \ell_X : G \to \mathbb{R}$, where for $g \in G$

$$\ell_X(g) = \inf_{x\in X} d_X(x, gx) = \min_{x\in X} d_X(x, gx).$$

It is well-known [22, 52, 10] that under some mild extra assumptions (which are satisfied, in particular, for the Outer space context discussed below), MLSRC holds, that is, knowing the function ℓ_X uniquely determines X and the action of G on X (up to a G-equivariant isometry).

For a free group F_N (where $N \geq 2$) the *Culler-Vogtmann Outer space* cv_N is an analog of the Teichmuller space of a hyperbolic surface. The space cv_N consists of minimal free discrete isometric actions of F_N on $\mathbb{R}$-trees, considered up to F_N-equivariant isometry. Every element $T \in \mathrm{cv}_N$ arises as the universal cover of a finite graph Γ, whose fundamental group is identified with F_N via a particular isomorphism, where edges of Γ are given positive real lengths and their lifts to T are given the same lengths. There is an important subset, $\mathrm{CV}_N \subseteq \mathrm{cv}_N$, consisting of all $T \in \mathrm{cv}_N$ such that the quotient metric graph T/F_N has volume 1. The space CV_N is the *projectivized Culler-Vogtmann Outer space*, which in fact was introduced first, in [23]. Both cv_N and CV_N play an important role in the study of $Out(F_N)$. We say that a subset $R \subseteq F_N$ is *spectrally rigid* if whenever $T_1, T_2 \in \mathrm{cv}_N$ are such that $||g||_{T_1} = ||g||_{T_2}$ for every $g \in R$ then $T_1 = T_2$ in cv_N. As noted above, $R = F_N$ is spectrally rigid and in fact it is enough to make R consist of representatives of all conjugacy classes in F_N. A surprising result of Smillie and Vogtmann [53] shows that there does not exist a finite spectrally rigid subsets of F_N, where $N \geq 3$. A result of Cohen, Lustig and Steiner [13] establishes the same fact for $N = 2$. In particular, it is proved in [53] that for any finite subset $R \subseteq F_N$, where $N \geq 3$, there exists a one-parametric family $(T_t)_{t\in[0,1]}$ of distinct points of CV_N such that for every $t \in [0, 1]$ the length functions $||.||_{T_0}$ and $||.||_{T_t}$ agree on R. A similar statement follows from the recent work of Duchin, Leininger, and Rafi for flat metrics on surfaces of finite type [26]. Moreover, the paper [26] gives a complete characterization of when a set of simple closed curves on a finite type surface is spectrally rigid with respect to the space of flat metrics on that surface.

In view of the results of [13, 53] it becomes interesting to look for infinite but "sparse" spectrally rigid subsets of F_N. The study of this topic was initiated by Kapovich in [38]. Namely, it is proved in [38] that if $A = \{a_1, \dots, a_N\}$ is a free basis of F_N then almost every trajectory of the simple non-backtracking random walk on F_N with respect to A yields a strongly spectrally rigid subset of F_N. In the present paper we obtain a very different class of examples of strongly spectrally rigid subsets of free groups. Our first main result is:

Theorem A. *Let $N \geq 3$ and let $H \leq Aut(F_N)$ be a subgroup whose image in $Out(F_N)$ is an infinite normal subgroup of $Out(F_N)$. Then for every $g \in F_N, g \neq 1$ the orbit $Hg = \{\varphi(g) : \varphi \in H\}$ is a strongly spectrally rigid subset of F_N.*

Theorem A applies, for example, to the cases where $H = \mathrm{Aut}(F_N)$ or where $H \leq \mathrm{Aut}(F_N)$ is the *Torelli subgroup*, that is, H is the set of all elements of $\mathrm{Aut}(F_N)$ that act as the identity map on the abelianization $\mathbb{Z}^N$ of F_N. Theorem A also immediately implies that any $\mathrm{Aut}(F_N)$-invariant subset of F_N with more than one element is strongly spectrally rigid in F_N.

The statement of Theorem A fails for $N = 2$. Indeed, for $F_2 = F(a, b)$ it is well-known that for every $\varphi \in \mathrm{Aut}(F_2)$ the element $\varphi([a, b])$ is conjugate to $[a, b]^{\pm 1}$ in F_2, which easily implies that $\mathrm{Aut}(F_2)[a, b]$ is not spectrally rigid in F_2. However, it turns out that this example is essentially the only obstruction for the conclusion of Theorem A to hold for $N = 2$. Thus we obtain the following companion statement for Theorem A:

Theorem B. *Let $F_2 = F(a,b)$. Let $H \leq Aut(F_N)$ be a subgroup whose image in $Out(F_N)$ is an infinite normal subgroup of $Out(F_N)$. Let $g \in F(a,b)$, $g \neq 1$ be such that g is not conjugate to a nonzero power of $[a,b]$ in F_2.*

Then Hg is strongly spectrally rigid in F_2.

It is well-known [55, 21] that every finite subgroup of $\mathrm{Out}(F_N)$ comes from a group of simplicial automorphisms of a finite connected graph without degree-one vertices and with fundamental group F_N. From here one can show that for $N \geq 3$ a normal subgroup $H \leq \mathrm{Out}(F_N)$ is infinite if and only if H is nontrivial. For the case $N = 2$ $\mathrm{Out}(F_2)$ does possess a finite nontrivial normal subgroup, namely the center of $\mathrm{Out}(F_2)$ which is the cyclic group of order two generated by a "hyper-elliptic involution" $a \mapsto a^{-1}, b \mapsto b^{-1}$.

The first step in the proof of Theorem A and of Theorem B is to establish Theorem 3.4 which says that the set $\mathcal{P}_N$ of all the primitive elements in F_N is a spectrally rigid subset in F_N for every $N \geq 2$. Recall that an element $g \in F_N$ is *primitive* if it belongs to some free basis of F_N. Thus $\mathcal{P}_N = \mathrm{Aut}(F_N)g$ where $g \in F_N$ is any primitive element. Theorem 3.4 is derived from the results of Francaviglia and Martino [29] about extremal Lipschitz distortions between two arbitrary points in cv_N. A key fact there is that for any $T, T' \in \mathrm{cv}_N$ the "extremal Lipschitz distortion" $D(T,T') := \sup_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T}$ is actually a maxiumum which is realized by an element g from some finite subset $\mathbf{U}_T \subseteq F_N$ depending only on T. Moreover, the explicit description of elements of $\mathbf{U}_T$ in [29] shows that they are all primitive, so that $\mathbf{U}_T \subseteq \mathcal{P}_N$. From here it is easy to see that if $||g||_T = ||g||_{T'}$ for every $g \in \mathcal{P}_N$ then $D(T,T') = D(T',T) = 1$ and hence $T = T'$ in cv_N. A more careful version of the above argument yields the following "relative rigidity" result:

Theorem C. *Let $T \in cv_N$ be arbitrary. There exists a finite set S (depending on T) of primitive elements in F_N with the following property: Whenever $T' \in cv_N$ is such that $||g||_{T'} = ||g||_T$ for every $g \in S$ then $T = T'$ in cv_N.*

In fact, the proof of Theorem C shows that we can take $S = \mathbf{U}_T$.

After Theorem 3.4 is established, we derive Theorem A and Theorem B from Theorem 3.4 using the machinery of *geodesic currents* on free groups, and particularly exploiting the *geometric intersection form* between trees and currents, constructed in [36, 40]. A geodesic current is a measure-theoretic analog of the notion of the conjugacy class in a (word-hyperbolic) group. Geodesic currents were introduced and studied by Bonahon [6, 7] in the context of hyperbolic surfaces and the Teichmuller space, where they turned out to be quite useful. A *geodesic current* on a free group F_N is a positive Radon measure μ on $\partial^2 F_N = \partial F_N \times \partial F_N - diag$ which is "flip"-invariant and F_N-invariant. The space $\mathrm{Curr}(F_N)$ of all geodesic currents on F_N comes equipped with a natural weak-* topology making it into a locally compact space, and with a natural left $\mathrm{Out}(F_N)$-action by linear transformations. The theory of geodesic currents on free groups has been actively developed in the last several years by Kapovich [35, 36, 37, 38] and Kapovich-Lustig [39, 40, 41, 42] (see also [5, 11, 32, 28, 44, 45, 46] for other recent applications of currents). The space $\mathrm{Curr}(F_N)$ turns out to be a natural counterpart for the Outer space cv_N, and, more generally, the closure $\overline{\mathrm{cv}}_N$ of cv_N. The closure $\overline{\mathrm{cv}}_N$ of cv_N (with respect to equivariant Gromov-Hausdorff convergence topology) is known to consist of all the minimal *very small* isometric actions of F_N on $\mathbb{R}$-trees. The Outer space cv_N is an open $\mathrm{Out}(F_N)$-invariant dense subset of $\overline{\mathrm{cv}}_N$. It is again well-known that any point $T \in \overline{\mathrm{cv}}_N$ is uniquely determined by its translation length function $||.||_T : F_N \to \mathbb{R}$.

