Algebraic and geometric solutions of hyperbolicity equations

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Abstract

In this paper, we study the differences between algebraic and geometric solutions of hyperbolicity equations for ideally triangulated 3-manifolds, and their relations with the variety of representations of the fundamental group of such manifolds into $\mathrm{PSL}(2,\mathbb{C})$.

We show that the geometric solutions of compatibility equations form an open subset of the algebraic ones, and we prove uniqueness of the geometric solutions of hyperbolic Dehn filling equations.

In the last section we study some examples, doing explicit calculations for three interesting manifolds.

Key words: Ideal Triangulations, Hyperbolic 3-Manifolds, Representations

1 Introduction

One of the most useful tools for studying the hyperbolic structures on 3-manifolds is the technique of ideal triangulations, introduced by Thurston in [16] to study the hyperbolic structures of the complement of the figure-eight knot. An ideal triangulation of an open 3-manifold M is a description of M as a disjoint union of copies of the standard tetrahedron with vertices removed (ideal tetrahedron), glued together by a given set of face-pairing maps. Once one has an ideally triangulated manifold M, the idea is to construct a hyperbolic structure on M by defining it on each tetrahedron and then by requiring that such structures are compatible with a global one on M. A complete finite-volume hyperbolic structure with totally geodesic faces on an oriented ideal tetrahedron is described by a complex number, called its modulus. Then, the properties of hyperbolic structures of M (if any) translate to algebraic equations on the moduli. We refer the reader to [16], [12], [1], [13], [14] for more details on ideal triangulations and hyperbolicity equations.

The main question is to decide whether a solution of the hyperbolicity equations actually defines a structure on M. The main problems arise when flat or negatively oriented tetrahedra appear. In terms of moduli a tetrahedron is flat (resp. negatively oriented) if the imaginary part of its modulus is zero (negative). A solution is called *positive* (*resp. partially flat*) if the moduli have positive (non negative) imaginary part. In this paper, we introduce the notion of *geometric solution* to describe those choices of moduli defining a structure on M (see Sections 3 and 4 for definitions). The main known results on the matter are:

- (Thurston [16]) Any positive solution of the hyperbolicity equations is geometric.
- (Petronio and Weeks [14]) Any partially flat solution of compatibility and completeness equations satisfying an additional condition on the angles is geometric.
- (Epstein-Penner decomposition [4]) Any noncompact, complete hyperbolic 3-manifold of finite volume admit an ideal triangulation with a geometric partially flat solution of compatibility and completeness equations.
- (Petronio and Porti [13]) Any solution sufficiently close to the Epstein-Penner decomposition is geometric.
- There exist examples of cusped manifolds admitting Dehn fillings which are hyperbolic and such that no ideal triangulations are known having a positive solution of the hyperbolic Dehn filling equations. For example the $(3,1)$-filling of the $m007$ SnapPea manifold ([17]).

In this work, we study the space of algebraic solutions of the compatibility equations, showing that near nondegenerate solutions it has the local structure of a branched covering of the space of representations $\mathrm{Hom}(\pi_1(M), \mathrm{Isom}^+(\mathbb{H}^3))$. We prove that the set of geometric solutions of the compatibility equations is an open subset of the set of algebraic ones; we prove the uniqueness of geometric solutions of the hyperbolic Dehn filling equations; we give examples of non-geometric solutions of compatibility and completeness equations. The paper is structured as follows.

In Section 2 we recall the definition of *ideal triangulation* and *modulus of a tetrahedron*.

In Sections 3 and 4 we give the systems of the *compatibility* and *hyperbolic Dehn filling* equations, and we give the definition of *geometric solution* of such systems.

Sections 5 and 6 are devoted respectively to the study of algebraic and geometric solutions of the above systems.

In Section 7 we do explicit calculations for some interesting examples. Namely, first we study two one-cusped manifolds admitting non-unique algebraic solutions for the compatibility and completeness equations and a (unique) geometric one. Then we study a non-hyperbolic manifold admitting a partially flat solution of the compatibility and completeness equations.

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2 Ideal triangulations with moduli

We fix here the class of manifolds we consider, namely the class of ideally triangulated cusped 3-manifolds.

Definition 1 (Cusped manifold) *A* cusped manifold *is an orientable* 3-*manifold M diffeomorphic to the interior of a compact manifold $\overline{M}$ whose boundary $\partial\overline{M}$ consists of a union of tori. A* cusp *of M is a closed regular neighborhood of a component of $\partial\overline{M}$.*

We denote by $\widehat{M}$ the compactification of M obtained by adding one point for each cusp of M. If $\widetilde{M}$ is the universal covering of M, we denote by $\widehat{\widetilde{M}}$ the space obtained by adding to $\widetilde{M}$ one point for each lift of each cusp of M.

The points added to M (or $\widetilde{M}$) are called ideal points. *To each ideal point p of M corresponds a torus T_p of the boundary of $\overline{M}$. We fix a smooth product structure $T_p \times [0,\infty)$ on the cusp corresponding to p. Such a structure induces a cone structure, obtained from $T_p \times [0,\infty]$ by collapsing $T_p \times \{\infty\}$ to p, on a neighborhood C_p of p in $\widehat{M}$. We lift such structures to the universal covering. If $\widetilde{p}$ is an ideal point of $\widetilde{M}$ that projects to the ideal point p of M, we denote by $N_{\widetilde{p}}$ the cone at $\widetilde{p}$. We regard $N_{\widetilde{p}}$ as the quotient obtained from $P_{\widetilde{p}} \times [0,\infty]$ by collapsing $P_{\widetilde{p}} \times \{\infty\}$ is collapsed to a point, and where $P_{\widetilde{p}} \times \{t\}$ projects to the torus $T_p \times \{t\}$ for $t < \infty$.*

For an ideal point p corresponding to T, the lifts of p to an ideal point of $\widehat{\widetilde{M}}$ correspond to the conjugates of $\pi_1(T)$ in $\pi_1(M)$ (via the correspondence $\widetilde{p} \leftrightarrow \mathrm{Stab}(\widetilde{p})$).

Definition 2 (Ideally triangulated manifold) *Let M be a cusped manifold. An ideal triangulation of M is a finite, smooth triangulation of $\widehat{M}$ hav-*

ing the set of ideal points as 0*-skeleton. An* ideally triangulated manifold *is a cusped manifold equipped with a finite smooth ideal triangulation* τ *which is compatible with the product structures of the cusps. Namely, for each cusp* C_p *we require* $\tau \cap (T_p \times \{0\})$ *to be a triangulation of* T_p *and the restriction to* C_p *of* τ *to be the product triangulation.*

It is well-known that any cusped manifold can be ideally triangulated. Indeed, ideal triangulations can be viewed as the dual of standard spines, and any cusped manifold admits a standard spine (see for example [1] and [11]).

We recall now the definition of modulus of an ideal tetrahedron. An ideal tetrahedron in $\overline{\mathbb{H}}^3$ is the convex hull of four distinct points in $\partial\mathbb{H}^3$. An orientation of an ideal tetrahedron is an ordering of its vertices, up to even permutations. When the four points do not lie in a 2-plane, these orientations correspond to the two orientations as a manifold. Using the model $\mathbb{C} \times \mathbb{R}^+$ for $\mathbb{H}^3$, we may apply an isometry to assume that the vertices are $(0, 1, \infty, z)$, where the *modulus* z is the cross-ratio $[v_1 : v_2 : v_3 : v_4]$. The cross sections by the horospheres $\mathbb{C} \times \{t\}$ are rescalings of the triangle $\{0, 1, z\}$ in $\mathbb{C}$, and it follows that the hyperbolic structure of the ideal tetrahedron is determined by the similarity structure of this horospherical triangle at a vertex. Changing the ordering of the vertices by an even permutation changes z to an element in the set $\{z, \frac{1}{1-z}, 1 - \frac{1}{z}\}$. This ambiguity may be avoided by fixing a preferred edge e of the tetrahedron and arranging the vertices (v_1, v_2, v_3, v_4) so that e joins v_1 to v_3. Choosing the edge opposite to e gives the same modulus.

In the sequel we tacitly assume that an orientation and an edge for each tetrahedron have been fixed.

Definition 3 *Let* τ *be an ideal triangulation of* M*. A choice of moduli* $\mathbf{z} = \{z_i,\ i \in I\}$ *for* τ *is a choice of a complex number* $z_i \neq 0, 1$ *for each tetrahedron* Δ_i *of* τ*. We write* $(\tau, \mathbf{z})$ *to mean an ideal triangulation* τ *with a choice of moduli* $\mathbf{z}$ *for* τ*.*

3 Compatibility equations, developing maps and holonomy

In this section, we recall some standard facts about ideal triangulations with moduli, we introduce the system $\mathcal{C}$ of compatibility equations and we give the definition of geometric solution of $\mathcal{C}$. We refer the reader to [16], [12], [1], [13], [14], [8] for more details. For this section, M will be an ideally triangulated manifold with a triangulation τ and a choice of moduli $\mathbf{z}$ for τ.

In the language of (X, G)-structures, an oriented (possibly incomplete) hyperbolic manifold is a space equipped with an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-atlas. Given $(\tau, \mathbf{z})$, the idea is to use the hyperbolic structures defined by $\mathbf{z}$ on the tetrahedra as local charts for an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-atlas for M. In order to succeed in this construction, a necessary condition is that for each edge e of τ the product of moduli around e is 1. Such conditions can be written as a system $\mathcal{C}$ of algebraic equations on the moduli, having the form

$$\pm \prod_i z_i^{\alpha_i}(1-z_i)^{\beta_i} = 1$$

where $\alpha_i, \beta_i \in \mathbb{Z}$ depend on the combinatorial data. These equations are called *compatibility equations.*

For a hyperbolic manifold N, it is well-known that there exist a developing map $D : \widetilde{N} \to \mathbb{H}^3$ and a holonomy representation $h : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that D is an h-equivariant local diffeomorphism (see for example [15]). A similar picture holds for ideally triangulated manifolds with a choice of moduli satisfying $\mathcal{C}$.

The following proposition is a basic fact about ideal triangulations with moduli, see for example [16], [1], [12], [8] for a proof and details.

Proposition 4 *Let M be an ideally triangulated manifold and let $\mathbf{z}$ be a choice of moduli. Then, equations $\mathcal{C}$ are satisfied if and only if there exist a map $D : \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ and a representation $h : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that:*

- *D maps each lift $\widetilde{\Delta}_i$ of each tetrahedron Δ_i of τ to an hyperbolic ideal tetrahedron of $\mathbb{H}^3$ with modulus z_i.*
- *D is h-equivariant, that is, for each $x \in \widehat{\widetilde{M}}$ and $\alpha \in \pi_1(M)$*

$$D(\alpha x) = h(\alpha)D(x)$$

  where $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations and on $\mathbb{H}^3$ via h.

Definition 5 *A map $D : \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ and a representation $h : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ satisfying the conditions of Proposition 4 are called respectively a* developing map for $\mathbf{z}$ *and a* holonomy *of* $\mathbf{z}$*. When we need to emphasize that D and h depend on $\mathbf{z}$, we write $D(\mathbf{z})$ and $h(\mathbf{z})$.*

Remark 6 *The maps D and h are not unique. Nevertheless, the conjugacy class of the holonomy depends only on $\mathbf{z}$.*

Definition 7 (Hyperbolic map) *Let N be a hyperbolic manifold with developing map $D_N : \widetilde{N} \to \mathbb{H}^3$. Let M be an ideally triangulated manifold with a choice of moduli $\mathbf{z}$. A map $f : M \to N$ is called* hyperbolic w.r.t. $\mathbf{z}$ *if it lifts to a map $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ such that $D_N \circ \widetilde{f}$ extends to a developing map for $\mathbf{z}$.*

$$
\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{N} \arrow[r, "D_N"] \arrow[d] & \mathbb{H}^3 \\
M \arrow[r, "f"] & N &
\end{tikzcd}
$$

Fig. 1. Hyperbolic map

Definition 8 (Geometric solution of $\mathcal{C}$) *Suppose M is an ideally triangulated manifold. We say that a choice of moduli $\mathbf{z}$ is a* geometric solution *of $\mathcal{C}$ if there exist a hyperbolic structure $\mathfrak{S}$ on M and a proper degree-one map $f : M \to M_{\mathfrak{S}}$ which is hyperbolic w.r.t. $\mathbf{z}$ (where $M_{\mathfrak{S}}$ means M with the structure $\mathfrak{S}$).*

The following facts are not hard to prove (see [8] for details).

