(1)

$$-1 + y'\sqrt{1+y} = t, \qquad y(0) = 1$$

$$y'\sqrt{1+y} = t+1, \qquad \int_1^{y(t)} \sqrt{1+u}\,du = \frac{t^2}{2} + t$$

$$\frac{2}{3}(1+u)^{3/2}\Big|_{u=1}^{u=y(t)} = \frac{t^2}{2} + t$$

$$\frac{2}{3}(1+y(t))^{3/2} - \frac{2}{3}\cdot 2^{3/2} = \frac{t^2}{2} + t$$

$$(1+y(t))^{3/2} = 2^{3/2} + \frac{3}{2}\left(\frac{t^2}{2} + t\right) = 2^{3/2} + \frac{3t^2}{4} + \frac{3}{2}t$$

$$\Rightarrow y(t) = \underbrace{\left[2^{3/2} + \frac{3}{4}t^2 + \frac{3}{2}t\right]}^{2/3} - 1$$

$$(\bar{t} > 0 \ \forall t \in \mathbb{R})$$

---

(2)

$$f(x) = x e^{-(x+x^2)}; \qquad f'(x) = e^{-(x+x^2)}\left[1 + x(-1-2x)\right]$$

$$= e^{-(x+x^2)}\left(1 - x - 2x^2\right) \geq 0$$

$$\Leftrightarrow -1 \leq x \leq \frac{1}{2}$$ (intervallo in cui f è crescente)

f è invece decrescente in $\left[\frac{1}{2}, +\infty\right[$ e in $]-\infty, -1]$

$$\lim_{x \to +\infty} x e^{-(x+x^2)} = (+\infty)\cdot e^{-\infty} = \lim_{x \to +\infty} \frac{x}{e^{x+x^2}} = 0$$

(Hopital)

$$\lim_{x \to -\infty} x e^{-(x+x^2)} = (-\infty)\cdot e^{-(+\infty)} = \infty \cdot 0 = \lim_{x \to -\infty} \frac{x}{e^{x+x^2}} = 0$$

③

$$\lim_{x\to 0} \frac{2-\sqrt{4+x\sin x}}{\sqrt{x^4+5^2(x^4+x^6)}} = \lim_{x\to 0} \frac{2-\sqrt{4+x\sin x}}{x^2\sqrt{1+5^2}}$$

$$\overset{(H)}{=} \frac{1}{\sqrt{1+5^2}} \lim \frac{-\frac{1}{2\sqrt{4+x\sin x}}\cdot(\sin x + x\cos x)}{2x}$$

$$= \frac{1}{\sqrt{1+5^2}} \lim_{x\to 0} \left(-\frac{1}{8}\right) \underbrace{\frac{\sin x + x\cos x}{x}}_{\to 2} = -\frac{1}{4\sqrt{1+5^2}}$$

$$\lim_{x\to 1+} \log\left(-x+\frac{3}{2}\right)\exp\left(\frac{1}{1-x^2}\right) = \log\left(\frac{1}{2}\right)\exp\left(\frac{1}{1-(1+)}\right)$$

$$= \log\left(\frac{1}{2}\right)\cdot\exp\left(\frac{1}{0-}\right) = 0-$$

---

④

$$\int_1^2 \sqrt{x}\exp\left(1+x^{3/2}\right)dx = \int_1^2 \frac{2}{3}\frac{d}{dx}\exp\left(1+x^{3/2}\right)dx$$

$$= \frac{2}{3}\left[\exp\left(1+2^{3/2}\right) - \exp(1)\right]$$

$$\int_a^b \sin^2(1+3x)\,dx = \frac{1}{2}\int_a^b \left[1-\cos(2+6x)\right]dx$$

$$= \frac{1}{2}(b-a) - \frac{1}{2}\cdot\frac{\sin(2+6x)}{6}\Big|_a^b$$

$$= \frac{1}{2}(b-a) - \frac{1}{12}\left(\sin(2+6b) - \sin(2+6a)\right)$$

(5) $F(x) = \int_0^x |1+t|\,dt$. Ricordo che $|1+t| = \begin{cases} 1+t & \text{se } t \geq -1 \\ -1-t & \text{se } t \leq -1 \end{cases}$

Se $x \geq -1$, ho $F(x) = \int_0^x (1+t)\,dt = x + \frac{x^2}{2}$

Se invece $x \leq -1$, allora posso usare l'additività

$$\int_0^x |1+t|\,dt = \int_0^{-1} |1+t|\,dt + \int_{-1}^{x} |1+t|\,dt = \int_0^{-1} (1+t)\,dt - \int_{-1}^{x} (1+t)\,dt$$

$$= \left(t + \frac{t^2}{2}\right)\Big|_0^{-1} - \left(t + \frac{t^2}{2}\right)\Big|_{-1}^{x} = -\frac{1}{2} - \left(x + \frac{x^2}{2}\right) - \frac{1}{2}$$

$$= -1 - x - \frac{x^2}{2}$$

Dunque

$$F(x) = \begin{cases} x + \frac{x^2}{2}, & \text{se } x \geq -1 \\ -1 - x - \frac{x^2}{2}, & \text{se } x \leq -1 \end{cases}$$