The interaction between $\overline{\text{cv}}_N$ and $\text{Curr}(F_N)$ is given by the *geometric intersection form*

$$\langle\cdot,\ \cdot\rangle : \overline{\text{cv}}_N \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0},$$

constructed in [40]. The intersection form has a number of useful properties, such as being continuous, $Out(F_N)$-invariant, $\mathbb{R}_{\geq 0}$-linear with respect to the second argument and $\mathbb{R}_{\geq 0}$-linear with respect to the first argument. Another key property of the intersection form, relating it to marked length spectra, is that for every $T \in \overline{\text{cv}}_N$ and every $g \in F_N, g \neq 1$ we have

$$\langle T, \eta_g\rangle = ||g||_T$$

where $\eta_g \in \text{Curr}(F_N)$ is the so-called "counting" current defined by g. This last property is crucial in deriving Theorem A and Theorem B from Theorem 3.4. The proof in [38] that a random trajectory of the simple non-backtracking random walk on F_N yields a spectrally rigid subset of F_N, was also based on using geodesic currents and the geometric intersection form. A key fact in that proof was that the $\mathbb{R}_{\geq 0}$-linear span of the set of counting currents η_{w_n}, where $(w_n)_{n\geq 1}$ was a random trajectory of the walk, is dense in $\text{Curr}(F_N)$. A similar line of reasoning cannot be used to prove Theorem A. Indeed, if A is a free basis of F_N and v is a freely reduced word over A, such that every freely reduced word of length 2 over A occurs in v as a subword, then v cannot be a subword of a cyclic word representing a primitive element in F_N. This implies that v has "weight 0" (see [36] for the relevant terminology) in any finite linear combination of counting currents of primitive elements. Hence, by continuity, v has "weight 0" in every current from the closure Z of the linear span of the counting currents of all primitive elements, and hence Z is a proper subset of $\text{Curr}(F_N)$.

In Section 7 we discuss a number of open problems motivated by the results of this paper.

We are grateful to Martin Lustig for useful conversations.

2. Preliminaries

2.1. Graphs and graph-related conventions.

Convention 2.1 (Graphs)**.** A *graph* is a 1-complex. The set of 0-cells of a graph Δ is denoted $V\Delta$ and its elements are called *vertices* of Δ. The closed 1-cells of a graph Δ are called *topological edges* of Δ. The set of all topological edges is denoted $E_{top}\Delta$.

The interior of every topological edge is homeomorphic to the interval $(0, 1) \subseteq \mathbb{R}$ and thus admits exactly 2 orientations (when considered as a 1-manifold). We call a topological edge endowed with the choice of an orientation on its interior an *oriented edge* of Δ. The set of all oriented edges of Δ is denoted $E\Delta$. For an oriented edge $e \in E\Delta$ changing its orientation to the opposite produces another oriented edge of Δ denoted e^{-1} and called the *inverse* of e. Thus $^{-1} : E\Delta \to E\Delta$ is a fixed-point-free involution.

For every oriented edge e of Δ there are naturally defined (and not necessarily distinct) vertices $o(e) \in V\Delta$, called the *origin* of e, and $t(e) \in E\Delta$, called the *terminus* of e, satisfying $o(e^{-1}) = t(e)$, $t(e^{-1}) = o(e)$.

An *orientation* on a graph Δ is a partition $E\Delta = E^+\Delta \sqcup E^-\Delta$, where for every $e \in E\Delta$ one of the edges e, e^{-1} belongs to $E^+\Delta$ and the other edge belongs to $E^-\Delta$.

Definition 2.2 (Paths)**.** A *simplicial path* or *edge-path* γ of *simplicial length* $n \geq 1$ in Δ is a sequence of oriented edges

$$\gamma = e_1, \dots, e_n$$

such that $t(e_i) = o(e_{i+1})$ for $i = 1, \dots, n-1$. We say that $o(\gamma) := o(e_1)$ is the *origin* of γ and that $t(\gamma) = t(e_n)$ is the *terminus* of γ. We also regard a $\gamma = v \in V\Delta$ as an simplicial path in Γ of simplicial length 0 with $o(\gamma) = t(\gamma) = v$. A simplicial path is called *reduced* if it does not contain a back-tracking, that is a path of the form ee^{-1}, where $e \in E\Delta$.

2.2. **Outer space.** The Culler-Vogtmann Outer space, introduced by Culler and Vogtmann in a seminal paper [23], is a free group analogue of the Teichmüller space of a closed surface of negative Euler characteristic. We briefly review some basics definitions and facts about the Outer space, and refer the reader to [23, 3, 8, 31, 54] for more detailed background information on the topic.

Definition 2.3 (Non-projectivized Outer Space)**.** Let F_N be a finitely generated free group of rank $N \geq 2$.

The *non-projectivized outer space* cv_N consists of all minimal free and discrete isometric actions of F_N on $\mathbb{R}$-trees. Two such trees T_1, T_2 are considered equal in cv_N if there exists an F_N-equivariant isometry between them. The space cv_N is endowed with the equivariant Gromov-Hausdorff convergence topology.

For $T \in \mathrm{cv}_N$ and $c > 0$ denote by $cT \in \mathrm{cv}_N$ the tree that coincides with T as a topological space and has the same F-action, but where the metric is multiplies by c.

It turns out that every $T \in \mathrm{cv}_N$ is uniquely determined by its *translation length function* $||.||_T : F_N \to \mathbb{R}$, where for every $g \in F_N$

$$||g||_T = \min_{x \in T} d_T(x, gx)$$

is the *translation length* of g.

A basic fact in the theory of Outer space states that:

Proposition 2.4. *Let $T_1, T_2 \in cv_N$ be such that $||.||_{T_1} = ||.||_{T_2}$, that is $||g||_{T_1} = ||g||_{T_2}$ for every $g \in F_N$. Then there exists an F_N-equivariant isometry between T_1 and T_2, so that $T_1 = T_2$ in cv_N.*

Note that $||g||_T = ||hgh^{-1}||_T$ for every $g, h \in F_N$. Thus $||.||_T$ can be thought of as a function on the set of conjugacy classes in F_N. The space cv_N comes equipped with a natural right $Out(F_N)$-action by homeomorphisms. At the length function level, if $\varphi \in Out(F_N)$, $T \in \mathrm{cv}_N$ and $g \in F_N$ we have

$$||g||_{T\varphi} = ||\varphi(g)||_T.$$

It is known that the equivariant Gromov-Hausdorff topology on cv_N coincides with the pointwise convergence topology at the level of length functions. Thus for $T_n, T \in \mathrm{cv}_N$ we have $\lim_{n\to\infty} T_n = T$ if and only if for every $g \in F$ we have $\lim_{n\to\infty} ||g||_{T_n} = ||g||_T$.

Definition 2.5 (Projectivized Outer Space)**.** Denote by CV_N the subset of cv_N consisting of all $T \in \mathrm{cv}_N$ such that the quotient graph T/F_N has volume 1.

The space CV_N is a closed $Out(F_N)$-invariant subset of cv_N and it is called the *projectivized Outer Space* of F_N.

It is known that CV_N is canonically homeomorphic to $\mathrm{cv}_N/\sim$, where $T_1 \sim T_2$ if there is $c > 0$ such that $T_2 = cT_1$ in cv_N. This fact justifies the term "projectivized Outer Space". For $T \in \mathrm{cv}_N$ we denote by $[T]$ the $\sim$-equivalence class of T and call $[T]$ *the projective class of* T.

Points of cv_N have a more explicit combinatorial description as "marked metric graph structures" on F:

Definition 2.6 (Metric graph)**.** A *metric graph* is a graph Δ endowed with a *metric structure* $\mathcal{L}$, that is, a function $\mathcal{L} : E\Delta \to (0, \infty)$ such that for every $e \in E\Delta$ we have $\mathcal{L}(e) = \mathcal{L}(e^{-1})$. The number $\mathcal{L}(e)$ is called the *length* of e with respect to $\mathcal{L}$.

For a metric graph $(\Delta, \mathcal{L})$ its *volume* is defined as

$$vol_{\mathcal{L}}(\Delta) = \frac{1}{2} \sum_{e \in E\Delta} \mathcal{L}(e) = \sum_{e \in E^+\Delta} \mathcal{L}(e),$$

where $E\Delta = E^+\Delta \sqcup E^-\Delta$ is any orientation on Δ.

Definition 2.7 (Marking)**.** Let F_N be a free group of finite rank $N \geq 2$. A *marking* or a *simplicial chart* on F_N is an isomorphism $\alpha : F_N \to \pi_1(\Gamma)$ where Γ is a finite connected graph with the first Betti number equal to N and such that Γ has no degree-1 and no degree-2 vertices.

Definition 2.8 (Marked metric graph)**.** A *marked metric graph* or a *marked metric graph structure* on F_N consists of a marking $\alpha : F_N \to \pi_1(\Gamma)$ on F_N together with a metric graph structure $\mathcal{L}$ on Γ.

Convention 2.9. Let $(\alpha, \mathcal{L})$ be a marked metric graph structure on F_N, where $\alpha : F_N \to \pi_1(\Gamma, p)$ is a marking and where $\mathcal{L}$ is a metric structure on Γ.

Then $(\alpha, \mathcal{L})$ defines a point $T \in \mathrm{cv}_N$ as follows. Topologically, let $T = \widetilde{\Gamma}$, with an action of F_N on T via α. We lift the metric structure $\mathcal{L}$ from Γ to T by giving every edge in T the same length as that of its projection in Γ. This makes T into an $\mathbb{R}$-tree equipped with a minimal free and discrete isometric action of F_N. (The assumption that Γ has no degree-1 vertices guarantees that the action of F_N on T is minimal). Thus $T \in \mathrm{cv}_N$ and in this situation we will sometimes use the notation $T = (\alpha, \mathcal{L}) \in \mathrm{cv}_N$. Note that $T/F = \Gamma$.