Proposition 9 *Any geometric solution of $\mathcal{C}$ is also an algebraic solution of equations $\mathcal{C}$.*

Proposition 10 *Let N be a hyperbolic manifold with holonomy h_N. Let $f : M \to N$ be a map hyperbolic w.r.t. $\mathbf{z}$ and let $f_* : \pi_1(M) \to \pi_1(N)$ be the induced homomorphism. Then $h(\mathbf{z}) = h_N \circ f_*$.*

4 Hyperbolic Dehn filling equations

For this section, $M \simeq \mathrm{int}(\overline{M})$ will be an cusped manifold with an ideal triangulation $\tau = (\{\Delta_i\})$. For each boundary torus T_n we fix a basis (μ_n, λ_n) of $H_1(T_n, \mathbb{Z})$. We denote by (p, q) a set $\{(p_n, q_n)\}$ of Dehn filling parameters, where each (p_n, q_n) is either a pair of coprime integers or the symbol ∞. The symbol $\mathbf{z}$ will denote a choice of moduli for τ satisfying $\mathcal{C}$.

In this section we introduce a system of equations on the moduli, called *hyperbolic Dehn filling equations*, which depend on a chosen set (p, q) of Dehn filling parameters. When the moduli have positive imaginary part, such equations imply that the completion of the hyperbolic structure defined by the moduli on M is the Dehn filling of M described by (p, q). The principal condition expressed by these equations is that if m is a loop in a boundary torus killed in homology by the filling, then the holonomy of m is trivial.

Hyperbolic Dehn filling equations can be written down without restrictions on the imaginary parts of the moduli, but in general there is not an obvious geometric interpretation of their solutions. For this reason, we distinguish between algebraic and geometric solutions of the equations.

First of all, we recall the definition of Dehn filling of a manifold.

Definition 11 (Dehn filling) *Let (p,q) be a set of Dehn filling parameters. For each n such that $(p_n,q_n) \neq \infty$, let L_n be an oriented solid torus, m_n be a meridian of $T'_n = \partial L_n$, l_n be a loop in T_n such that $[l_n] = p_n\mu_n + q_n\lambda_n$ and $\varphi_n : T_n \to T'_n$ be an orientation-reversing homeomorphism such that $\varphi_n(l_n) = m_n$. The Dehn filling of M with parameters (p,q) is the manifold*

$$M_{(p,q)} = \mathrm{int}\Big(\overline{M} \sqcup \{L_n\} \big/ \{\varphi_n\}\Big)$$

The tori L_n are called filling tori.

We notice that not all the boundary tori are filled in $M_{(p,q)}$. Namely, a torus T_n is filled if and only if $(p_n,q_n) \neq \infty$. If $(p_n,q_n) = \infty$ for all n , then $M_{(p,q)} = M$.

Consider now a complete hyperbolic manifold N, so $\widetilde{N} = \mathbb{H}^3$, and let γ be an oriented geodesic in N. Since γ is oriented, for any lift $\tilde{\gamma} \subset \mathbb{H}^3$ the endpoint of $\tilde{\gamma}$ is well-defined.

Definition 12 *Let N, γ be as above. Let $f : M \to N$ be a hyperbolic map w.r.t* $\mathbf{z}$. *If $\widetilde{f} : \widetilde{M} \to \mathbb{H}^3$ is a lift of f, we say that f* spirals around *γ near an ideal point v if $\widetilde{f}$ carries any lift of v to the endpoint of a lift of γ.*

Let $T \subset \partial\overline{M}$ be a boundary torus. Consider the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ and a developing map D such that the vertex corresponding to T is lifted to a vertex mapped to ∞ by D. Then, the group $h(\pi_1(T))$ consists of maps which fix ∞. By considering the restriction to $\partial\mathbb{H}^3 \equiv \mathbb{CP}^1$ of the elements of $h(\pi_1(T))$, we obtain a representation $h_T : \pi_1(T) \to \mathrm{Aff}(\mathbb{C})$. Since h is well-defined up to conjugation, then the dilation component of h_T is well-defined, and it is a representation $\rho_T : \pi_1(T) \to \mathbb{C}^*$.

Since $\pi_1(T)$ is abelian, its image $h_T(\pi_1(T))$ consists of maps which commute with each other. Therefore, it is easy to see that either they are all translations, or they have a common fixed point. In the former case we have $\rho_T \equiv 1$. In the latter case, up to conjugation, we can suppose that the fixed point is 0. Thus we get $h_T = \rho_T$, in the sense that for all $\alpha \in \pi_1(T)$ and $\zeta \in \mathbb{C}$, we have $h_T(\alpha)(\zeta) = \rho_T(\alpha) \cdot \zeta$.

Remark 13 *In the following, by writing $\rho_T \equiv 1$ we mean that $h_T(\pi_1(T))$ consists of translations and by $h_T = \rho_T$ we mean that $h_T(\pi_1(T))$ consists of maps which fix 0.*

To write the equations, we need to work with $\log(\rho_T)$. In the following definition we fix a suitable determination of the logarithm of ρ_T.

Definition 14 (Logarithm of Dilation component) *Let D be a develop-*

ing map for $\mathbf{z}$*. Let* $T \subset M$ *be a boundary torus of* $\overline{M}$*, pushed a little inside* M*, and let* $\widetilde{T} \subset \widetilde{M}$ *one if its lifts. Consider the model* $\mathbb{C} \times \mathbb{R}^+$ *of* $\mathbb{H}^3$ *so that the ideal point corresponding to* $\widetilde{T}$ *is mapped to* ∞*. Suppose that* $h_T = \rho_T$ *and suppose that the following condition holds:*

The developed image of $\widetilde{T}$ *does not intersect the line* $(0, \infty)$*.*

Then we choose a determination of $\log(\rho_T)$ *as follows: let* H *be the universal covering of* $\mathbb{H}^3 \setminus (0, \infty)$ *made by using the covering* $\exp : \mathbb{C} \to \mathbb{C}^*$*. Let* x_0 *and* $\widetilde{x}_0$ *be base-points in* T *and* $\widetilde{T}$*. Let* $\eta : [0, 1] \to T$ *be a loop at* x_0 *and* $\widetilde{\eta}$ *be its lift starting from* $\widetilde{x}_0$*. Let* $\widetilde{\alpha} : [0, 1] \to \mathbb{C}^*$ *be the horizontal component of* $D \circ \widetilde{\eta}$*. As* $D \circ \widetilde{\eta}$ *lifts to* H*, the path* $\widetilde{\alpha}$ *lifts to a path* $\overline{\alpha} : [0, 1] \to \mathbb{C}$*. Since* $h_T = \rho_T$*, then* $\widetilde{\alpha}(1) = \rho_T([\eta]) \cdot \widetilde{\alpha}(0)$*, and then* $\overline{\alpha}(1) = \log(\rho_T([\eta])) + \overline{\alpha}(0)$*.*

The points $\overline{\alpha}(0)$ *and* $\overline{\alpha}(1)$ *depend only on the homotopy class of* η *and on the choice of the base-points. If we change the base-points, the determination of* $\log(\rho_T([\eta]))$ *does not change.*

We are now ready to give the hyperbolic Dehn filling equations.

Definition 15 (Dehn filling equations) *Let* $(p, q) = \{(p_n, q_n)\}$ *be a set of Dehn filling parameters. For each* n*, let* $\rho_n(\mathbf{z})$ *be the dilation component of the holonomy of the* n*-th boundary torus* T_n*, when* $\mathbf{z}$ *varies on the space of solutions of the compatibility equations. We say that* $\mathbf{z}$ *is an* algebraic *solution of the* (p, q)*-equations if for each* n *we have:*

- *If* $(p_n, q_n) = \infty$*, then* $\rho_n(\mathbf{z}) \equiv 1$*.*
- *If* $(p_n, q_n) \neq \infty$*, then* $h_{T_n}(\mathbf{z}) = \rho_n(\mathbf{z})$*, the condition of Definition 14 holds, and*
$$p_n \log(\rho_n(\mathbf{z})[\mu_n]) + q_n \log(\rho_n(\mathbf{z})[\lambda_n]) = 2\pi i.$$

We say that z *is a* geometric *solution of the* (p, q)*-equations if, denoting by* $N = M_{(p,q)}$ *the Dehn filling of* M *with parameters* $\{(p_n, q_n)\}$*, we have:*

a) N *is complete hyperbolic and the cores of the filling tori are disjoint geodesics* $\{\gamma_n\}$*.*
b) There exists a proper map $f : M \to N \setminus \{\gamma_n\} \subset N$ *of degree 1, which is hyperbolic w.r.t.* $\mathbf{z}$*.*
c) For each n *with* $(p_n, q_n) \neq \infty$*, if* v_n *denotes the ideal point corresponding to* T_n*, then* f *spirals around* γ_n *near* v_n*, where* γ_n *has the orientation induced by the Dehn filling parameters* (p_n, q_n)*.*

Remark 16 *When all the coefficients* (p_n, q_n) *are* ∞*, then the system of the* (p, q)*-equations is nothing but the classical system* $\mathcal{M}$ *of the so-called* completeness equations*. When the moduli have positive imaginary part, equations* $\mathcal{M}$ *imply that the hyperbolic structure defined by the moduli on* M *is complete* *(of finite volume).*

The following fact is not hard to prove (see [8] for details).

Proposition 17 *For any* (p,q)*, each geometric solution of the* (p,q)*-equations is also algebraic.*

Remark 18 *It is well-known that any algebraic solution of* (p,q)*-equations such that each* z_n *has positive imaginary part is geometric. In Section 7 we give examples of algebraic solutions that are not geometric.*

5 Algebraic solutions of hyperbolicity equations

In this section we study the space of algebraic solutions of compatibility equations. We show that there is a one-to-finite correspondence between representations of the fundamental group of a given ideally triangulated 3-manifold and (algebraic) solutions of $\mathcal{C}$ for such a manifold. This gives another way to see the space of generalized Dehn filling coefficients.

For this section we keep the notation fixed in Section 4. When $\mathbf{z}$ is a solution of $\mathcal{C}$, $h(\mathbf{z})$ will denote its holonomy. To simplify notations, we often omit to indicate the base-points for the fundamental groups. For any boundary torus T_n, we assume that a representative $\pi_1(T_n) < \pi_1(M)$ of the conjugacy class of its fundamental group has been fixed. For each isometry $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3)$ let $\mathrm{Fix}(\gamma)$ denote the set of the points of $\overline{\mathbb{H}}^3$ fixed by γ. For a subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ we set $\mathrm{Fix}(\Gamma) = \cap_{\gamma\in\Gamma}\mathrm{Fix}(\gamma)$.

It is easily checked that the following fact holds (see for example [8]).

Lemma 19 *For any abelian subgroup* Γ *of* $\mathrm{Isom}^+(\mathbb{H}^3)$*,* $\mathrm{Fix}(\Gamma)$ *is not empty. Moreover,*

(1) $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *is infinite if and only if* $\Gamma = \{\mathrm{Id}\}$.
(2) $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3 = \emptyset$ *if and only if* Γ *is a dihedral group generated by two rotations of angle* π *around orthogonal axes.*
(3) $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *contains a single point if and only* Γ *contains only parabolic isometries.*
(4) Otherwise $\mathrm{Fix}(\Gamma) \cap \partial\mathbb{H}^3$ *contains exactly two points.*

For any representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$, we denote by $\mathcal{D}_\rho$ the set of all ρ-equivariant maps from the ideal points of $\widetilde{M}$ to $\partial\mathbb{H}^3$. Because of equivariance, if $D \in \mathcal{D}_\rho$ and q is an ideal point of $\widetilde{M}$, then $D(q) \in \rho(\mathrm{Stab}(q))$. Moreover, the elements of D_ρ can be constructed as in the proof of the next proposition.