Per il calcolo del limite ~~bastava~~ si può usare il confronto appena fatto

$$\lim_{x \to -\infty} \frac{F(x)}{1-x^2} = \lim_{x \to -\infty} \frac{-1-x-\frac{x^2}{2}}{1-x^2} = \frac{1}{2}$$

oppure Hopital $\left(\lim_{x \to -\infty} \frac{|1+x|}{-2x} = \lim_{x \to -\infty} \frac{-1-x}{-2x} = \frac{1}{2} \right)$

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(6) $$z = \frac{1}{-i+2i^3-2} = \frac{1}{-3i-2} = \frac{1}{-3i-2} \cdot \frac{-2+3i}{-2+3i} = \frac{-2+3i}{4+9}$$

$$= -\frac{2}{13} + \frac{3i}{13} \quad \Rightarrow \quad \mathrm{Re}\, z = -\frac{2}{13}$$

$$\mathrm{Im}\, z = \frac{3}{13}$$

(1)

$$\frac{\partial f}{\partial v}\left(0,\frac{\pi}{4}\right) = \lim_{t\to 0} \frac{f\left((0,\frac{\pi}{4}) + t(1,0)\right) - f(0,\pi/4)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,\pi/4) - f(0,\pi/4)}{t} =$$

$$= \lim_{t\to 0} \frac{1}{t}\left\{ e^{t^2\pi/4}\cos\left(t+\frac{\pi}{2}\right) - e^{0\cdot\frac{\pi}{4}}\cos\left(\frac{\pi}{2}\right)\right\}$$

$$= \lim_{t\to 0} e^{t^2\pi/4}\,\frac{\cos(t+\frac{\pi}{2})}{t} \overset{\text{Hop}}{=} e^{0} \lim_{t\to 0} \frac{-\sin(t+\frac{\pi}{2})}{1} = -1$$

(0/0)

Formula del gradiente:

$$\nabla f(x,y) = \left(2xy\, e^{x^2y}\cos(x+2y) - e^{x^2y}\sin(x+2y),\right.$$

$$\left. x^2 e^{x^2y}\cos(x+2y) - 2e^{x^2y}\sin(x+2y)\right)$$

$$\nabla f\left(0,\frac{\pi}{4}\right) = \left(-\sin\frac{\pi}{2}, -2\sin\left(\frac{\pi}{2}\right)\right) = (-1,-2)$$

Dunque

$$\left\langle \nabla f\left(0,\frac{\pi}{4}\right), (1,0)\right\rangle = \langle (-1,2),(1,0)\rangle = -1$$

---

(2) $f(x,y) = x^2+y^2 - \log\left(1+\sqrt{x^2+y^2}\right)$

È una funzione radiale e, scritto $|(x,y)| = r$, vale

$f(x,y) = g(r)$ dove $g(r) = r^2 - \log(1+r)$, con $r \geq 0$

Vale $g(0) = 0$, $\lim_{r\to\infty} g(r) = +\infty$

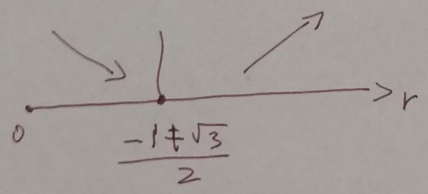
Inoltre possiamo analizzare il andamento di g calcolando $g'(r) = 2r - \frac{1}{1+r} > 0$ $(\Leftrightarrow)$

$\Leftrightarrow \quad 2r > \frac{1}{1+r} \quad \Leftrightarrow \quad$ (poiché $1+r>0$)

$\Leftrightarrow \begin{cases} 2r(1+r) > 1 \\ r \geq 0 \end{cases} \quad \Leftrightarrow \quad r \geq \frac{-1+\sqrt{3}}{2}$

Dunque g è decrescente su $\left[0, \frac{-1+\sqrt{3}}{2}\right]$

crescente su $\left[\frac{-1+\sqrt{3}}{2}, +\infty\right[$



Pertanto

$\min f = g\left(\frac{-1+\sqrt{3}}{2}\right)$

$= \left(\frac{-1+\sqrt{3}}{2}\right)^2 - \log\left(\frac{1+\sqrt{3}}{2}\right) = \min f$

e $\sup f = +\infty$ (non esiste $\max f$)