Moreover, it is not hard to see that every point of cv_N arises in this fashion and that cv_N is exactly the set of all those $T = (\alpha, \mathcal{L}) \in \mathrm{cv}_N$ where $(\alpha, \mathcal{L})$ is a marked metric graph structure on F_N with $vol_{\mathcal{L}}(\Gamma) = 1$.

For this reason we will also think of elements $T \in \mathrm{cv}_N$ as metric graphs, and the default assumption will be that every vertex of T has degree ≥ 3.

The following useful proposition is an immediate corollary of the definitions:

Proposition 2.10. *Let $T \in cv_N$ be realized by a marked metric graph structure $(\alpha : F_N \to \pi_1(\Gamma), \mathcal{L})$ on F_N, so that $T = (\widetilde{\Gamma}, d_{\mathcal{L}})$. Let $g \in F_N, g \neq 1$. Let γ_g be the unique immersed circuit in Γ obtained by reducing and cyclically reducing the edge-path $\alpha(g)$ in Γ.*

Then $||g||_T$ is equal to the $\mathcal{L}$-length of γ_g.

The outer space cv_N has a natural closure $\overline{\mathrm{cv}}_N$ with respect to the equivariant Gromov-Hausdorff convergence topology (or, equivalently, with respect to the length function topology). It is known [3, 12, 30] that $\overline{\mathrm{cv}}_N$ consists precisely of (the F_N-equivarinat isometry classes of) all the *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees. The $\mathrm{Out}(F_N)$-action on cv_N naturally extends to an action on $\overline{\mathrm{cv}}_N$ by homeomorphisms. Moreover, the projectivization $\overline{\mathrm{CV}}_N$ of $\overline{\mathrm{cv}}_N$ is compact and contains (a copy of) CV_N as an open dense $\mathrm{Out}(F_N)$-invariant subset. The space $\overline{\mathrm{CV}}_N$ is sometimes called the *Thurston compactification* of CV_N.

3. Extremal Lipschitz distortions and rigidity of the set of primitive elements

For $N \geq 2$ we denote by $\mathcal{P}_N$ the set of all primitive elements in F_N.

Notation 3.1. Let $T \in \text{cv}_N$ and $T' \in \overline{\text{cv}}_N$. Denote

$$D(T,T') := \sup_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T}. \tag{‡}$$

Notation 3.2 (Almost simple curves)**.** Let $T \in \text{cv}_N$ and let $\Gamma = T/F_N$ be the quotient metric graph, with the metric structure $\mathcal{L}$ coming from T. Thus F_N is naturally identified with $\pi_1(\Gamma)$ via an isomorphism $\alpha : F_N \to \pi_1(\Gamma)$ and $T = (\alpha, \mathcal{L})$ in cv_N.

Let $\mathbf{U}_T \subseteq F_N$ be the set of all elements of F_N corresponding (under α) to the closed curves γ in Γ of one of the following types:

(1) γ is a nontrivial simple closed circuit in Γ;
(2) $\gamma = \gamma_1\gamma_2$ is a concatenation of two nontrivial simple closed circuits γ_1, γ_2, each beginning and ending at a common vertex v, and such that γ_1, γ_2 do not contain any common topological edges. We refer to such γ as *figure-eight curves* in T/F_N.
(3) γ is a "barbell" circuit, that is $\gamma = \gamma_1\beta\gamma_2\beta^{-1}$ where γ_i is a nontrivial simple closed circuit at a vertex v_i of Γ with $v_1 \neq v_2$, where β is a simple edge-path from v_1 to v_2 in Γ and where γ_1, β, β_2 do not have any common topological edges. We refer to such γ as *barbell curves* in T/F_N.

Note that, by construction, every element of $\mathbf{U}_T$ is primitive in F_N and the set $\mathbf{U}_T$ is finite. Moreover $\#\mathbf{U}_T \leq K(N)$ for some constant $K(N)$ depending only on N. We call elements of $\mathbf{U}_T$ *almost simple curves* for T/F_N.

We need the following fact established by Francaviglia and Martino in [29]:

Proposition 3.3. *Let $T, T' \in cv_N$ be arbitrary. Then*

$$D(T,T') = \max_{g \in \mathbf{U}_T} \frac{||g||_{T'}}{||g||_T}.$$

Thus we see that the supremum in the definition of $D(T,T')$ in (‡) is a maxiumum and it is achieved on one of elements from the finite subset $\mathbf{U}_T \subseteq \mathcal{P}_N$.

Proposition 3.3 quickly implies that the set of all primitive elements is spectrally rigid:

Theorem 3.4. *Let $N \geq 2$. Then the set $\mathcal{P}$ of all primitive elements in F_N is spectrally rigid in F_N.*

Proof. Let $T, T' \in \text{cv}_N$ be such that $||g||_T = ||g||_{T'}$ for every $g \in \mathcal{P}_N$.

Then

$$D(T,T') = \sup_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T} = \max_{g \in \mathbf{U}_T} \frac{||g||_{T'}}{||g||_T} = 1$$

since $\mathbf{U}_T \subseteq \mathcal{P}_N$. Similarly,

$$D(T',T) = \sup_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T} = \max_{g \in \mathbf{U}_{T'}} \frac{||g||_T}{||g||_{T'}} = 1$$

so that

$$\inf_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T} = \sup_{g \in F_N, g \neq 1} \frac{||g||_{T'}}{||g||_T} = 1.$$

Thus $||g||_T = ||g||_{T'}$ for every $g \in F_N$ and hence $T = T'$ in cv_N, as required. ∎

We now state a more precise (compared to Proposition 3.3) statement summarizing some of the results of [29]:

Proposition 3.5. *Let $T, T' \in cv_N$ be arbitrary. Let $L := D(T,T')$. Then there exists an F_N-equivariant L-Lipschitz map $f : T \to T'$ with the following properties:*

(1) *On each edge e of T the map f is a linear map with constant stretch $D_e \geq 0$ (note that, because of F_N-equivariance, $D_{e_1} = D_{e_2}$ whenever e_1 and e_2 are in the same F_N-orbit of edges of T).*

(2) *There exists $h \in \mathbf{U}_T$ such that for every edge e in the axis A_h of h in T we have $D_e = L$ and the restriction of f to A_h is injective (so that $f|_{A_h}$ is an L-homothety).*

We can now establish Theorem C from the Introduction:

Theorem 3.6. *Let $T, T' \in cv_N$ be such that $||g||_T = ||g||_{T'}$ for every $g \in \mathbf{U}_T$. Then $T' = T$ in cv_N.*

Proof. Let $T, T' \in \mathrm{cv}_N$ be such that $||g||_T = ||g||_{T'}$ for every $g \in \mathbf{U}_T$. Hence by Proposition 3.3 $D(T,T') = 1$. Let $f : T \to T'$ be an F_N-equivariant 1-Lipschitz map provided by Proposition 3.5. In particular, $D_e \leq 1$ for every edge e of T.

We claim that $D_e = 1$ for every edge e of T. Indeed, suppose not, so that there exists an edge e_0 of T with $D_{e_0} < 1$. Let y_0 be the edge of T/F_N which is the projection of e_0 to T/F_N. Then there exists an immersed circuit γ_0 in T/F_N passing through the edge y_0 such that γ_0 is an almost simple curve in F_N/T. Let $g_0 \in F_N$ correspond to γ_0, so that $g_0 \in \mathbf{U}_T$. The fact that $D_{e_0} < 1$ and that $D_e \leq 1$ for every edge e of T implies that $||g_0||_{T'} < ||g_0||_T$. Again, this contradicts our assumptions on T, T' and the fact that $g_0 \in \mathbf{U}_T$.

We now claim that f is injective. Indeed, suppose not. Then there exist two distinct oriented edges e_1, e_2 of T with a common initial vertex v such that f "folds" a non-degenerate initial segment of e_1 and a non-degenerate initial segment of e_2. Let y_1, y_2 be the edges of T/F_N which are the projections of e_1 and e_2 respectively to T/F_N. From the definition of an almost simple curve for T/F_N it follows that there exists an immersed circuit γ in T/F_N containing $y_1^{-1}y_2$ as a subpath such that γ is an almost simple curve in T/F_N. Thus γ corresponds to $g \in \mathbf{U}_T$. By F_N-equivariance of f we may assume that $e_1^{-1}e_2$ is a subpath of the axis of g in T. Since $D_e = 1$ for every edge e of T and since f folds nondegenerate initial segments of e_1 and e_2, it follows that $||g||_{T'} < ||g||_T$. This contradicts the fact that $g \in \mathbf{U}_T$ and that by assumption $||g||_T = ||g||_{T'}$. Hence f is injective as claimed.

Thus the map f is injective and is isometric on every edge of T. Therefore $f : T \to T'$ is an isometric embedding. Since the actions of F_N on T and T' are minimal, it follows that $f(T) = T'$, so that $f : T \to T'$ is an F_N-equivariant isometry. Hence $T = T'$ in cv_N, as required.

□

4. Geodesic currents

4.1. **Basic facts.** We will only state a few basic facts and definitions about currents, and refer the reader to [50, 35, 36, 37, 39, 40, 41] for more detailed background information regarding geodesic currents.