Proposition 20 *Let $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation. Suppose that for any boundary torus T_n, $\rho(\pi_1(T_n))$ is not dihedral. Then the set $\mathcal{D}_\rho$ is not empty. If, in addition, the ρ-images of the fundamental groups of all the boundary tori are not trivial, then $\mathcal{D}_\rho$ is finite. Moreover, $\mathcal{D}_\rho$ consists of one element if and only if the ρ-images of the fundamental groups of all the boundary tori are parabolic.*

Proof. We prove the first claim by showing how to construct an element D of $\mathcal{D}_\rho$. Let q be an ideal point of $\widetilde{M}$. The stabilizer $\mathrm{Stab}(q)$ of q in $\pi_1(M)$ is conjugate to the fundamental group of some boundary torus. It follows that $\rho(\mathrm{Stab}(q))$ is not dihedral, so by Lemma 19 it has at least one fixed point x in $\partial\mathbb{H}^3$. Define $D(q) = x$ and extend D to the $\pi_1(M)$-orbit of q by equivariance. Do the same for the remaining ideal points.

Now, let T_n be a boundary torus. By Lemma 19, if $\rho(\pi_1(T_n))$ is not trivial, then it has one or two fixed points in $\partial\mathbb{H}^3$. Thus, in the construction of D, when one has to choose the image of an ideal point, one has at most two possibilities. Since the ideal points of M are finite in number, then in $\widetilde{M}$ there is only a finite number of $\pi_1(M)$-orbits of ideal points. Therefore, one has to make only a finite number of choices. The last claim directly follows from point (3) of Lemma 19. □

In the sequel, let the symbol $*$ denote the *degenerate* modulus, with the meaning that an ideal tetrahedron has modulus $*$ if and only if it is a degenerate tetrahedron (it has two or more coincident vertices).

Theorem 21 (Representations determine moduli) *Suppose that $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a representation such that for any boundary torus T_n, $\rho(\pi_1(T_n))$ is not dihedral. Then, each element D of $\mathcal{D}_\rho$ induces a choice $\mathbf{z}_D$ of moduli in $(\mathbb{C}\setminus\{0,1\})\cup\{*\}$. Moreover, if $\mathbf{z}_D$ contains no $*$-moduli, then it is an algebraic solution of $\mathcal{C}$ with holonomy ρ.*

Proof. The choice of moduli $\mathbf{z}_D$ is defined simply by taking, for each Δ_i of τ, the modulus of the convex hull of the D-image of the vertices of any lift $\widetilde{\Delta}_i$ of Δ_i, setting the modulus to $*$ if D is not injective on the vertices of $\widetilde{\Delta}_i$. This definition is unambiguous because of the equivariance of D. If $\mathbf{z}_D$ contains no $*$-moduli then, by induction on the n-skeleta, one can easily construct a developing map for $\mathbf{z}_D$ that extends D. Thus by Proposition 4, $\mathbf{z}_D$ is a solution of $\mathcal{C}$. The holonomy of $\mathbf{z}_D$ is ρ because of the ρ-equivariance of D. □

Remark 22 *If $\varphi \in \mathrm{Isom}^+(\mathbb{H}^3)$ and $\rho' = \varphi \circ \rho \circ \varphi^{-1}$, then a natural correspondence between $\mathcal{D}_\rho$ and $\mathcal{D}_{\rho'}$ is defined by mapping $D \in \mathcal{D}_\rho$ to the element $\varphi \circ D \in \mathcal{D}_{\rho'}$. Note that $\mathbf{z}_D = \mathbf{z}_{\varphi\circ D}$.*

We give now a topological description of the sets $\mathcal{D}_\rho$ when ρ varies in the space

$\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)$. Let $p_1,\dots,p_k$ be the ideal points of M, and for all $n=1,\dots,k$ let q_n be a lift of p_n. Let $\mathcal{D}$ be the topological subspace of $\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)\times(\partial\mathbb{H}^3)^k$ defined by

$$\mathcal{D}=\bigcup_\rho\Big\{\{\rho\}\times\mathrm{Fix}\Big(\rho(\mathrm{Stab}(q_1))\Big)\times\cdots\times\mathrm{Fix}\Big(\rho(\mathrm{Stab}(q_k))\Big)\Big\}$$

and let p be the projection $p:\mathcal{D}\to\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)$.

By Proposition 20, for any representation ρ, a bijection between $p^{-1}(\rho)$ and $\mathcal{D}_\rho$ is well-defined by mapping $(\rho,x_1,\dots,x_k)$ to the element D of $\mathcal{D}_\rho$ such that $D(q_n)=x_n$. In the sequel we identify $\mathcal{D}_\rho$ with $p^{-1}(\rho)$.

The space $\mathcal{D}$ is strictly related to the space of generalized Dehn filling coefficients. We briefly recall some results in this field, referring the reader to [16], [3] and [2] for a detailed discussion.

Let $R(M)=\mathrm{Hom}(\pi_1(M),\mathrm{SL}(2,\mathbb{C}))$ be the variety of representations of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbb{C})$ and let $\chi(M)=R(M)//\mathrm{SL}(2,\mathbb{C})$ be its variety of characters. For $\rho\in R(M)$, its character χ_ρ is its projection to $\chi(M)$ and can be viewed as the map $\chi_\rho:\pi_1(M)\to\mathbb{C}$ defined by $\chi_\rho(\gamma)=\mathrm{trace}(\rho(\gamma))$.

For each $j=1,\dots,k$ let s_j be a slope in T_j. If χ_0 is the character of the holonomy of the complete structure of M (if any), then (see for example [2]) there exists a branched covering

$$\overline{p}:V\subset\mathbb{C}^k\to W\subset\chi(M)\tag{1}$$

where V and W are neighborhoods respectively of 0 and χ_0 such that, if $\chi_\rho=\overline{p}(u_1,\dots,u_k)$, then $2\cosh(u_j/2)=\pm\mathrm{trace}(\rho(s_j))$. Thus, the $\overline{p}$-fiber of a point is a finite set with a 2-to-1 choice for each $u_j\neq 0$.

We show now that also the projection $p:\mathcal{D}\to\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)$ has a branched covering structure which is strictly related to the one of $\overline{p}$. We denote by *parabolic order* of ρ the number $P(\rho)$ of boundary tori where ρ is parabolic:

$$P(\rho)=\#\{n\in\{1,\dots,k\}:\rho(\pi_1(T_n))\text{ is parabolic}\}.$$

The parabolic order stratifies $\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)$ as follows. Let

$$\mathrm{Par}^{(l)}(M)=\{\rho\in\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big):P(\rho)\leq l\};$$

then

$$\mathrm{Hom}\Big(\pi_1(M,x_0),\mathrm{Isom}^+(\mathbb{H}^3)\Big)=\bigcup_{l=0}^k\mathrm{Par}^{(l)}(M).$$

Proposition 23 *Let $\rho_0 : \pi_1(M, x_0) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be a representation such that $\rho_0(\pi_1(T_n))$ is not dihedral nor trivial for any of the T_n. Then there exists a neighborhood U of ρ_0 in $\mathrm{Hom}\Big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\Big)$ such that the restriction of p to $p^{-1}(U)$ is a branched covering whose branched locus is stratified by the parabolic order. More precisely, if $U^{(l)}$ denotes $U \cap \mathrm{Par}^{(l)}(M)$, then for each l*

$$p : p^{-1}\Big(U^{(l)} \setminus U^{(l-1)}\Big) \longrightarrow U^{(l)} \setminus U^{(l-1)}$$

is a finite covering which branches at $U^{(l+1)}$. Moreover, if there exists $D_0 \in \mathcal{D}_{\rho_0}$ such that $\mathbf{z}_{D_0}$ contains no $*$-moduli, then U can be choosen such that for each path $\alpha : [0,1] \to U$, with $\alpha(0) = \rho_0$, and each lift $\widetilde{\alpha} : [0,1] \to \mathcal{D}$ with $\widetilde{\alpha}(0) = D_0$, no $*$-moduli appear in $\mathbf{z}_{\widetilde{\alpha}(t)}$ for $t \in [0,1]$.

Proof. Since for $\rho_0(\pi_1(T_n))$ to be trivial or dihedral is a closed condition, we may choose a neighborhood U of ρ_0 in which each $\rho(\pi_1(T_n))$ is nontrivial and not dihedral for any of the T_n. Suppose that $\rho_0 \in U^{(l)} \setminus U^{(l-1)}$. It is not restrictive to assume that $\rho_0(\pi_1(T_n))$ is parabolic for $n = 1, \ldots, l$. Thus, since $\rho_0 \in U^{(l)}$, $\rho_0(\pi_1(T_n))$ is not parabolic for $n > l$, and the same holds for any $\rho \in U^{(l)}$.

By Proposition 20, for $\rho \in U^{(l)}$ the set $\mathcal{D}_\rho$ consists of a finite number of points. Let now $\alpha : [0,1] \to U^{(l)} \setminus U^{(l-1)}$ be a continuous path with $\alpha(0) = \rho_0$. The sets $\mathrm{Fix}(\rho(\mathrm{Stab}(q_n)))$ depend continuously on ρ. Moreover, since $\alpha(t) \in U^{(l)} \setminus U^{(l-1)}$, the cardinality of the sets $\mathrm{Fix}\Big(\alpha(t)(\mathrm{Stab}(q_n))\Big)$ depends continuously on t. It follows that for any $D_0 \in \mathcal{D}_{\rho_0}$ there exists a unique lift $\widetilde{\alpha} : [0,1] \to \mathcal{D}$ with $\widetilde{\alpha}(0) = D_0$ and $p(\widetilde{\alpha}(t)) = \alpha(t)$, and this proves that p is a finite covering of $U^{(l)} \setminus U^{(l-1)}$.

If $\rho \in U^{(l)} \setminus U^{(l-1)}$ approaches a representation $\overline{\rho} \in U^{(l+1)} \setminus U^{(l)}$, then there is a torus, say T_{l+1}, such that $\rho(\pi_1(T_{l+1}))$, which is not parabolic, approaches a parabolic group. Then the two points of $\mathrm{Fix}(\rho(\pi_1(T_{l+1})))$ converge to the same point, which is $\mathrm{Fix}(\overline{\rho}(\pi_1(T_{l+1})))$. Thus, two fibers of the covering glue together, and this shows that there is an effective branch at $U^{(l+1)}$. The last claim follows since $\mathbf{z}_{\widetilde{\alpha}(t)}$ depends continuously on t. $\square$

When a cuspidal group becomes parabolic, the type of branching of the map $p : \mathcal{D} \to \mathrm{Hom}(\pi_1(M), \mathrm{Isom}^+(\mathbb{H}^3))$ is of the type $\mathbb{C} \ni z \to z^2$. This is exactly the branch-type of the covering $\bar{p} : V \subset \mathbb{C}^k \to W \subset \chi(M)$.

To see the analogy with the space of generalized Dehn filling coefficients, consider the character-map $\chi : \mathrm{Hom}\Big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\Big) \to \chi(M)$

$$\chi : \rho \mapsto \chi_\rho.$$

Suppose that ρ_0 is the holonomy of the complete hyperbolic structure of M

(if any). Let U be a neighborhood as in Proposition 23 and let V, W be as in (1). It is not restrictive to assume $W = \chi(U)$. Then, using the fact that the coverings p and $\overline{p}$ have the same behavior at the branch locus, one can see that the map χ lifts to a map

$$\widetilde{\chi} : p^{-1}(U) \subset \mathcal{D} \to V \subset \mathbb{C}^k$$

such that $\chi \circ p = \overline{p} \circ \widetilde{\chi}$.

The following diagram resumes the correspondences between $\mathcal{D}$, the space of algebraic solutions, $\mathrm{Hom}\Big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\Big)$ and $\chi(M)$.

$$\begin{tikzcd}
\{\mathbf{z}_D,\ D \in \mathcal{D}_\rho\} \arrow[d, shift right, "\substack{\text{If no}\\ *\text{-moduli}}"'] & & \mathcal{D} \arrow[ll, "\text{Theorem 21}"'] \arrow[r, "\widetilde{\chi}"] \arrow[d, "p"] & \mathbb{C}^k \arrow[d, "\overline{p}"] \\
\left\{ \begin{array}{c} \text{Algebraic} \\ \text{solutions} \end{array} \right\} \arrow[u, shift right, "\text{Inclusion}"'] \arrow[rr, "\text{holonomies}"] & & \mathrm{Hom}\Big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\Big) \arrow[r, "\chi"] & \chi(M)
\end{tikzcd}$$

Fig. 2. The space $\mathcal{D}$

6 Geometric solutions of hyperbolicity equations

In this section we study the space of geometric solutions of compatibility and hyperbolic Dehn filling equations. We show that the set of geometric solutions of $\mathcal{C}$ is an open subset of the set of algebraic ones. We also show that the geometric solutions of hyperbolic Dehn filling equations are unique.

For this section we keep the notation fixed at the beginning of Section 5.

We show now that the set of geometric solutions of $\mathcal{C}$ is an open subset of the set of algebraic ones. We recall that, for each cusp C_n, we fixed a product structure on the lift $N_n \cong P_n \times [0, \infty]$ of C_n, where P_n covers the torus T_n and $P_n \times \{\infty\} \sim q_n$ (see Definition 1).

Lemma 24 *Let h_0 be the holonomy of a geometric solution of $\mathcal{C}$. Then there exists a neighborhood U of h_0 in $\mathrm{Hom}\Big(\pi_1(M, x_0), \mathrm{Isom}^+(\mathbb{H}^3)\Big)$ such that $p|_{p^{-1}(U)}$ is a branched covering and, for each $\rho \in U$ and $D \in \mathcal{D}_\rho$, there exists a local diffeomorphism $D_\rho : \widetilde{M} \to \mathbb{H}^3$ such that:*

(1) D_ρ is a developing map for a (possibly incomplete) hyperbolic structure $\mathfrak{S}_\rho$ on M with holonomy ρ.
(2) The map D_ρ "extends" D. More precisely, in each N_n, D_ρ maps all the sets of the form $\{x\} \times [0, \infty]$ to geodesic rays ending at $D(q_n)$.

(3) The maps D_ρ can be chosen continuously in $\mathcal{D}$ w.r.t. the compact C^1-topology of maps $\widetilde{M} \to \mathbb{H}^3$.

Proof. This is nothing but Lemma 1.7.2 of [3] or Lemma B.1.10 of [2]. These Lemmas are stated and proved starting from the holonomy of a complete hyperbolic structure of M, but it is not hard to see that they hold if one starts from the holonomy of a geometric solution of $\mathcal{C}$, the proofs remaining substantially the same. $\square$

Theorem 25 (Geometric solutions are an open set) *The set of geometric solutions of $\mathcal{C}$ is open in the set of algebraic solutions of $\mathcal{C}$.*

Proof. Let $\mathbf{z}_0$ be a geometric solution of $\mathcal{C}$ and let h_0 be its holonomy. By Definition 8, there exists a hyperbolic structure $\mathfrak{S}_0$ on M with holonomy h_0, a developing map D_0 for $\mathfrak{S}_0$ and a map $f : M \to M$ such that, if $\widetilde{f}$ is a lift of f, $D_0 \circ \widetilde{f}$ is a developing map for $\mathbf{z}_0$ (Figure 3).

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{M} \arrow[r, "D_0"] \arrow[d] & \mathbb{H}^3 \\
M \arrow[r, "f"] & M &
\end{tikzcd}$$

Fig. 3. The hyperbolic map f

Let U be a neighborhood of h_0 such that the conclusions of Proposition 23 and Lemma 24 hold for U. Then for any algebraic solution $\mathbf{z}$ of $\mathcal{C}$ such that $h(\mathbf{z}) \in U$ there exists a hyperbolic structure $\mathfrak{S}_{\mathbf{z}}$ on M and a developing map $D_{\mathbf{z}}$ for $\mathfrak{S}_{\mathbf{z}}$ such that, if $g_z = D_z \circ \widetilde{f}$ (see Figure 4), then

$$\mathbf{z}_{g_{\mathbf{z}}} = \mathbf{z}$$

where we used the symbol $g_{\mathbf{z}}$ also for the restriction of $g_{\mathbf{z}}$ to the ideal points. Moreover, since $h(\mathbf{z})$ depends continuously on $\mathbf{z}$, $D_{\mathbf{z}}$ depends continuously on $\mathbf{z}$. To show that $\mathbf{z}$ is a geometric solution of $\mathcal{C}$, we construct a hyperbolic map $f_{\mathbf{z}}$ from M to $(M, \mathfrak{S}_{\mathbf{z}})$ by perturbing the initial hyperbolic map f.

Let $\varphi_{\mathbf{z}} : \widehat{\widetilde{M}} \to \mathbb{H}^3$ be a developing map for $\mathbf{z}$ which coincides with $g_{\mathbf{z}}$ on the ideal points and depends continuously on $\mathbf{z}$ (Figure 4). Moreover, we require $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f}$.

Such a $\varphi_{\mathbf{z}}$ can be easily constructed by straightening $g_{\mathbf{z}}$. Moreover, using convex combinations along geodesics in $\mathbb{H}^3$, an $h(\mathbf{z})$-equivariant homotopy $H_{\mathbf{z}} : \widetilde{M} \times [0,1] \to \mathbb{H}^3$ can be constructed such that

$$H_{\mathbf{z}}(x, 0) = g_{\mathbf{z}}(x) \qquad H_{\mathbf{z}}(x, 1) = \varphi_{\mathbf{z}}(x).$$

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "D_{\mathbf{z}}"] \\ \widetilde{M} \arrow[ur, "\widetilde{f}"] \arrow[r, shift left, "g_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}"] \arrow[r, shift right, "\varphi_{\mathbf{z}}"'] & \mathbb{H}^3 \end{tikzcd}$$

Fig. 4. The maps $g_{\mathbf{z}}$ and $\varphi_{\mathbf{z}}$

More precisely, for any $x \in \widetilde{M}$, the map $[0,1] \ni t \mapsto H_{\mathbf{z}}(x,t)$ parametrizes the geodesic segment from $g_{\mathbf{z}}(x)$ and $\varphi_{\mathbf{z}}(x)$ (the parameter being a multiple of the arc-length depending of the distance between $g_{\mathbf{z}}(x)$ and $\varphi_{\mathbf{z}}(x)$). The fact that $\varphi_{\mathbf{z}}$ is a developing map does not imply in general that $\mathbf{z}$ is geometric. The problem is that $\varphi_{\mathbf{z}}$ should be the lift of a map $M \to M$, and this may not happen if, for example, looking at the restriction of $\varphi_{\mathbf{z}}$ to a cusp, one sees that its image intersects the axis of the holonomy of the cusp.

With Figure 4 in mind, the idea to rule out pathologies is to try to lift the homotopy $H_{\mathbf{z}}$ to a homotopy of $\widetilde{f}$, that is, to a map $F_{\mathbf{z}} : \widetilde{M} \times [0,1] \to \widetilde{M}$ such that

$$F_{\mathbf{z}}(x,0) = \widetilde{f}(x) \qquad \text{and} \qquad H_{\mathbf{z}}(x,t) = D_{\mathbf{z}} \circ F_{\mathbf{z}}(x,t).$$

At the 0-level, clearly we set $F_{\mathbf{z}}(x,0) = \widetilde{f}(x)$. Since $D_{\mathbf{z}}$ is a local diffeomorphism, $H_{\mathbf{z}}$ can be locally lifted a little near the 0-level. Since $\widetilde{M}$ is not compact, it is not clear a priori how long $H_{\mathbf{z}}$ lifts, and how this depends on the point x.

For any $x, \mathbf{z}$ define

$$\varepsilon_{x,\mathbf{z}} = \sup\Big\{ s \in [0,1] : H_{\mathbf{z}} \text{ continuously lifts if restricted to } \{x\} \times [0,s] \Big\}.$$

Since $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f}$, the homotopy $H_{\mathbf{z}_0}$ is constant in t, that is $H_{\mathbf{z}_0}(x,t) = \varphi_{\mathbf{z}_0}(x)$. Therefore $\varepsilon_{x,\mathbf{z}_0} = 1$.

Lemma 26 *For every compact set $E \subset \widetilde{M}$ there exists a neighborhood B of $\mathbf{z}_0$ such that for all $\mathbf{z} \in B$ and $x \in E$, we have $\varepsilon_{x,\mathbf{z}} = 1$.*

Proof. We only sketch the proof, which can be found with all details in [8].

Since the local diffeomorphisms $D_{\mathbf{z}}$ converge to D_0 when $\mathbf{z}$ goes to $\mathbf{z}_0$, for any $y \in \widetilde{M}$ there exists a neighborhood $A(y)$ of y in $\widetilde{M}$ and a neighborhood B_y of $\mathbf{z}_0$ such that for any $\mathbf{z} \in B_y$ the map $D_{\mathbf{z}}$ is a diffeomorphism with the image when restricted to $A(y)$. Moreover, the neighborhoods B_y's can be chosen in such a way that they are intersection of the space of solutions of $\mathcal{C}$ with balls of $\mathbb{C}^k$ centered at $\mathbf{z}_0$, and one can prove (see [8]) that the neighborhoods $A(y)$'s and B_y's can be chosen in such a way that radii of the balls B_y are lower semicontinuous in y. Define now

$$R(x) = \sup\{ s \in \mathbb{R} : |\mathbf{z} - \mathbf{z}_0| < s \Rightarrow \varepsilon_{x,\mathbf{z}} = 1 \}$$

Since $D_{\mathbf{z}} \to D_0$ as $\mathbf{z} \to \mathbf{z}_0$, and since $\varphi_{\mathbf{z}_0} = D_0 \circ \widetilde{f} = g_{\mathbf{z}_0}$, the maps $\varphi_{\mathbf{z}}$ and $g_{\mathbf{z}}$ become closer and closer as $\mathbf{z} \to \mathbf{z}_0$. It follows that for every $x \in \widetilde{M}$ if $|\mathbf{z} - \mathbf{z}_0|$ is small enough, then the whole geodesic segment joining $\varphi_{\mathbf{z}}(x)$ to $g_{\mathbf{z}}(x)$ is completely contained in $D_{\mathbf{z}}(A(\widetilde{f}(x)))$. It follows that for all $x \in \widetilde{M}$, $R(x) > 0$, and it turns out (see [8]) that there is no converging sequence $(x_n) \subset \widetilde{M}$ such that

$$\lim R(x_n) = 0.$$

Then, the function

$$\underline{R}(x) = \sup \left\{ \xi : \widetilde{M} \to \mathbb{R} \text{ lower semicontinuous s.t. } \xi(x) \leq R(x) \right\}$$

is lower semicontinuous and strictly positive.

For any compact set E, by lower semicontinuity, the function $\underline{R}$ has a minimum in E, which is strictly positive. It follows that there exists a neighborhood B of $\mathbf{z}_0$ such that for all $\mathbf{z} \in B$ and $x \in E$ we have $\varepsilon_{x,\mathbf{z}} = 1$. $\square$

In particular, we choose E as follows. Let M_0 be the closure of M minus the cusps (so $M_0 \simeq \overline{M}$), let $\widetilde{M}_0$ be its lift and let E be a fundamental domain of $\widetilde{M}_0$ for the action of $\pi_1(M)$.

Thus, for $\mathbf{z} \in B$, the homotopy $H_{\mathbf{z}}$ lifts to $F_{\mathbf{z}}$ on the points of E, and $F_{\mathbf{z}}$ extends to the whole $\widetilde{M}_0$ by equivariance. For any $x \in \widetilde{M}_0$ we set

$$\widetilde{f}_{\mathbf{z}}(x) = F_{\mathbf{z}}(x, 1).$$

Clearly, $\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}_{\mathbf{z}}$ on $\widetilde{M}_0$, and we will show in Lemma 27 that $\widetilde{f}_{\mathbf{z}}$ extends to the whole $\widehat{\widetilde{M}}$, keeping the property that

$$\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}_{\mathbf{z}}.$$

By equivariance, $\widetilde{f}_{\mathbf{z}}$ projects to a map $f_{\mathbf{z}} : M \to M$ which is hyperbolic w.r.t. $\mathbf{z}$ because $\varphi_{\mathbf{z}}$ is a developing map for $\mathbf{z}$. Moreover the degree of $f_{\mathbf{z}}$ continuously depends on $\mathbf{z}$, so it is constant 1. Then each $\mathbf{z} \in B$ is a geometric solution of $\mathcal{C}$. This completes the proof of Theorem 25. $\square$

Lemma 27 *The map $\widetilde{f}_{\mathbf{z}}$ extends to the whole $\widehat{\widetilde{M}}$, keeping the property that*

$$\varphi_{\mathbf{z}} = D_{\mathbf{z}} \circ \widetilde{f}_{\mathbf{z}}.$$

Proof. For each $n = 1, \ldots, k$, the map $\widetilde{f}_{\mathbf{z}}$ is defined on $P_n \times \{0\}$ ($N_n \cong P_n \times [0, \infty]$ is the product structure on the nth cups, see Definition 1). Moreover, since $\varphi_{\mathbf{z}}$ is a developing map for $\mathbf{z}$, it is not restrictive to suppose that it maps sets of the form $\{x\} \times [0, \infty] \subset N_n$ to geodesic rays ending at $g_z(q_n)$. By Property 2 of Lemma 24, such rays lift to $\widetilde{M}$. It follows that φ_z lifts on the cusps to a map extending $\widetilde{f}_{\mathbf{z}}$. □

We prove now the uniqueness of the geometric solutions of hyperbolic Dehn filling equations.