For the free group F_N define its "double boundary" $\partial^2 F_N$ as

$$\partial^2 F_N := \partial F_N \times \partial F_N - diag = \{(\xi, \zeta) \in \partial F_N \times \partial F_N : \xi \neq \zeta\}.$$

The space $\partial^2 F_N$ comes equipped with a natural topology, inherited from and $\partial F_N \times \partial F_N$ natural translation action of F_N by homeomorphisms. There is also a natural "flip" map $\partial^2 F_N \to \partial^2 F_N$, $(\xi, \zeta) \mapsto (\zeta, \xi)$, interchanging the two coordinates on $\partial^2 F_N$.

Recall that a *geodesic current* on F_N is a positive Radon measure μ on $\partial^2 F_N$ which is F_N-invariant and flip-invariant. Here the "flip" map $\partial^2 F_N \to \partial^2 F_N$, $(X,Y) \mapsto (Y,X)$ interchanges the two coordinates of $\partial^2 F_N$. The space $\mathrm{Curr}(F_N)$ of all geodesic currents on F_N comes equipped with a natural weak-* topology and a natural left $\mathrm{Aut}(F_N)$-action by $\mathbb{R}_{\geq 0}$-linear homeomorphisms. The group of inner automorphisms of F_N is contained in the kernel of this action, and hence the $\mathrm{Aut}(F_N)$ action naturally factors through to the action of $\mathrm{Out}(F_N)$ on $\mathrm{Curr}(F_N)$.

Every nontrivial element $g \in F_N$ defines a *counting* current $\eta_g \in \mathrm{Curr}(F_N)$, which turns out to depend only on the conjugacy class $[g]$ of g in F_N. Although the explicit definition of η_g is not directly relevant for this paper, we briefly recall one of the equivalent definitions of η_g here. Suppose first that $g \in F_N$ is a nontrivial element which is not a proper power in F_N. There are two well-defined distinct "poles" $g^\infty, g^{-\infty} \in \partial F_N$ where

$$g^\infty = \lim_{n\to\infty} g^n, \quad g^{-\infty} = \lim_{n\to\infty} g^{-n}$$

where the convergence is understood in the sense of the standard hyperbolic compactification $F_N \cup \partial F_N$ of F_N. Then $(g^{-\infty}, g^\infty) \in \partial^2 F_N$. Let $[g]$ denote the conjugacy class of g in F_N. Then

$$\eta_g := \sum_{h \in [g] \cup [g^{-1}]} \delta_{(h^{-\infty}, h^\infty)}.$$

Now if $g \in F_N$ is an arbitrary nontrivial element, g can be uniquely written as $g = g_0^m$ where $m \geq 1$ and $g_0 \in F_N$ is not a proper power. Then η_g is defined as $\eta_g := m\eta_{g_0}$. We summarize the following basic facts about counting currents (see [36]):

Proposition 4.1. *Let $N \geq 2$ and let $g \in F_N$, $g \neq 1$. Then*

(1) *We have $\eta_g = \eta_{g^{-1}}$.*
(2) *For every $n \in \mathbb{Z}$, $n \neq 1$, we have $\eta_{g^n} = n\eta_g$.*
(3) *For every $h \in [g]$ we have $\eta_g = \eta_h$. Thus η_g depends only on the conjugacy class of g, so we also use the notation $\eta_{[g]} := \eta_g$.*
(4) *For every $\varphi \in Out(F_N)$ we have $\varphi\eta_{[g]} = \eta_{\varphi([g])}$.*

The scalar multiples of counting currents are called *rational* currents. An important basic fact states that the set $\mathcal{R}_N := \{r\eta_g | r \geq 0, g \neq 1, g \in F_N\}$ of all rational currents is dense in $\mathrm{Curr}(F_N)$. The space $\mathrm{Curr}(F_N)$ has a natural projectivization $\mathbb{P}Curr(F_N)$ consisting of all equivalence classes $[\mu]$ where $\mu \in Curr(F_N)$, $\mu \neq 0$. Here two currents μ_1, μ_2 are equivalent if there exists $r > 0$ such that $\mu_2 = r\mu_1$. The equivalence class $[\mu]$ is also called the *projective class* of μ. The space $\mathbb{P}Curr(F_N)$ is compact and it inherits a left action of $\mathrm{Out}(F_N)$ by homeomorphisms. A basic fact about geodesic currents states (see [36]):

Proposition 4.2. *Let $N \geq 2$ and let $g, h \in F_N$ be nontrivial elements. Then $[\eta_g] = [\eta_h]$ in $\mathbb{P}Curr(F_N)$ if and only if there exist $u \in F_N$, $m, n \in \mathbb{Z}$ such that $[g] = [u^m]$ and $[h] = [u^n]$.*

4.2. **The geometric intersection form.** A key object connecting the Outer space and the space of geodesic currents is the so-called *geometric intersection form*, constructed in [40]:

Proposition 4.3. *Let $N \geq 2$. There exists a unique continuous map*

$$\langle \cdot, \cdot \rangle : \overline{cv}_N \times Curr(F_N) \to \mathbb{R}_{\geq 0}$$

with the following properties:

(1) $\langle T, c_1\mu_1 + c_2\mu_2\rangle = c_1\langle T, \mu_1\rangle + c_2\langle T, \mu_2\rangle$ *for any* $T \in \overline{cv}_N$, $\mu_1, \mu_2 \in Curr(F_N)$, $c_1, c_2 \geq 0$.
(2) $\langle cT, \mu\rangle = c\langle T, \mu\rangle$ *for any* $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$ *and* $c \geq 0$.
(3) $\langle \varphi T, \varphi\mu\rangle = \langle T, \mu\rangle$ *for any* $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$ *and* $\varphi \in Out(F_N)$.
(4) $\langle T, \eta_g\rangle = ||g||_T$ *for every* $T \in \overline{cv}_N$ *and* $g \in F_N, g \neq 1$.

The value $\langle T, \mu\rangle$ is called the *geometric intersection number* of $T \in \overline{\text{cv}}_N$ and $\mu \in Curr(F_N)$.

4.3. **The minimal set.**

Definition 4.4 (Minimal set $\mathbb{M}_N$)**.** Let $N \geq 2$. Denote by $\mathbb{M}_N$ the closure in $\mathbb{P}Curr(F_N)$ of the set

$$\{[\eta_g] : g \in F_N \text{ is primitive.}\}$$

It is easy to see that $\mathbb{M}_N \subseteq \mathbb{P}Curr(F_N)$ is a closed Out(F_N)-equivariant subset. It turns out [39] that for $N \geq 3$ this is the minimal such subset:

Proposition 4.5. *Let* $N \geq 2$. *Then:*
(1) [50] *For every element* $[\mu] \in \mathbb{M}_N$ *the subset Out*$(F_N)[\mu]$ *is dense in* $\mathbb{M}_N$.
(2) [39] *Let* $N \geq 3$. *Then* $\mathbb{M}_N \subseteq \mathbb{P}Curr(F_N)$ *is the unique minimal closed Out*(F_N)*-equivariant nonempty subset. This means that whenever* $Z \subseteq \mathbb{P}Curr(F_N)$ *is a closed Out*(F_N)*-equivariant nonempty subset then* $\mathbb{M}_N \subseteq Z$.

For $N \geq 3$ part (1) of Proposition 4.5 follows directly from part (2). For $N = 2$ part (1) of Proposition 4.5 follows from the results of Reiner Martin [50] who showed that $\mathbb{M}_2$ is homeomorphic to the circle and that the action of Out$(F_2) = GL(2, \mathbb{Z})$ on $\mathbb{M}_2$ can be identified with the standard action of $GL(2, \mathbb{Z})$ on $\mathbb{S}^1 = \partial H^2$.

The conclusion of part (2) of Proposition 4.5 is false for $N = 2$. Indeed, for $F_2 = F(a, b)$, it is easy to see that η_g for $g = [a, b]$ is a fixed point for the action of Out(F_2) on Curr(F_2) and hence $[\eta_g]$ is fixed by the action of Out(F_2) on $\mathbb{P}Curr(F_2)$. However, we will see later that a weaker version of Proposition 4.5 is true for $N = 2$ and that version will be sufficient for our purposes.

4.4. **Stable and unstable currents.** Let $N \geq 2$. An element $\varphi \in$ Out(F_N) is called *fully irreducible* or *iwip* (for "irreducible with irreducible powers") if there is no integer $n \geq 1$ such that φ^n preserves the conjugacy class of a proper free factor of F_N.

For an element $\varphi \in$ Out(F_N) the conjugacy class $[g]$, where $g \in F_N$, $g \neq 1$, is *periodic* if there exists $n \geq 1$ such that $\varphi^n[g] = [g]$. An element $\varphi \in$ Out(F_N) is called *atoroidal* if φ does not have any periodic conjugacy classes.

It is well-known that all non-atoroidal iwips in Out(F_N) come from homeomorphisms of compact surfaces with a single boundary component:

Proposition 4.6. [2] *Let* $N \geq 2$ *and* $\varphi \in Out(F_N)$ *be an iwip. Then the following hold:*
(1) *The automorphism* φ *is not atoroidal if and only if there exists an isomorphism* $\alpha : F_N \to \pi_1(\Sigma)$, *where* Σ *is a connected compact surface with exactly one boundary component, such that* φ *is induced by a homeomorphism of* Σ.
(2) *Let* $\varphi \in Out(F_N)$ *be a non-atoroidal iwip, let* Σ *be as in (1) and let* $[h]$ *be the conjugacy class in* F_N *given by the boundary of* Σ.