Proposition 28 *Suppose that the Dehn filling $N = M_{(p,q)}$ is hyperbolic. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two finite-volume, complete hyperbolic structures on N such that the cores γ_n of the filling tori are geodesics for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Then there exists an orientation-preserving isometry $\alpha : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ such that $\alpha(\gamma_n) = \gamma_n$ for all n.*

Proof. By rigidity, the identity $\mathrm{Id} : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ is homotopic to an isometry α. Thus for each n the loop γ_n is freely homotopic to $\alpha(\gamma_n)$. By hypothesis γ_n is geodesic for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Since α is an isometry it follows that $\alpha(\gamma_n)$ is a geodesic for $\mathfrak{S}_2$. Hence γ_n and $\alpha(\gamma_n)$ are geodesics for $\mathfrak{S}_2$ and they are freely homotopic, so they must coincide. □

Lemma 29 *If the Dehn filling coefficients (p, q) are such that there exists a geometric solution of the (p, q)-equations, then $M_{(p,q)}$ has finite volume.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equations. By definition, $M_{(p,q)}$ is complete hyperbolic. Let $\mathrm{vol}(z_i)$ be the volume of a hyperbolic ideal tetrahedron of modulus z_i, with $\mathrm{vol}(z_i) < 0$ if $\Im(z_i) < 0$. Since by definition of geometric solution there exists a proper degree-one map $f : M \to M_{(p,q)} \setminus \{\gamma_n\}$ which is hyperbolic w.r.t. $\mathbf{z}$, then

$$\mathrm{vol}(M_{(p,q)}) = \mathrm{vol}(\mathrm{Im}(f)) \leq \sum |\mathrm{vol}(z_i)| < \infty.$$

□

Lemma 30 *Let (p, q) be a set of Dehn filling coefficients and let $\mathbf{z}$ and $\mathbf{w}$ be two geometric solutions of the (p, q)-equations. Then there exists $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $h(\mathbf{w}) = \psi \circ h(\mathbf{z}) \circ \psi^{-1}$.*

Proof. Let $N = M_{(p,q)}$ be the (p, q)-Dehn filling of M endowed with its hyperbolic structure, and let $\imath : M \to N$ be one of the inclusions $M \to M_{(p,q)}$. Then, both representations $h(\mathbf{z})$ and $h(\mathbf{w})$ split along $\imath_* : \pi_1(M) \to \pi_1(N)$ giving representations of $\pi_1(N)$ of maximal volume. By Lemma 29, the rigidity theorem for representations of fundamental groups of finite-volume hyperbolic manifolds (see [7, Theorem 1.4]) applies, and $h(\mathbf{z})$ and $h(\mathbf{w})$ are conjugate. □

Theorem 31 *For any Dehn filling coefficient (p, q) there exists at most one geometric solution of the (p, q)-equations.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equations. By Proposition 17, $\mathbf{z}$ is also an algebraic solution of the (p, q)-equations. In particular, $h(\mathbf{z})(\pi_1(T_n))$ is not dihedral for any boundary torus T_n. If $D_{\mathbf{z}}$ is the restriction of a developing map for $\mathbf{z}$ to the ideal points, then $D_{\mathbf{z}} \in \mathcal{D}_{h(\mathbf{z})}$ and $\mathbf{z} = \mathbf{z}_{D_{\mathbf{z}}}$. By Proposition 20, if for all n we have $(p_n, q_n) = \infty$, then D is the unique element of $\mathcal{D}_{h(\mathbf{z})}$. Otherwise, D is the unique element of $\mathcal{D}_{h(\mathbf{z})}$ that satisfies condition c) of Definition 15. If $\mathbf{w}$ is another geometric solution of the (p, q)-equations, then by Lemma 30 there exists $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $h(\mathbf{w}) = \psi \circ h(\mathbf{z}) \circ \psi^{-1}$. As above, and by Proposition 28, $D_{\mathbf{w}}$ is completely determined as an element of $\mathcal{D}_{h(\mathbf{w})}$, and $D_{\mathbf{w}} = \psi \circ D_{\mathbf{z}} \in \mathcal{D}_{\psi \circ h(\mathbf{z}) \circ \psi^{-1}} = \mathcal{D}_{h(\mathbf{w})}$. Finally, by Remark 22

$$\mathbf{z} = \mathbf{z}_{D_{\mathbf{z}}} = \mathbf{z}_{\psi \circ D_{\mathbf{z}}} = \mathbf{z}_{D_{\mathbf{w}}} = \mathbf{w}.$$

□

Remark 32 *Theorem 31 in particular implies the uniqueness of geometric solutions of* $\mathcal{C}+\mathcal{M}$*, where* $\mathcal{M}$ *is the system of completeness equations recalled in Remark 16. We notice that Lemma 30 can be proved using a version of Mostow's rigidity for cusped manifold (see for example [8]) instead of the rigidity of representations.*

7 Examples

In this section we explicitly compute the solutions of the compatibility and completeness equations for some particular one cusped 3-manifolds.

- We study two bundles over S^1, called LR^3 and L^2R^3, with fiber a punctured torus. These manifolds admit non-unique algebraic solutions and a (unique) geometric one.
- We study a manifold with non-trivial JSJ decomposition, obtained by gluing a Seifert manifold to the complement of the figure-eight knot. This manifold is not hyperbolic but it admits a partially flat solution of the compatibility and completeness equations.

The manifolds LR^3 and L^2R^3 are interesting because on one hand they show that the algebraic solutions are not unique, and on the other hand they provides examples of algebraic solutions which are not geometric (Proposition 33). We notice that these "bad" solutions do not involve flat tetrahedra, and have a good behavior on the boundary. Namely, the boundary torus inherits an intrinsic Euclidean structure (up to scaling). This fact is surprising because the geometry of a finite-volume hyperbolic 3-manifold is strictly related to the geometry of its boundary. In fact, the equations on the moduli have an interpretation as conditions on the geometry of the boundary. More precisely, any ideal triangulation of M induces a triangulation of the boundary tori, by considering the manifold with boundary obtained by chopping off an open regular neighborhood of the ideal vertices. A modulus for the hyperbolic structure of an ideal tetrahedron determines a modulus for the similarity structure of the triangles obtained as horospherical sections near the vertices. So an ideal triangulation with moduli of M induces a triangulation with moduli of the boundary tori. The compatibility equations express the fact that the moduli for the triangles lead to similarity structures on the tori. The completeness equations express the fact that the structures of the boundary tori are Euclidean. Moreover, when the imaginary part of the moduli is not negative, the control of the geometry of the boundary implies a control of the one of the whole M. For example, in order to have a complete finite-volume hyperbolic structure on M, it suffices to check that the boundary tori have Euclidean structures.

In [6] it is shown that any algebraic solution of the compatibility and completeness equations for the similarity structure of a triangulated torus leads to a Euclidean structure, even if there are negative triangles, provided that the algebraic sum of the areas of the triangles is not zero. So the example of LR^3 shows that the Euclidean situation in dimension 2 and the hyperbolic one in dimension 3 become quite different when we allow the moduli to have negative imaginary part.

The manifold with non-trivial JSJ decomposition that we study in the last example is a manifold that admits an ideal triangulation with a positive, partially flat solution of the compatibility and completeness equations. Such a solution cannot be geometric as the manifold is not hyperbolic. This seems to contradict [14] (see the introduction). Actually there is no contradiction because in our example the conditions on the angles are not satisfied (see below for details). This example shows that such conditions play a central role for a solution to be geometric.

7.1 Notation

To begin with, we fix some notation. Let L and R be the following matrices of $\mathrm{SL}(2,\mathbb{Z})$:

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Each element A of $\mathrm{SL}(2,\mathbb{Z})$ is conjugate to a product $A = \pm\prod_{i=1}^{n} A_i^{n_i}$, with $A_i \in \{L, R\}$ and $n_i \in \mathbb{N}$.

Let S be the punctured torus $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$. Then each element $A \in \mathrm{SL}(2,\mathbb{Z})$ induces a homeomorphism φ_A of S. Given $A = \prod A_i^{n_i}$, we call $\prod A_i^{n_i}$ the manifold obtained from $S \times [0,1]$ by gluing $(x,0)$ to $(\varphi_A(x),1)$. For such a manifold, using the algorithm described in [5], one easily obtains an ideal triangulation with $\sum n_i$ tetrahedra.

We notice that the complement of the figure-eight knot is the manifold LR, and its standard ideal triangulation with two tetrahedra is the one obtained according to [5].

We use the following notation for labeling simplices. For each vertex v of a tetrahedron X, we write X_v for the triangle obtained by chopping off the vertex v from X, and X^v for the face of X opposite to v. Given a tetrahedron X and two vertices v, w of X, by abuse of notation, we use the label w also for the edge of the triangle X_v corresponding to the face X^w. A modulus for a tetrahedron X is named z_X and we will specify the edge to which it is referred.

7.2 The manifold LR^3

Let M be the manifold LR^3, i.e. the manifold obtained as described above by using the element $LR^3 = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$. Using the algorithm described in [5], we get the ideal triangulation τ of M with four tetrahedra, labeled A, B, C, D and pictured in Figure 5.

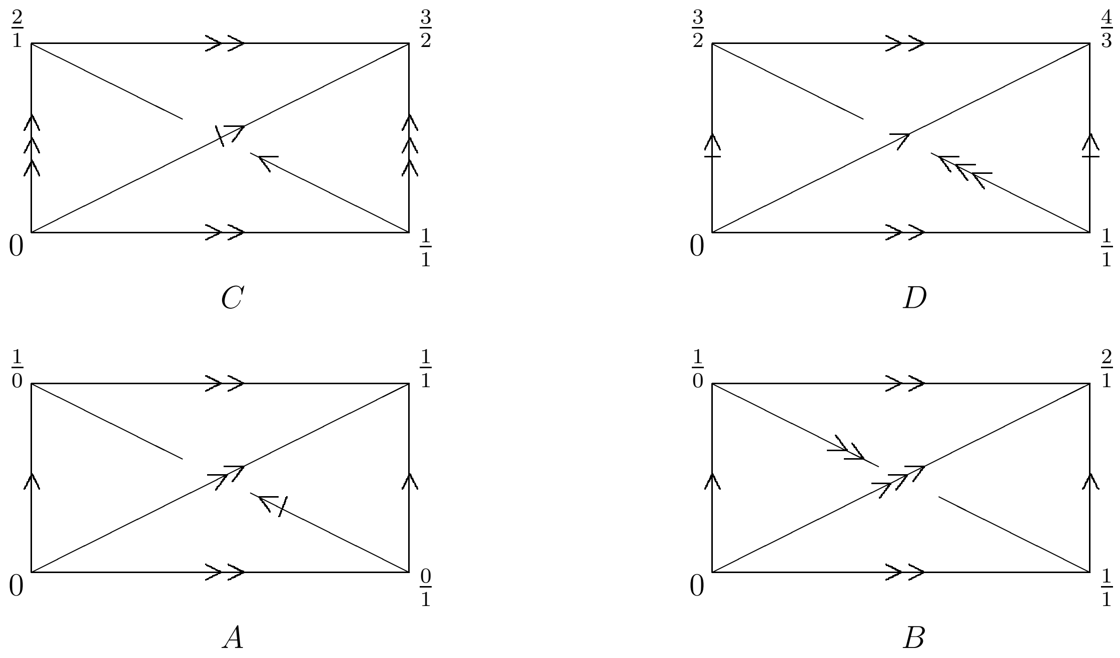


Fig. 5. Ideal triangulation of M

We label the vertices of the tetrahedra as in Figure 5 (we use such labels because they are natural using the algorithm of [5]). The moduli are referred to the edge $\overline{0\,\frac{1}{1}}$ (note that this edge is common to all the tetrahedra).

The face-pairing rules of τ are, according to the arrows in the picture:

$$A^{\frac{0}{1}} \longleftrightarrow B^{\frac{2}{1}} \quad B^{\frac{1}{0}} \longleftrightarrow C^{\frac{3}{2}} \quad C^{\frac{2}{1}} \longleftrightarrow D^{\frac{4}{3}} \quad D^{\frac{3}{2}} \longleftrightarrow A^{\frac{1}{1}}$$
$$A^{\frac{1}{0}} \longleftrightarrow B^{0} \quad B^{\frac{1}{1}} \longleftrightarrow C^{0} \quad C^{\frac{1}{1}} \longleftrightarrow D^{0} \quad D^{\frac{1}{1}} \longleftrightarrow A^{0}$$

The induced triangulation on the boundary torus is described in Figure 6.

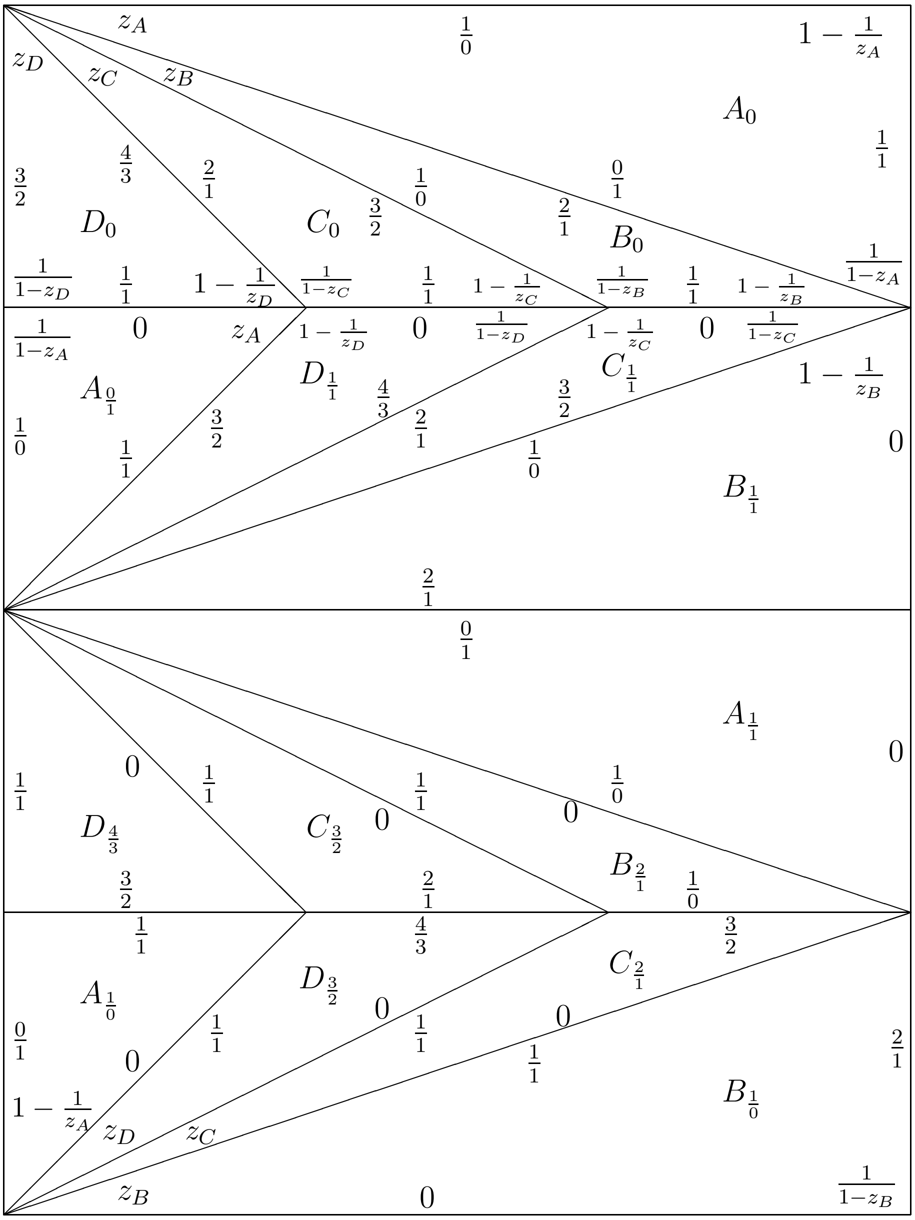


Fig. 6. The triangulation of the boundary torus

We can now write down the compatibility and completeness equations. It is easy to check that $\mathcal{C}+\mathcal{M}$ is equivalent to the system (2).

$$\mathcal{C}\begin{cases} \mathcal{C}_1. & z_A(1-\dfrac{1}{z_A})^2 z_D^2 z_C^2 z_B^2 \dfrac{1}{1-z_B} = 1 \\[2ex] \mathcal{C}_2. & (\dfrac{1}{1-z_A})^2 \dfrac{1}{1-z_D}(1-\dfrac{1}{z_B})^2 \dfrac{1}{1-z_C} = 1 \\[2ex] \mathcal{C}_3. & (1-\dfrac{1}{z_D})^2 \dfrac{1}{1-z_C} z_A = 1 \\[2ex] \mathcal{C}_4. & (1-\dfrac{1}{z_C})^2 \dfrac{1}{1-z_D}\dfrac{1}{1-z_B} = 1 \end{cases} \tag{2}$$

$$\mathcal{M}. \qquad z_D z_C z_B (1-z_A) = 1$$

Moreover, the product of the four equations $\mathcal{C}$ is exactly the square of the product of all the moduli, and so it is 1. So if three equations are satisfied, then the remaining one must be. It follows that we can discard one of the $\mathcal{C}$'s.

We discard $\mathcal{C}_2$. Using $\mathcal{M}$ in $\mathcal{C}_1$ and then $\mathcal{C}_1$ in $\mathcal{C}_4$ and $\mathcal{M}$ we obtain the following system of equations, equivalent to $\mathcal{C}+\mathcal{M}$:

$$\begin{cases} \mathcal{M}. & z_D z_C (1-z_A)^2 = -z_A \\ \mathcal{C}_1. & z_A(1-z_B) = 1 \\ \mathcal{C}_3. & (\frac{z_D-1}{z_D})^2 \frac{z_A}{1-z_C} = 1 \\ \mathcal{C}_4. & (\frac{z_C-1}{z_C})^2 \frac{z_A}{1-z_D} = 1 \end{cases} \tag{3}$$

Solving the system, one finds four non-degenerate solutions; one completely positive, giving the hyperbolic structure of M, and one with two negative tetrahedra, and their conjugates (i.e. the same situations but with inverted orientations). The following table contains numerical approximations of the solutions. Note that even if the modulus z_B is different from the modulus z_A, equation $\mathcal{C}_1$ implies that the geometric versions of A and B are isometric.

Solution 1		Volumes
z_A	$0.4275047 + i1.5755666$	0.9158907
z_B	$0.8395957 + i0.5911691$	0.9158907
z_C	$0.7271548 + i0.2284421$	0.5786694
z_D	$0.7271548 + i0.2284421$	0.5786694
Solution 2		**Volumes**
z_A	$1.0724942 + i0.5921114$	0.8144270
z_B	$0.2854042 + i0.3945194$	0.8144270
z_C	$-1.7271548 - i0.6779619$	-0.2398640
z_D	$-1.7271548 - i0.6779619$	-0.2398640

In Figures 7 and 8, we describe what the triangulation of the boundary torus of M looks like when we choose the moduli of Solution 2. There are two types of triangles, the positive ones, relative to the tetrahedra A and B and the negative ones, relative to C and D. In Figure 7 the four triangles of the top quarter of the triangulation of Figure 6 are pictured.

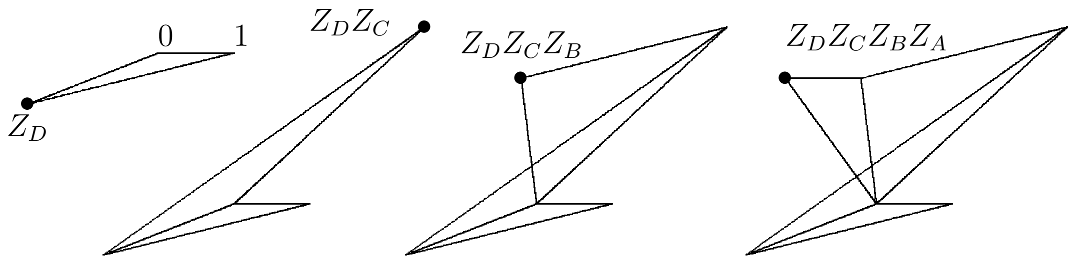


Fig. 7. The triangles $D_0,\ C_0,\ B_0,\ A_0$ with the moduli of Solution 2.

The two parts of Figure 8 are the top and bottom part of the triangulation of Figure 6.

Now we look at the algebraic expression of the solutions. A simple calculation

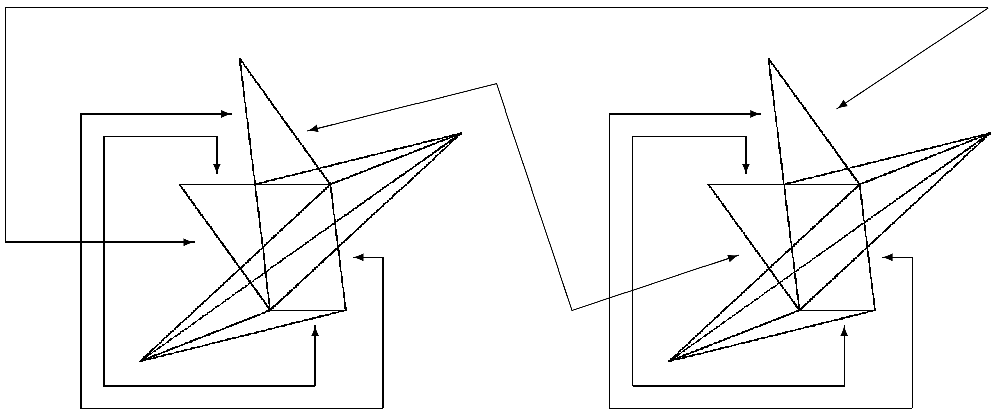


Fig. 8. Geometric triangulation of the boundary torus, Solution 2.

shows that the moduli can be expressed by equations (4):

$$\begin{cases} z_C = z_D = w \\ z_A = \dfrac{w^2}{1-w} \\ z_B = 1 - \dfrac{1}{z_A} = \dfrac{w^2+w-1}{w^2} \\ w^4 + 2w^3 - w^2 - 3w + 2 = 0 \end{cases} \tag{4}$$

The four solutions correspond to the four roots $w_1, \overline{w_1}, w_2, \overline{w_2}$ of the polynomial $P(w) = w^4 + 2w^3 - w^2 - 3w + 2$. Note that looking at the reduction (mod 2) of P, one can see that P is irreducible over $\mathbb{Z}$, and then also over $\mathbb{Q}$.

The holonomy representation can be explicitly calculated as a function of w. Let us fix a fundamental domain F for M obtained by taking one copy of each tetrahedron and then performing the gluings:

$$A^{\frac{1}{0}} \longleftrightarrow B^0 \quad B^{\frac{1}{1}} \longleftrightarrow C^0 \quad C^{\frac{1}{1}} \longleftrightarrow D^0$$

Consider now the geometric version of F, i.e. a developed image of F. The holonomy is generated by the isometries corresponding to the remaining face-pairing rules. We consider the upper half-space model of $\mathbb{H}^3$ with coordinates in which the points $0, 1, \infty$ of $\partial\mathbb{H}^3$ are the vertices of D labeled respectively $\frac{3}{2}, 0, \frac{4}{3}$. Calculations show that in this model the holonomy is generated by the elements of $\mathrm{PSL}(2,\mathbb{C})$ represented by the matrices:

$$\begin{pmatrix} 1 & \frac{w^2}{w^2+w-1} \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -w \\ \frac{1}{w} & -w-1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -w^2 \\ -1 & w^2+w-1 \end{pmatrix}$$

that respectively correspond to the face-pairing rules

$$A^0 \longrightarrow D^{\frac{1}{1}} \quad C^{\frac{2}{1}} \longrightarrow D^{\frac{4}{3}} \quad B^{\frac{2}{1}} \longrightarrow A^{\frac{0}{1}}$$

What is important is that the entries of such matrices are numbers belonging to $\mathbb{Q}(w)$ (and this can be proved even without the explicit calculations).