Then the only periodic conjugacy classes of φ *in* F_N *are those of the form* $[h^m]$, *where* $m \in \mathbb{Z}, m \neq 0$. *The conjugacy class* $[h]$ *is called the* peripheral curve *of* φ. *(Note that for an non-atoroidal iwip* φ *the peripheral curve* $[h]$ *is defined uniquely* *up to inversion. Namely, if $[g]$ is a periodic conjugacy class of φ such that $g \in F_N$ is not a proper power then $[g] = [h^{\pm 1}]$.)*

(3) *Let $N = 2$ and $F_2 = F(a, b)$. Let $u = [a, b] \in F_2$. Let $\varphi \in Out(F_2)$ be an iwip. Then φ is not atoroidal and $[u]$ is the peripheral curve of φ.*

For an element $\varphi \in \mathrm{Out}(F_N)$ a current $\mu \in \mathrm{Curr}(F_N)$ is called an *eigencurrent* of φ if $\mu \neq 0$ and $\varphi\mu = \lambda\mu$ for some $\lambda \geq 0$. In that case the number λ is called the associated *eigenvalue* of μ for φ. Thus for $\mu \in \mathrm{Curr}(F_N)$, $\mu \neq 0$ is an eigencurrent of φ if and only if $[\mu] \in \mathbb{P}\mathrm{Curr}(F_N)$ is a fixed point of φ. If $[\mu] \in \mathbb{P}\mathrm{Curr}(F_N)$ is a fixed point of φ, we also refer to the eigenvalue of μ for φ as the eigenvalue of $[\mu]$ for φ, and we sometimes refer to $[\mu]$ as the eignecurrent of φ.

Recall that $\mathbb{M}_N \subseteq \mathbb{P}\mathrm{Curr}(F_N)$ is a closed $\mathrm{Out}(F_N)$-invariant subset. As proved by Reiner Martin [50], if $\varphi \in \mathrm{Out}(F_N)$ is an iwip, then φ has the "North-South" dynamics on the minimal set $\mathbb{M}_N \subseteq \mathbb{P}\mathrm{Curr}(F_N)$ and, moreover, if φ is an atoroidal iwip, then φ has the "North-South" dynamics on $\mathbb{P}\mathrm{Curr}(F_N)$. We only need the following weak version of Martin's result:

Proposition 4.7 (Stable and unstable eigencurrents of iwips)**.** [50]
Let $N \geq 2$ and let $\varphi \in Out(F_N)$ be an iwip.
Then the following holds:

(1) *The element φ has exactly two distinct fixed points in $\mathbb{M}_N$. One of these fixed points, called the* stable eigencurrent *of φ, and denoted by $[\mu_+] = [\mu_+(\varphi)] \in \mathbb{M}_N$, has eigenvalue > 1 for φ, and the other fixed point, called the* unstable eigencurrent *of φ and denoted by $[\mu_-] = [\mu_-(\varphi)] \in \mathbb{M}_N$, has eigenvalue < 1 for φ. Thus $\varphi\mu_+ = \lambda_+\mu_+$ for $\lambda_+ > 1$ and $\varphi\mu_- = \frac{1}{\lambda_-}\mu_-$ for $\lambda_- > 1$.*

(2) *If φ is both atoroidal and fully irreducible then for every $g \in F_N$, $g \neq 1$ we have*
$$\lim_{n\to\infty} \varphi^n[\eta_g] = [\mu_+(\varphi)], \qquad \lim_{n\to\infty} \varphi^{-n}[\eta_g] = [\mu_-(\varphi)].$$

(3) *If φ is an iwip which is not atoroidal and if $[u]$ is the peripheral curve for φ then for every nontrivial $g \in F_N$ such that g is not conjugate to u^m, $m \in \mathbb{Z}$ we have*
$$\lim_{n\to\infty} \varphi^n[\eta_g] = [\mu_+(\varphi)], \qquad \lim_{n\to\infty} \varphi^{-n}[\eta_g] = [\mu_-(\varphi)].$$

(4) *For any $\theta \in Out(F_N)$ the element $\varphi' := \theta\varphi\theta^{-1}$ is again an iwip and $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)]$, $[\mu_-(\varphi')] = \theta[\mu_-(\varphi)]$.*

5. Consequences of spectral rigidity of the set of primitive elements

Recall that $\mathcal{P}_N$ denotes the set of all primitive elements in F_N.

A key tool in proving our main results is the following:

Proposition 5.1. *Let $N \geq 2$, let $H \leq Aut(F_N)$ and let $g \in F_N, g \neq 1$. Suppose that the closure of the set $H[\eta_g]$ in $\mathbb{P}Curr(F_N)$ contains the set $\mathbb{M}_N$.*

Then
$$Hg = \{\varphi(g) | \varphi \in H\} \subseteq F_N$$
is a spectrally rigid subset of F_N.

Proof. Let $T, T' \in \mathrm{cv}_N$ be such that $||\varphi(g)||_T = ||\varphi(g)||_{T'}$ for every $\varphi \in H$. We need to show that $T = T'$ in cv_N.

Let Z be the closure of the set $H[\eta_g]$ in $\mathbb{P}\mathrm{Curr}(F_N)$. By assumption we have $\mathcal{M}_N \subseteq Z$. Hence for every primitive element $a \in F_N$ there exists a sequence $\varphi_n \in H$ and a sequence $c_n \geq 0$ such that

$$\lim_{n\to\infty} c_n \varphi_n \eta_g = \lim_{n\to\infty} c_n \eta_{\varphi_n(g)} = \eta_a \qquad \text{in } Curr(F_N).$$

Proposition 4.3 then implies that

$$||a||_T = \langle T, \eta_a\rangle = \langle T, \lim_{n\to\infty} c_n \eta_{\varphi_n(g)}\rangle = \lim_{n\to\infty} c_n \langle T, \eta_{\varphi_n(g)}\rangle =$$
$$\lim_{n\to\infty} c_n ||\varphi_n(g)||_T = \lim_{n\to\infty} c_n ||\varphi_n(g)||_{T'} = \lim_{n\to\infty} c_n \langle T', \eta_{\varphi_n(g)}\rangle =$$
$$\langle T', \lim_{n\to\infty} c_n \eta_{\varphi_n(g)}\rangle = \langle T', \eta_a\rangle = ||a||_{T'}.$$

Thus $||a||_T = ||a||_{T'}$ for every primitive element $a \in F_N$. Theorem 3.4 now implies that $T = T'$ in cv_N. □

6. Spectrally rigid automorphic orbits

6.1. The case $N \geq 3$.

Lemma 6.1. *Let $N \geq 2$ and let $H \leq Out(F_N)$ be an infinite normal subgroup. Then H contains a fully irreducible element φ.*

Proof. By a result of Handel and Mosher [34] either H contains a fully irreducible element or there exist a subgroup H_1 of finite index in H and a proper free factor B of F_N such that every element of H_1 leaves the conjugacy class of B invariant.

Suppose that the latter case occurs, so that some subgroup H_1 of finite index in H preserves the conjugacy class of a proper free factor B of F_N.

It is well known (see, for example, [9]) that $\mathrm{Out}(F_N)$ is virtually torsion-free and hence there are no infinite torsion subgroups in $\mathrm{Out}(F_N)$. Thus if we show that every element of H_1 has finite order, this will imply that H_1 is finite, yielding a contradiction with the assumption that H is infinite and that H_1 has finite index in H.

Note that if $\theta \in \mathrm{Out}(F_N)$ is arbitrary, then $\theta H_1 \theta^{-1}$ leaves the conjugacy class $[\theta(B)]$ invariant. Since H is normal in $\mathrm{Out}(F_N)$, it follows that $\theta H_1 \theta^{-1}$ has finite index in $\theta H \theta^{-1} = H$ and hence $H_\theta := H_1 \cap \theta H_1 \theta^{-1}$ has finite index in H_1. Thus every element of H_1 has a positive power belonging to H_θ. Similarly, for any finite collection of elements $\theta_1, \dots, \theta_m \in \mathrm{Out}(F_N)$ the subgroup

$$H_{\theta_1,\dots,\theta_m} := H_1 \cap \theta_1 H_1 \theta_1^{-1} \cap \dots \cap \theta_m H_1 \theta_m^{-1}$$

has finite index in H_1. Moreover, every element of $H_{\theta_1,\dots,\theta_m}$ leaves invariant each of the conjugacy classes $[B], [\theta_1(B)], \dots [\theta_m(B)]$ and every element of H_1 has a positive power that belongs to $H_{\theta_1,\dots,\theta_m}$.

Let $\psi \in H_1$ be arbitrary. Choose a free basis $A = \{a_1, \dots, a_N\}$ of F_N such that for some $1 \leq k \leq N-1$ the set $\{a_1, \dots, a_k\}$ is a free basis of B. Let $\Psi \in \mathrm{Aut}(F_N)$ be a lift of ψ to $\mathrm{Out}(F_N)$. Since ψ preserves the conjugacy class of B, for every $n \in \mathbb{Z}$ the cyclically reduced form of $\Psi^n(a_1)$ over A does not involve $a_N^{\pm 1}$. For each $m = 2, \dots, N-1$ let θ_m be the automorphism of F_N induced by the permutation of A which interchanges a_m and a_N and leaves the other elements of A fixed. Thus $\theta_m(B)$ is generated by a subset of A that does not involve a_m. Let $n \geq 1$ be such that Ψ^n belongs to $H_{\theta_2,\dots,\theta_{N-1}}$. Then the image $\Psi^n(a_1)$ is a freely reduced word over A whose cyclically reduced form does not involve $a_2^{\pm 1}, \dots, a_N^{\pm 1}$, so that $\Psi(a_1)$ is conjugate to $a_1^{\pm 1}$ in F_N. By taking a larger finite collection of automorphisms θ of F_N we can find a positive power Ψ which sends every a_i to a conjugate of a_i in F_N, where $i = 1, \dots, N$.

Moreover, by considering the automorphisms θ of F_N corresponding to all the elementary Nielsen transformations on A, we can find an even bigger positive power Ψ^M of Ψ with the property that $\Psi^M(u)$ is conjugate to u in F_N for every freely reduced word u of length ≤ 2 over A. It is well known and easy to see that this implies that Ψ^M is an inner automorphism of F_N, so that $\psi^M = 1$ in $\mathrm{Out}(F_N)$. Thus we have shown that every element of H_1 has finite order in $\mathrm{Out}(F_N)$, which, since $\mathrm{Out}(F_N)$ is virtually torsion-free, implies that H_1 is finite. However, this contradicts the assumption that H is infinite and that H_1 has finite index in H.

Thus H contains a fully irreducible element, as required.

□

Lemma 6.2. *Let $N \ge 3$ and let $H \le Out(F_N)$ be an infinite normal subgroup. Then for every nontrivial element $g \in F_N$ there exists $\psi \in H$ such that $\psi(g)$ is not conjugate $g^{\pm 1}$ in F_N.*

Proof. Let $g \in F_N$, $g \ne 1$. Suppose that for every $\psi \in H$ $\psi(g)$ is conjugate to g or g^{-1} in F_N. Thus $H[\eta_g] = [\eta_g]$. Hence for every $\theta \in \mathrm{Out}(F_N)$ the subgroup $\theta H \theta^{-1}$ fixes $\theta[\eta_g] = [\eta_{\theta(g)}]$ in $\mathbb{P}\mathrm{Curr}(F_N)$. Since H is normal in $\mathrm{Out}(F_N)$, it follows that H fixes $\theta[\eta_g] = [\eta_{\theta(g)}]$ for every $\theta \in \mathrm{Out}(F_N)$.

Choose an atoroidal iwip $\psi \in \mathrm{Out}(F_N)$. Thus H fixes $\psi^n[\eta_g]$ for every $n \ge 1$. Hence, by Proposition 4.7, we have $H[\mu_+(\psi)] = [\mu_+(\psi)]$. However, by a result of [43], for an atoroidal iwip ψ the stabilizer of $[\mu_+(\psi)]$ in $\mathrm{Out}(F_N)$ is virtually cyclic and contains $\langle \psi \rangle$ as a subgroup of finite index. Hence H is virtually cyclic, which contradicts the assumption that H is normal in $\mathrm{Out}(F_N)$.

□

We are now ready to prove the main result (Theorem A from the Introduction) for the case $N \ge 3$:

Theorem 6.3. *Let $N \ge 3$. Let $H \le Aut(F_N)$ be a subgroup such that the image of H in $Out(F_N)$ is an infinite normal subgroup of $Out(F_N)$.*

Then for every $g \in F_N$, $g \ne 1$, the automorphic orbit

$$Hg \subseteq F_N$$

is a spectrally rigid subset of F_N.

Proof. Denote by $\overline{H} \le \mathrm{Out}(F_N)$ the image of H in $\mathrm{Out}(F_N)$. Thus $\overline{H}$ is an infinite normal subgroup of $\mathrm{Out}(F_N)$.

Let $g \in F_N$, $g \ne 1$ be arbitrary. Let Z be the closure of the set $H[\eta_g]$ in $\mathbb{P}\mathrm{Curr}(F_N)$.

Claim. We have $\mathbb{M}_N \subseteq Z$.

Lemma 6.1 implies that $\overline{H}$ contains some iwip element φ.

Case 1. Suppose first that the iwip φ is atoroidal.

By Proposition 4.7 we have

$$\lim_{n\to\infty} \varphi^n[\eta_g] = [\mu_+(\varphi)].$$

Therefore $[\mu_+(\varphi)] \in Z$. Recall also that $[\mu_+(\varphi)] \in \mathbb{M}_N$.

Let $\theta \in \mathrm{Out}(F_N)$ be arbitrary. Since $\overline{H}$ is normal in $\mathrm{Out}(F_N)$, it follows that $\theta\varphi\theta^{-1} \in \overline{H}$. The element $\varphi' = \theta\varphi\theta^{-1}$ is again an iwip. Hence the same argument as above implies that $\mu_+(\theta\varphi\theta^{-1}) \in Z$.

Since $\mu_+(\theta\varphi\theta^{-1}) = \theta[\mu_+(\varphi)]$, we see that $\theta[\mu_+(\varphi)] \in Z$ for every $\theta \in \mathrm{Out}(F_N)$. By Proposition 4.5 the subset $\mathrm{Out}(F_N)[\mu_+(\varphi)] \subseteq \mathbb{M}_N$ is dense in $\mathbb{M}_N$. Since Z is closed, this implies that $\mathbb{M}_N \subseteq Z$, as claimed.

Case 2. Suppose that the iwip φ is not atoroidal. Let $[u]$ be the peripheral curve of $[\varphi]$.

Since $Hg = H\Psi(g)$ for every $\Psi \in H$, Lemma 6.2 implies that, after possibly replacing g by $\Psi(g)$ for some $\Psi \in H$, we may assume that $[\eta_g] \neq [\eta_u]$. Then Proposition 4.7 implies that

$$\lim_{n\to\infty} \varphi^n[\eta_g] = [\mu_+(\varphi)].$$

Thus again we see that $[\mu_+(\varphi)] \in Z$.

Let $\theta \in \mathrm{Out}(F_N)$ be arbitrary and let $\varphi' = \theta\varphi\theta^{-1}$. Thus $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)]$ and $\theta(u)$ is the peripheral curve of φ'.

If $[\eta_g] \neq [\eta_{\theta(u)}]$ then again Proposition 4.7 implies that $\lim_{n\to\infty}(\varphi')^n[\eta_g] = [\mu_+(\varphi')] = \theta[\mu_+(\varphi)]$, so that $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)] \in Z$.

Suppose now that $[\eta_g] = [\eta_{\theta(u)}]$. Then Lemma 6.2 again implies that there is some $\psi \in \overline{H}$ such that $\psi[\eta_g] = [\eta_{\psi(g)}] \neq [\eta_{\theta(u)}]$. Note that $\overline{H}[g] = \overline{H}[\psi(g)]$ since $\psi \in \overline{H}$. Proposition 4.7 now implies that

$$\lim_{n\to\infty} (\varphi')^n[\eta_{\psi(g)}] = [\mu_+(\varphi')].$$

Thus again $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)] \in Z$.

We have shown that $\theta[\mu_+(\varphi)] \in Z$ for every $\theta \in \mathrm{Out}(F_N)$. Since $\mathrm{Out}(F_N)[\mu_+(\varphi)]$ is dense in $\mathbb{M}_N$ and the set Z is closed, it follows that $\mathcal{M}_N \subseteq Z$.

Thus the Claim is verified, so that $\mathbb{M}_N \subseteq Z$. Proposition 5.1 now implies that Hg is a spectrally rigid subset of F_N.

□

6.2. **The case $N = 2$.** Fix a free basis $A := \{a, b\}$ of F_2, so that $F_2 = F(a, b)$.

We need the following weaker version of Proposition 4.5 for the case $N = 2$:

Proposition 6.4. *Let $g \in F_2, g \neq 1$ be such that g is not conjugate to any integer power of $[a, b]$. Let $H \leq Out(F_2)$ be an infinite normal subgroup. Let Z be any closed H-invariant subset of $\mathbb{P}Curr(F_2)$ such that $[\eta_g] \in Z$. Then $\mathbb{M}_2 \subseteq Z$.*

Proof. Let Z be the closure of the set $H[\eta_g] = \{[\eta_{\varphi(g)}] : \varphi \in H\}$ in $\mathbb{P}Curr(F_2)$.

Lemma 6.1 implies that there exists an iwip element $\varphi \in H$. Recall that in this case $u = [a, b]$ is the peripheral curve for φ and, moreover, powers of $[a, b]$ are the only periodic conjugacy classes for φ in F_2.

Since by assumption g is not conjugate to any integer power of $[a, b]$, it follows from Proposition 4.7 that

$$\lim_{n\to\infty} \varphi^n[\eta_g] = \lim_{n\to\infty}[\eta_{\varphi^n(g)}] = [\mu_+(\varphi)].$$

Therefore $[\mu_+(\varphi)] \in Z$. For every $\theta \in \mathrm{Out}(F_2)$ $\varphi' = \theta\varphi\theta^{-1}$ is again an iwip with $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)]$. The subgroup H is normal in $\mathrm{Out}(F_N)$, and therefore $\varphi' \in H$. Now Proposition 4.7 again implies that $[\mu_+(\varphi')] \in Z$. Since $[\mu_+(\varphi')] = \theta[\mu_+(\varphi)]$, we have shown that $\theta[\mu_+(\varphi)] \in Z$ for every $\theta \in \mathrm{Out}(F_2)$. By Proposition 4.5, every $\mathrm{Out}(F_2)$-orbit of a point of $\mathbb{M}_2$ is dense in $\mathbb{M}_2$, and therefore $\mathbb{M}_2 \subseteq Z$, as required. □

Remark 6.5. Suppose g is conjugate in $F_2 = F(a, b)$ to $[a, b]^k$ for some $k \in \mathbb{Z}$. Then $\mathrm{Aut}(F_2)g \subseteq F_2$ is not spectrally rigid; hence any subset of $\mathrm{Aut}(F_2)g$ is not spectrally rigid.