Proposition 33 *The Solution 2 is not geometric.*

Proof. This obviously follows from the uniqueness of geometric solutions, but we also give an alternative proof. Let w_1 (resp. w_2) be the root of P relative to Solution 1 (resp. 2) of $\mathcal{C} + \mathcal{M}$. So w_1 gives the hyperbolic structure of M. Let $h_j : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ be the holonomy representation relative to w_j for $j = 1, 2$. Since P is irreducible and the entries of the holonomy matrices are in $\mathbb{Q}(w)$, it follows that a relation between elements holds for h_1 if and only if it holds for h_2. Since h_1 is the holonomy of the complete hyperbolic structure of M, it is faithful, and it follows that also h_2 is faithful.

The image of h_2 cannot be discrete because otherwise $\mathbb{H}^3/h_2$ would be a hyperbolic manifold M' with too small a volume. We notice that by the rigidity of representations (see [7]) it follows that to obtain a contradiction, it is sufficient to show that $\mathrm{vol}(h_2) \neq \mathrm{vol}(h_1)$. By Proposition 10 the holonomy of any geometric solution is discrete, so Solution 2 cannot be geometric. $\square$

From the fact that h_2 is not discrete and Proposition 10 it follows that there is no map, which is hyperbolic w.r.t. Solution 2, from LR^3 to any hyperbolic manifold. Finally, we show that the image of h_2 is dense in $\mathrm{PSL}(2, \mathbb{C})$. We need the following standard fact about $\mathrm{PSL}(2, \mathbb{C})$ (see for example [10] or [9]).

Lemma 34 *Let G be a non-elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *and suppose that G is not discrete. Then the closure of G is either* $\mathrm{PSL}(2, \mathbb{C})$ *or it is conjugate to* $\mathrm{PSL}(2, \mathbb{R})$ *or to a $\mathbb{Z}_2$-extension of* $\mathrm{PSL}(2, \mathbb{R})$.

Proposition 35 *The image of the holonomy relative to Solution* 2 *is dense in* $\mathrm{PSL}(2, \mathbb{C})$.

Proof. It is easy to check that the image of h_2 is a non-elementary subgroup of $PSL(2, \mathbb{C})$. Suppose that its closure is conjugate to $PSL(2, \mathbb{R})$ or to a $\mathbb{Z}_2$-extension of $PSL(2, \mathbb{R})$. Then there exist a line in $\mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$ which is h_2-invariant. Looking at the parabolic elements of h_2, it is easy to see that such a line does not exist. The conclusion follows by Lemma 34. $\square$

The example discussed so far is interesting for several reasons. On one hand it shows that an algebraic solution of $\mathcal{C} + \mathcal{M}$ can be non-geometric. On the other hand it shows that there is no uniqueness of the algebraic solutions.

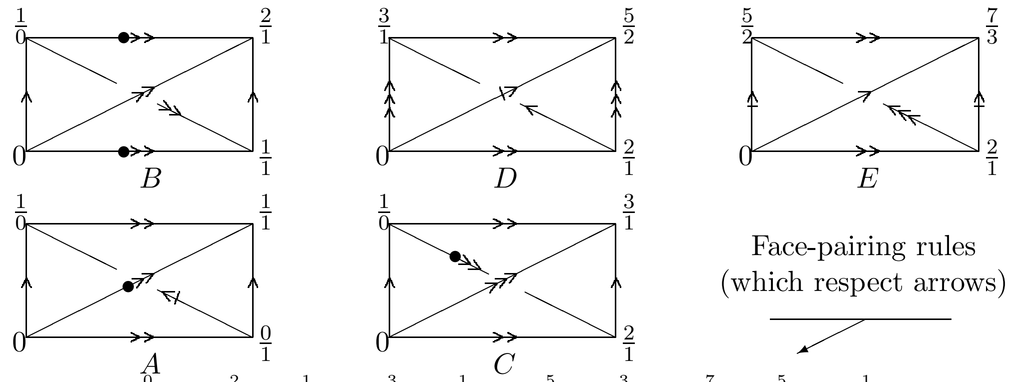
Moreover this example does not involve flat tetrahedra, so it is quite "regular." Finally, the bad solution of $\mathcal{C}+\mathcal{M}$ of LR^3 has the property that "everything works OK at the boundary," namely, the triangulation with moduli induced on the boundary torus defines on it a Euclidean structure (up to scaling). Roughly speaking, this means that the cusp of LR^3 would like to have a complete hyperbolic structure of finite volume according to the bad solution of $\mathcal{C}+\mathcal{M}$, but the rest of the manifold does not agree.

7.3 The manifold L^2R^3

In this section we do calculations for the manifold L^2R^3.

$$\mathrm{L}^2\mathrm{R}^3 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}7&2\\3&1\end{pmatrix}.$$

Using the algorithm described in [5], we get the ideal triangulation τ of M with five tetrahedra, labeled A, B, C, D, E and pictured in Figure 9.



$A^{\frac{0}{1}} \leftrightarrow B^{\frac{2}{1}}$ $\quad B^{\frac{1}{1}} \leftrightarrow C^{\frac{3}{1}}$ $\quad C^{\frac{1}{0}} \leftrightarrow D^{\frac{5}{2}}$ $\quad D^{\frac{3}{1}} \leftrightarrow E^{\frac{7}{3}}$ $\quad E^{\frac{5}{2}} \leftrightarrow A^{\frac{1}{1}}$

$A^{\frac{1}{0}} \leftrightarrow B^{0}$ $\quad B^{\frac{1}{0}} \leftrightarrow C^{0}$ $\quad C^{\frac{2}{1}} \leftrightarrow D^{0}$ $\quad D^{\frac{2}{1}} \leftrightarrow E^{0}$ $\quad E^{\frac{2}{1}} \leftrightarrow A^{0}$

Fig. 9. Ideal triangulation of M

We label the vertices of the tetrahedra as in Figure 9. The moduli z_A and z_B are referred to the edge $\overline{0\,\frac{1}{0}}$ while z_C, z_D, z_E to the edge $\overline{0\,\frac{2}{1}}$. The induced triangulation on the boundary torus is that of Figure 10.

It is easy to see that the system of compatibility and completeness equations

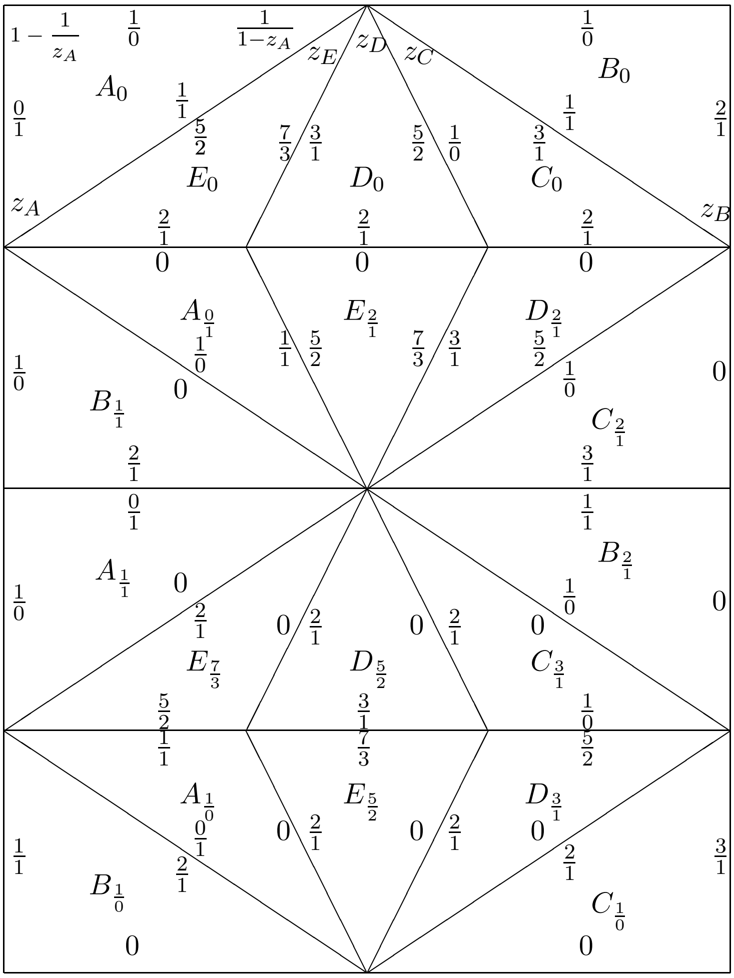


Fig. 10. Triangulation of the boundary torus

$\mathcal{C} + \mathcal{M}$ is equivalent to the following one:

$$\begin{cases} z_A z_B = z_C z_D z_E \\ z_C(1 - z_A) = 1 \\ (1 - z_D)^2 z_E^2 = (1 - z_E)^2 z_D^2 \\ (z_A - 1)^2 = z_A^2 (1 - z_B)^2 \\ (1 - \dfrac{1}{z_E})^2 \dfrac{1}{1 - z_D} (1 - \dfrac{1}{z_A}) = 1 \end{cases}$$

Solving this system, we have found eight solutions. The following tables contain numerical approximations of the solutions. Note that even if the modulus z_A is different from the modulus z_C, the second equation implies that the geometric versions of A and C are isometric.

	Solution 1	volume	Solution 2	volume
z_A	$0.75 + i0.6614378$	0.9626730	$0.75 - i0.6614378$	-0.9626730
z_B	$1.25 + i0.6614378$	0.7413987	$1.25 - i0.6614378$	-0.7413987
z_C	$0.5 + i1.3228756$	0.9626730	$0.5 - i1.3228756$	-0.9626730
z_D	1	$*$	1	$*$
z_E	1	$*$	1	$*$

	Solution 3	volume	Solution 4	volume
z_A	1.588633261	0	1.127804076	0
z_B	1.370528159	0	1.113321168	0
z_C	-1.69885025	0	-7.824476637	0
z_D	0.3783840018	0	0.2518509745	0
z_E	-3.387066549	0	-0.6371698130	0

	Solution 5	volume	Solution 6	volume
z_A	$0.4950484 + i0.3298695$	0.7399514	$0.4950484 - i0.3298695$	-0.7399514
z_B	$0.6011109 + i0.9321327$	1.0089809	$0.6011109 - i0.9321327$	-1.0089809
z_C	$1.3880304 + i0.9067580$	0.7399514	$1.3880304 - i0.9067580$	-0.7399514
z_D	$0.5022247 + i0.2691269$	0.6433681	$0.5022247 - i0.2691269$	-0.6433681
z_E	$0.6077815 + i0.3441339$	0.7596486	$0.6077815 - i0.3441339$	-0.7596486

	Solution 7	volume	Solution 8	volume
z_A	$0.1467328 + i1.2472524$	0.9386051	$0.1467328 - i1.2472524$	-0.9386051
z_B	$1.9069644 + i0.7908171$	0.4782906	$1.9069644 - i0.7908171$	-0.4782906
z_C	$0.3736330 + i0.5461534$	0.9386051	$0.3736330 - i0.5461534$	-0.9386051
z_D	$1.1826577 - i2.5849142$	-0.7155138	$1.1826577 + i2.5849142$	0.7155138
z_E	$-0.5956636 + i1.2429350$	0.7019645	$-0.5956636 - i1.2429350$	-0.7019645

Solutions 1 and 2 contain degenerate tetrahedra. We notice that the non-degenerate moduli of such solutions are exactly those that give the hyperbolic structure on the manifold obtained by removing the tetrahedra D and E and adding the gluing rules:

$$C^{\frac{1}{0}} \leftrightarrow A^{\frac{1}{1}} \text{ via } (0, \tfrac{3}{1}, \tfrac{2}{1}) \leftrightarrow (0, \tfrac{1}{0}, \tfrac{0}{1})$$
$$C^{\frac{2}{1}} \leftrightarrow A^{0} \text{ via } (0, \tfrac{1}{0}, \tfrac{3}{1}) \leftrightarrow (\tfrac{0}{1}, \tfrac{1}{0}, \tfrac{1}{1}).$$

Now we look at the algebraic expression of Solutions 3-8. Let $P(x) = x^6 + 4x^5 + 3x^4 + 3x^3 - 4x^2 + 2$. A simple calculation shows that the moduli can be expressed in terms of roots of P by the following expressions:

$$\begin{cases} z_A = \frac{1}{22}(5w^5 + 19w^4 + 9w^3 + 6w^2 - 8w + 17) \\ z_B = \frac{1}{44}(10w^5 + 49w^4 + 62w^3 + 34w^2 - 16w + 34) \\ z_C = \frac{1}{11}(-12w^5 - 39w^4 - 4w^3 - 10w^2 + 72w - 32) \\ z_D = \frac{1}{22}(-4w^5 - 13w^4 + 6w^3 + 15w^2 + 2w + 4) \\ z_E = w \\ P(w) = 0 \end{cases}$$

Solutions $3, 4, 7, 8$ are not geometric because of uniqueness of geometric solutions. Moreover, as in the case of LR^3, the polynomial P is irreducible, and the argument of Proposition 33 works in the present case.