Indeed, let T_A and T_B be the Cayley graphs of $F(a,b)$ with respect to $A=\{a,b\}$ and $B=\{a,ab\}$ accordingly. As usual, we give all edges of T_A, T_B length 1. Then $T_A, T_B \in \mathrm{cv}_2$ and $T_A \neq T_B$ in cv_2.

It is well-known that for any $\varphi \in \mathrm{Aut}(F(a,b))$, the element $\varphi([a,b])$ is conjugate to $[a,b]^{\pm 1}$ in $F(a,b)$. It follows that for any free bases A and B of $F_2 = F(a,b)$, and for any $\varphi \in \mathrm{Aut}(F_2)$ we have $||\varphi(g)||_A = ||\varphi(g)||_B = 4|k|$. The Cayley graphs T_A and T_B of F_2 with respect to A and B respectively are both points in cv_2. Thus we see that for every $\varphi \in \mathrm{Aut}(F_2)$ we have $||\varphi(g)||_{T_A} = ||\varphi(g)||_{T_B} = 4|k|$. Since we chose A and B so that $T_A \neq T_B$ in cv_2, this shows that the orbit $\mathrm{Aut}(F(a,b))g$ is not spectrally rigid in $F(a,b)$.

It turns out that Remark 6.5 provides the only obstruction to extending Theorem 6.3 to the case $N=2$, and we obtain Theorem B from the Introduction:

Theorem 6.6. *Let $F_2 = F(a,b)$ and let $g \in F_2, g \neq 1$ be such that g is not conjugate to a power of $[a,b]$ in $F(a,b)$.*

Let $H \leq Aut(F_2)$ be a subgroup such that its image in $Out(F_2)$ is an infinite normal subgroup. Then Hg is a spectrally rigid subset of $F(a,b)$.

Proof. Suppose that $g \in F(a,b)$, $g \neq 1$ is such that g is is not conjugate to a power of $[a,b]$ in $F(a,b)$. Let Z be the closure in $\mathbb{P}Curr(F_N)$ of the set $H[\eta_g]$. Proposition 6.4 implies that $\mathcal{M}_2 \subseteq Z$. Proposition 5.1 now implies that Hg is a spectrally rigid subset of F_2. □

7. Open problems

One can define a more restrictive notion of spectral rigidity than the one considered in this paper. Namely, call a subset $S \subseteq F_N$ *strongly spectrally rigid* if whenever $T, T' \in \overline{\mathrm{cv}}_N$ are such that $||g||_T = ||g||_{T'}$ for every $g \in S$ then $T = T'$ in $\overline{\mathrm{cv}}_N$.

Problem 7.1. For $N \geq 3$ is it true that every nontrivial $\mathrm{Aut}(F_N)$ orbit is strongly spectrally rigid in F_N? Is it true that the set $\mathcal{P}$ of all the primitive elements is strongly spectrally rigid in F_N?

In [38] it is proved that for the nonbacktracking simple random walk on F_N almost every trajectory of that walk gives a strongly spectrally rigid subset of F_N. However, the proofs of Theorem 6.3 and Theorem 6.6 in the present paper do not imply strong spectral rigidity, primarily because our proof of spectral rigidity of the set $\mathcal{P}_N$ in Theorem 3.4 only works for the interior points of the Outer space. Nevertheless, there is some positive evidence that the set $\mathcal{P}_N$ of all the primitive elements may indeed be strongly spectrally rigid in F_N. Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal iwip (or fully irreducible) element and let $T_\varphi \in \mathrm{cv}_N$ be the "stable tree" of an atoroidal fully irreducible $\varphi \in \mathrm{Out}(F_N)$ (in particular $T_\varphi \varphi = \lambda T_\varphi$, where $\lambda > 1$ is the Perron-Frobenius eigenvalue of a train-track representative of φ). We can prove that whenever $T' \in \mathrm{cv}_N$ is such that the lengths functions of T_φ and of T' agree on all primitive elements of F_N then $T' = T_\varphi$ in $\overline{\mathrm{cv}}_N$. Namely, in this case one can show that the Bestvina-Feighn-Handel "legal" lamination $L_{BFH}(\varphi)$ of φ is contained in the dual algebraic lamination $L^2(T')$ of T'. This implies, for instance by the results of [41], that $[T_\varphi] = [T']$ in $\overline{\mathrm{CV}}_N$, and it is then not hard to deduce that $T' = T_\varphi$. We refer the reader to [4, 41, 14, 15] for the background on dual algebraic laminations and on laminations associated to iwip automorphisms.

Problem 7.2. Does there exist a subset $S \subseteq F_N$ such that S is spectrally rigid but not strongly spectrally rigid?

Problem 7.3. Let $N \geq 2$ and let $S \subseteq \mathcal{P}_N$ be an arbitrary subset of the set $\mathcal{P}_N$ of all primitive elements in F_N.

Is it true that S is spectrally rigid in F_N if and only if the closure of $\{[\eta_g] : g \in S\}$ in $\mathbb{P}\mathrm{Curr}(F_N)$ is equal to $\mathbb{M}_N$?

It is easy to show, using the intersection form, that if the closure of $\{[\eta_g] : g \in S\}$ in $\mathbb{P}\mathrm{Curr}(F_N)$ is equal to $\mathbb{M}_N$ then S is spectrally rigid. However, the converse implication appears to be quite difficult. A recent result of Duchin, Leininger, and Rafi [26] establish a similar result to that suggested in Problem 7.3 in the context of singular flat metrics on surfaces.

We have seen in Theorem 6.3 that for $N \geq 3$ if $H \leq \mathrm{Aut}(F_N)$ is a subgroup that projects to an infinite normal subgroup of $\mathrm{Out}(F_N)$ then the $\mathrm{Aut}(F_N)$-orbit of every nontrivial element g of F_N is spectrally rigid. As noted in [38], one can combine the original Smillie-Vogtmann argument from [53] with some results about train tracks to show that if $\varphi \in \mathrm{Aut}(F_N)$ is a fully irreducible (also known as iwip) automorphism then for every $g \in F_N, g \neq 1$ the orbit $\langle\varphi\rangle g$ of the cyclic group $\langle\varphi\rangle \leq \mathrm{Aut}(F_N)$ is not spectrally rigid. This fact, together with Theorem 6.3, suggests the following:

Problem 7.4. Let $N \geq 3$ and let $H \leq \mathrm{Aut}(F_N)$ be an arbitrary subgroup. Is it true that either for every nontrivial $g \in F_N$ the orbit $Hg \subseteq F_N$ is spectrally rigid or that for every nontrivial $g \in F_N$ the orbit $Hg \subseteq F_N$ is not spectrally rigid?

A positive answer to the above question would mean that rigidity or non-rigidity of the orbit Hg (where $g \in F_N, g \neq 1$) depends only on the subgroup $H \leq \mathrm{Aut}(F_N)$ and not on the choice of a nontrivial element $g \in F_N$. We believe that in the case of cyclic subgroups of $\mathrm{Aut}(F_N)$ their orbits are never spectrally rigid:

Conjecture 7.5. *Let $N \geq 2$ and let $\varphi \in Aut(F_N)$ be arbitrary. Put $H = \langle\varphi\rangle \leq Aut(F_N)$. Then for every $g \in F_N$ the orbit $Hg \subseteq F_N$ is not spectrally rigid.*

Theorem C motivates the following question:

Problem 7.6 (Relatively strongly rigid finite sets)**.** Given $T \in \overline{\mathrm{cv}}_N$, does there exist a finite subset $S \subseteq \mathcal{P}$ such that whenever $T' \in \overline{\mathrm{cv}}_N$ is such that $||g||_T = ||g||_{T'}$ for all $g \in S$ then $T = T'$ in $\overline{\mathrm{cv}}_N$? What if we just require S to be a finite subset of F_N (and not necessarily of $\mathcal{P}$)?

As we noted in the introduction, by a result of Cohen-Lustig-Steiner [13], there does not exist a finite spectrally rigid subset of F_2. However, the argument in [13] involves looking at trees $T \in \mathrm{cv}_2$ with variable volume of the quotient metric graph T/F_2. On the other hand, the Smillie-Vogtmann construction [23] for $N \geq 3$ only uses trees with co-volume 1, that is, points of CV_N. In fact, for $N = 2$ the situation is quite different when restricting trees with quotient graphs of volume 1, that is, to CV_2. Elaborating the arguments from [24] we can show that for $F_2 = F(a,b)$ the set $S_0 := \{a, b, ab, ab^{-1}, [a,b]\}$ is "CV_2-rigid", that is, knowing the $||.||_T$-lengths, for an arbitrary $T \in \mathrm{CV}_2$, of elements of S_0, uniquely determines T. This naturally leads to the following question:

Problem 7.7. Does there exist a finite CV_2-rigid set of primitive elements in F_2?