7.4 A manifold with non-trivial JSJ decomposition

The manifold we consider in this section is obtained by gluing to the boundary torus of the complement of the figure-eight knot a Seifert manifold. The resulting manifold, which we call M, clearly is not hyperbolic because it contains an incompressible torus (the old boundary torus).

This example is interesting because the manifold M admits an ideal triangulation with four tetrahedra such that there exists a positive, partially flat solution of $\mathcal{C} + \mathcal{M}$. Obviously such a solution cannot be geometric, as M is not hyperbolic. We remark that in the present example the moduli do not satisfy the equations on the angles. Namely, when a modulus is positive, the tetrahedron has well-defined dihedral angles at its edges, in such a way that the sum of angles of any horospherical triangle is always 2π. Then in addiction to equations $\mathcal{C}$ one can require that the sum of the angles around any edge is exactly 2π. Such equations are called $\mathcal{C}^*$. In [14] is proved that every partially flat solution of $\mathcal{C}^* + \mathcal{M}$ is geometric. Here we produce a non-geometric, partially flat solution of $\mathcal{C} + \mathcal{M}$ that does not satisfy $\mathcal{C}^*$. This shows that the equations $\mathcal{C}^*$ play a fundamental role in order to have hyperbolicity.

We describe now our manifold M. We use the techniques of standard spines to construct an ideal triangulation of M, referring to [11] for details on the theory of spines. Let A be the following subset of $\mathbb{C}$:

$$A = \{z \in \mathbb{C} : |z| \leq 4,\ |z-2| > 1, |z+2| > 1\}.$$

A is a disc with two holes. Let $I \subset A$ be the set of points with zero real part. Let S be the space obtained from $A \times [0,1]$ by gluing $(z,0)$ to $(-z,1)$ and let L be the Möbius strip coming from I. The manifold S is the Seifert manifold we want to glue to complement of the figure-eight knot. We will refer to the external and internal components of ∂S as C_e and C_i. Note that $\partial L \subset C_e$.

We glue C_e to the boundary torus of the complement of the figure-height knot. To do this, we specify where we glue the boundary of the Möbius strip. We use the classical triangulation of the complement of the figure-eight knot. If one imagines looking from the cusp inside the complement of the figure-eight knot, one gets the following picture:

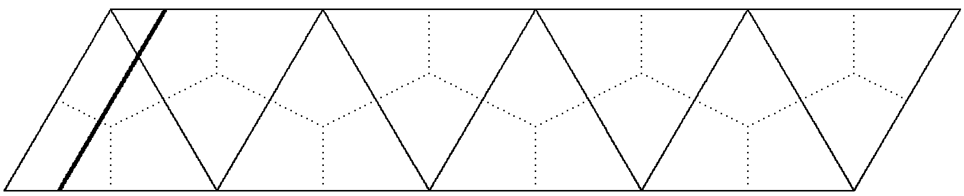


Fig. 11. The boundary of the complement of the figure-eight knot

There, the eight equilateral triangles of the boundary are pictured. The dashed lines represent the standard spine dual of the ideal triangulation, and the marked line is where we glue ∂L.

Since S retracts to $C_e \cup L$, a spine of M is obtained simply by gluing a Möbius strip to the spine of the complement of the figure-eight knot as in Figure 11. Such a spine has a vertex more than the old one, but is not standard. Performing a *lune* move (see [11]) along the Möbius strip we obtain a standard spine of M with five vertices. As the new spine is standard, its dual is an ideal triangulation with five tetrahedra. Such a triangulation can be simplified with an *MP*-move (this is the T-move of [11]), replacing the three new tetrahedra with an equivalent pair of tetrahedra. At the end, we get the triangulation of M sketched in Figure 12.

The tetrahedra labeled A and B are the old ones (those of the complement of

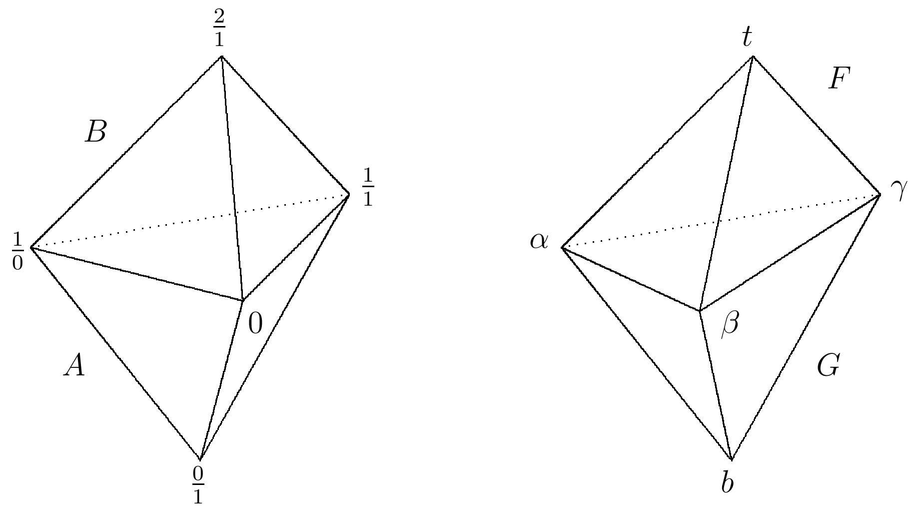


Fig. 12. The ideal triangulation of M

the figure-eight knot). The gluing rules are the following:

$$
\begin{array}{ll}
A^{\frac{0}{1}} \leftrightarrow B^{\frac{2}{1}} : \ (0, \tfrac{1}{0}, \tfrac{1}{1}) \leftrightarrow (0, \tfrac{1}{0}, \tfrac{1}{1}) & A^{\frac{1}{0}} \leftrightarrow B^{0} : \ (0, \tfrac{0}{1}, \tfrac{1}{1}) \leftrightarrow (\tfrac{1}{0}, \tfrac{1}{1}, \tfrac{2}{1}) \\
A^{\frac{1}{1}} \leftrightarrow B^{\frac{1}{0}} : \ (0, \tfrac{0}{1}, \tfrac{1}{0}) \leftrightarrow (0, \tfrac{1}{1}, \tfrac{2}{1}) & A^{0} \leftrightarrow F^{\gamma} : \ (\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{1}{0}) \leftrightarrow (t, \alpha, \beta) \\
B^{\frac{1}{1}} \leftrightarrow G^{\gamma} : \ (0, \tfrac{1}{0}, \tfrac{2}{1}) \leftrightarrow (b, \beta, \alpha) & F^{t} \leftrightarrow G^{b} : \ (\alpha, \beta, \gamma) \leftrightarrow (\alpha, \beta, \gamma) \\
F^{\alpha} \leftrightarrow G^{\beta} : \ (\beta, \gamma, t) \leftrightarrow (\gamma, \alpha, b) & F^{\beta} \leftrightarrow G^{\alpha} : \ (\alpha, t, \gamma) \leftrightarrow (\gamma, b, \beta)
\end{array}
$$

The moduli z_A and z_B are referred to the edge $\overline{0\frac{1}{1}}$ and z_F, z_G to $\overline{\alpha\beta}$. The triangulation of the boundary torus is that of Figure 13. It is readily checked that the system of compatibility and completeness equations is equivalent to:

$$
\begin{cases} \dfrac{1}{1-z_A} \cdot \dfrac{1}{z_B} \cdot \dfrac{z_F}{z_G} = 1 \\ z_G z_F = 1 \end{cases}
\qquad
\begin{cases} \dfrac{(1-z_A)^2}{z_A} \cdot \dfrac{z_B^2}{1-z_B} = 1 \\ z_B(1-z_A) = 1 \end{cases}
$$

From this, we easily get $z_G = z_F$ and $z_F^2 = 1$. Since we are looking for non-degenerate solutions, we have $z_F = z_G = -1$. Using this we get $z_A = z_B$ and $z_A^2 - z_A + 1 = 0$. Therefore, $z_A = z_B = (1 \pm i\sqrt{3})/2$. That is, the ideal tetrahedra F and G are flat but not degenerate, while A and B are regular, exactly as in the complement of the figure-eight knot. We notice that the space obtained by gluing together the geometric versions of the tetrahedra A, B, F, G is not a manifold.

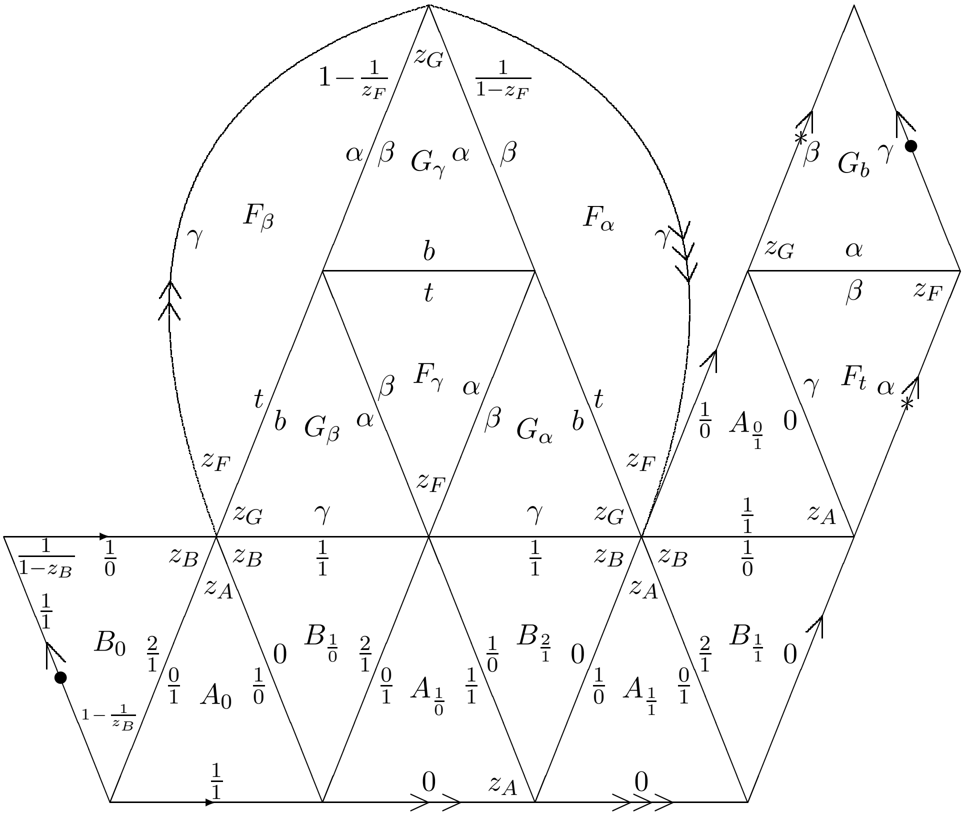


Fig. 13. Triangulation with moduli of the boundary torus

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