References

[1] M. Bestvina and M. Feighn, *The topology at infinity of $Out(F_n)$*. Invent. Math. **140** (2000), no. 3, 651–692

[2] M. Bestvina, and M. Handel, *Train tracks and automorphisms of free groups.* Ann. of Math. (2) **135** (1992), no. 1, 1–51

[3] M. Bestvina and M. Feighn, *Outer Limits*, preprint, 1993; http://andromeda.rutgers.edu/˜feighn/papers/outer.pdf

[4] M. Bestvina, M. Feighn, and M. Handel, *Laminations, trees, and irreducible automorphisms of free groups.* Geom. Funct. Anal. **7** (1997), no. 2, 215–244

[5] M. Bestvina, M. Feighn, *A hyperbolic $Out(F_n)$ complex*, Groups Geom. Dyn. **4** (2010), no. 1, 31- 58

[6] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3. Ann. of Math. (2) **124** (1986), no. 1, 71–158

[7] F. Bonahon, *The geometry of Teichmüller space via geodesic currents.* Invent. Math. **92** (1988), no. 1, 139–162

[8] M. Bridson, and K. Vogtmann, *The symmetries of Outer space.* Duke Math. J. **106** (2001), no. 2, 391–409.

[9] M. Bridson, and K. Vogtmann, *Automorphism groups of free groups, surface groups and free abelian groups.* Problems on mapping class groups and related topics, 301Ð316, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., 2006

[10] I. Chiswell, *Introduction to* Λ*-trees*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001

[11] M. Clay and A. Pettet, *Currents twisting and nonsingular matrices*, to appear in Commentarii Mathematici Helvetici

[12] M. Cohen and M. Lustig, *Very small group actions on R-trees and Dehn twist automorphisms.* Topology **34** (1995), no. 3, 575–617

[13] M. Cohen, M. Lustig, and M. Steiner, *R-tree actions are not determined by the translation lengths of finitely many elements.* Arboreal group theory (Berkeley, CA, 1988), 183–187, Math. Sci. Res. Inst. Publ., 19, Springer, New York, 1991

[14] T. Coulbois, A. Hilion, and M. Lustig, *$\mathbb{R}$-trees and laminations for free groups I: Algebraic laminations,* J. Lond. Math. Soc. (2) **78** (2008), no. 3, 723–736

[15] T. Coulbois, A. Hilion, and M. Lustig, *$\mathbb{R}$-trees and laminations for free groups II: The dual lamination of an $\mathbb{R}$-tree,* J. Lond. Math. Soc. (2) **78** (2008), no. 3, 737–754

[16] T. Coulbois, A. Hilion, and M. Lustig, *$\mathbb{R}$-trees and laminations for free groups III: Currents and dual $\mathbb{R}$-tree metrics,* J. Lond. Math. Soc. (2) **78** (2008), no. 3, 755–766

[17] Ch. B. Croke. *Rigidity for surfaces of nonpositive curvature.* Comment. Math. Helv. **65** (1990), no. 1, 150–169

[18] Ch. B. Croke, *Rigidity theorems in Riemannian geometry.* Geometric methods in inverse problems and PDE control, 47–72, IMA Vol. Math. Appl., 137, Springer, New York, 2004.

[19] C. Croke, P. Eberlein, and B. Kleiner, *Conjugacy and rigidity for nonpositively curved manifolds of higher rank.* Topology **35** (1996), no. 2, 273–286.

[20] C. Croke, A. Fathi, and J. Feldman. *The marked length-spectrum of a surface of nonpositive curvature.* Topology, **31** (1992), no. 4, 847–855

[21] M. Culler, *Finite groups of outer automorphisms of a free group.* Contributions to group theory, 197 207, Contemp. Math., 33, Amer. Math. Soc., Providence, 1984

[22] M. Culler, and J. W. Morgan, *Group actions on $\mathbb{R}$-trees.* Proc. London Math. Soc. (3) **55** (1987), no. 3, 571Ð604

[23] M. Culler, K. Vogtmann, *Moduli of graphs and automorphisms of free groups.* Invent. Math. **84** (1986), no. 1, 91–119

[24] M. Culler, K. Vogtmann, *The boundary of outer space in rank two*, Arboreal group theory (Berkeley, CA, 1988), 189–230, Math. Sci. Res. Inst. Publ., 19, Springer, New York, 1991

[25] F. Dal'Bo, and I. Kim, *Marked length rigidity for symmetric spaces.* Comment. Math. Helv. **77** (2002), no. 2, 399–407

[26] M. Duchin, C. J. Leininger, and K. Rafi, *Length spectra and degeneration of flat metrics*, Invent. Math. **182** (2010), no. 2, pp. 231–277

[27] A. Fathi, F. Laudenbach, and V. Poenaru *Travaux de Thurston sur les surfaces.* Astérisque, **66-67**. Société Mathématique de France, Paris, 1979

[28] S. Francaviglia, *Geodesic currents and length compactness for automorphisms of free groups*, Trans. Amer. Math. Soc. **361** (2009), no. 1, 161–176.

[29] S. Francaviglia and A. Martino, *Metric Properties of Outer Space*, Publicacions Matemátiques **55** (2011), no. 2, 433–473

[30] V. Guirardel, *Approximations of stable actions on R-trees.* Comment. Math. Helv. **73** (1998), no. 1, 89–121

[31] V. Guirardel, *Dynamics of* $\mathrm{Out}(F_n)$ *on the boundary of outer space.* Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 4, 433–465

[32] U. Hamenstädt, *Lines of minima in Outer space*, preprint, November 2009; arXiv:0911.3620

[33] S. Hersonsky and F. Paulin. *On the rigidity of discrete isometry groups of negatively curved spaces.* Comment. Math. Helv. **72** (1997), vol. 3, pp 349–388

[34] M. Handel and L. Mosher, *Subgroup classification in* $Out(F_n)$, preprint, arXiv:0908.1255

[35] I. Kapovich, *The frequency space of a free group,* Internat. J. Alg. Comput. **15** (2005), no. 5-6, 939–969

[36] I. Kapovich, *Currents on free groups,* Topological and Asymptotic Aspects of Group Theory (R. Grigorchuk, M. Mihalik, M. Sapir and Z. Sunik, Editors), AMS Contemporary Mathematics Series, vol. 394, 2006, pp. 149-176

[37] I. Kapovich, *Clusters, currents and Whitehead's algorithm*, Experimental Mathematics **16** (2007), no. 1, pp. 67-76

[38] I. Kapovich, *Random length-spectrum rigidity for free groups*, Proceedings of AMS, to appear; arXiv:1001.1729

[39] I. Kapovich and M. Lustig, *The actions of* $Out(F_k)$ *on the boundary of outer space and on the space of currents: minimal sets and equivariant incompatibility.* Ergodic Theory Dynam. Systems **27** (2007), no. 3, 827–847

[40] I. Kapovich and M. Lustig, *Geometric Intersection Number and analogues of the Curve Complex for free groups*, Geometry & Topology **13** (2009), 1805–1833

[41] I. Kapovich and M. Lustig, *Intersection form, laminations and currents on free groups*, Geom. Funct. Analysis (GAFA), **19** (2010), no. 5, 1426–1467

[42] I. Kapovich and M. Lustig, *Domains of proper dicontinuity on the boundary of Outer space*, Illinois J. Math. **54** (2010), no. 1, pp. 89–108, special issue dedicated to Paul Schupp

[43] I. Kapovich and M. Lustig, *Stabilizers of* $\mathbb{R}$*-trees with free isometric actions of* F_N, J. Group Theory, to appear; ArXiv:0904.1881

[44] I. Kapovich and T. Nagnibeda, *The Patterson-Sullivan embedding and minimal volume entropy for Outer space*, Geom. Funct. Anal. (GAFA) **17** (2007), no. 4, 1201–1236

[45] I. Kapovich and T. Nagnibeda, *Geometric entropy of geodesic currents on free groups*, Dynamical Numbers: Interplay Between Dynamical Systems and Number Theory, Contemporary Mathematics series, American Mathematical Society, 2010, pp. 149-176

[46] I. Kapovich and T. Nagnibeda, *Generalized geodesic currents on free groups*, preprint, 2011; arXiv:1105.5742

[47] Inkang Kim, *Ergodic theory and rigidity on the symmetric space of non-compact type.* Ergodic Theory Dynam. Systems **21** (2001), no. 1, 93–114

[48] Inkang Kim, *Marked length rigidity of rank one symmetric spaces and their product,* Topology **40** (2001), no. 6, 1295–1323

[49] Inkang Kim, *Rigidity on symmetric spaces,* Topology **43** (2004), no. 2, 393Ð405.

[50] R. Martin, *Non-Uniquely Ergodic Foliations of Thin Type, Measured Currents and Automorphisms of Free Groups*, PhD Thesis, 1995

[51] J.-P. Otal. *Le spectre marque des longueurs des surfaces a courbure negative.* Ann. of Math.(2) **131**(1990), no. 1, 151–162

[52] F. Paulin, *The Gromov topology on R-trees.* Topology Appl. **32** (1989), no. 3, 197–221

[53] J. Smillie and K. Vogtmann, *Length functions and outer space*, Michigan Math. J. **39** (1992), no. 3, 485–49

[54] K. Vogtmann, *What is... outer space?* Notices Amer. Math. Soc. **55** (2008), no. 7, 784 786.

[55] B. Zimmermann, *On homeomorphisms of n-dimensional handlebodies and on finite extensions of Schottky groups*, Comment. Math. Helv. **56** (1981), no. 3, 474 -486

SST/IRMP, Chemin du Cyclotron 2, bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
E-mail address: mathieu.carette@uclouvain.be

Dipartimento di Matematica of the University of Bologna, Pizza Porta S. Donato 5, 40126 Bologna, Italy
E-mail address: francavi@dm.unibo.it

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu

School of Mathematics, University of Southampton; Highfield, Southampton, SO17 1BJ, United Kingdom
E-mail address: A.Martino@soton.ac.uk