

A Frobenius-type theorem for singular Lipschitz distributions*

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Abstract

We prove a Frobenius-type theorem for singular distributions generated by a family of locally Lipschitz continuous vector fields satisfying almost everywhere a quantitative finite type condition.

1 Introduction and main result

Consider a family $\mathcal{P} := \{Y_1, \dots, Y_q\}$ of vector fields in \mathbb{R}^n . For all $x \in \mathbb{R}^n$ let $P_x := \text{span}\{Y_{1,x}, \dots, Y_{q,x}\}$, where $Y_{j,x} := Y_j(x)$ denotes the vector field Y_j evaluated at $x \in \mathbb{R}^n$.

As a consequence of the classical Frobenius theorem, it is known that if the vector fields are smooth, the rank $p_x := \dim P_x$ is constant in \mathbb{R}^n and if the family satisfies the *involutivity condition*

$$[Y_i, Y_j](x) \in P_x \quad \text{for all } x \in \mathbb{R}^n, i, j \in \{1, \dots, q\}, \quad (1.1)$$

then for all $x \in \mathbb{R}^n$ there is a smooth immersed submanifold M^x containing x and with $T_y M^x = P_y$ for all $y \in M^x$; see [Che46]. The result still holds if one removes the constant-rank assumption, but the involutivity assumption (1.1) does not suffice. Hermann [Her62] has shown that a sufficient condition for smooth distributions is the *finite-type involutivity condition*

$$[Y_i, Y_j] = \sum_{1 \leq k \leq q} c_{ij}^k Y_k \quad \text{for all } i, j \in \{1, \dots, q\}, \quad (1.2)$$

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where the coefficients c_{ij}^k are locally bounded.

In modern terms, a “maximal” choice the integral manifold M^x can be identified as the following *orbit* or *leaf*:

$$\mathcal{O}^x := \{y : d(x, y) < \infty\}, \quad (1.3)$$

where d denotes the Carnot–Carathéodory distance

$$d(x, y) := \inf \left\{ T \geq 0 : \exists \gamma \text{ subunit on } [0, T] \text{ with } \gamma(0) = x, \gamma(T) = y \right\}. \quad (1.4)$$

Recall that a path $\gamma : [0, T] \rightarrow \mathbb{R}^n$ is subunit if it is Lipschitz and satisfy for a.e. t the ODE $\dot{\gamma}(t) = \sum_{j=1}^q u_j(t) Y_j(\gamma(t))$ with $u \in L^\infty((0, T), \mathbb{R}^q)$ and $|u(t)| \leq 1$ for a.e. t . We agree that $d(x, y) = \infty$ if there are no subunit paths connecting x and y .

Generalizations of the Frobenius theorem hold for nonsmooth vector fields. Hartman [Har70], Simic [Sim96] and Rampazzo [Ram07] have shown that, given a constant-rank family of locally Lipschitz vector fields which satisfies (1.1) almost everywhere or as a set-valued commutator [RS07], then integral manifolds are $C^{1,1}$ smooth.

In applications it is sometimes necessary a Frobenius theorem for singular distributions, in which the dimension of the subspace P_x may vary with x . This happens for example when one considers the singular distribution associated with the Hamiltonian vector fields on a Poisson manifold. For a good account on this application we refer to [AMV04, Section 3.4] and [LM87, Appendix 3].

In this note, under an assumption similar to (1.2), we show that a regularity result for orbits of Lipschitz vector fields holds in the nonconstant rank case. Here is our main result. Below τ_d denotes the topology associated with the control distance d defined in (1.4). Let also for $x \in \mathbb{R}^n$ and $r > 0$, $P_x^r := \sum_{j=1}^q c_j Y_{j,x} : |c| \leq r$.

Theorem 1.1. *Let $\mathcal{P} := \{Y_1, \dots, Y_q\}$ be a family of locally Lipschitz continuous vector fields in \mathbb{R}^n . Write $Y_j =: g_j \cdot \nabla$ and assume that for any bounded open set $\Omega \subset \mathbb{R}^n$ there is $C = C_\Omega > 0$ such that*

$$[Y_j, Y_k]_x := (Y_j g_k(x) - Y_k g_j(x)) \cdot \nabla \in P_x^{C_\Omega} \quad \text{for a.e. } x \in \Omega. \quad (1.5)$$

Then for all $x \in \mathbb{R}^n$ the orbit \mathcal{O}^x with topology τ_d has a structure of connected $C^{1,1}$ immersed submanifold of \mathbb{R}^n with $T_y \mathcal{O}^x = P_y$ for all $y \in \mathcal{O}^x$.

Remark 1.2. Concerning the statement, note that assumption (1.5) is meaningful in view of the Rademacher theorem. Moreover note the following facts:

- (a) Our orbits are defined in a different way from the usual Sussmann's orbits. Indeed, the Sussmann's orbit associated with \mathcal{P} is the set of points in \mathbb{R}^n which are reachable from x via a piecewise integral curve of vector fields in \mathcal{P} . It is trivially $\mathcal{O}_{\text{Sussmann}}^x \subset \mathcal{O}^x$. It is known that $\mathcal{O}_{\text{Sussmann}}^x = \mathcal{O}^x$ in more regular situations. However, it is not known to the authors whether these sets agree if the vector fields are only Lipschitz continuous.
- (b) If one removes the quantitative assumption (1.5), the statement of Theorem 1.1 fails. This can be seen in the well known example $\mathcal{P} = \{\partial_1, \exp(-1/x_1^2)\partial_2\}$, where $\mathcal{O}^x = \mathbb{R}^2$ for all choices of x . Therefore, equality $T_y\mathcal{O}^x = P_y$ fails at any y in the x_2 -axis. Note that in such example (1.1) holds, but (1.5) is not satisfied.
- (c) One could ask whether or not any orbit \mathcal{O} of a finite family \mathcal{P} of locally Lipschitz vector fields is a submanifold (without requiring that $T_y\mathcal{O}^x = P_y$ for all $y \in \mathcal{O}$). In the smooth case this is true by Sussmann's theorem [Sus73]. We do not know whether a version of Sussmann's theorem for Lipschitz vector fields holds.
- (d) In the constant rank case, orbits are $C^{1,1}$ leaves of a foliation, [Ram07, HT07].
¹ In our case this is in general not possible, because the dimension of different orbits can be different. See Example 1.3.
- (e) Given a family $\mathcal{P} = \{Y_1, \dots, Y_q\}$ of vector fields, condition (1.5) is not necessary to have the conclusions of Theorem 1.1. See Example 1.4 below.
- (f) In the constant rank case, Rampazzo [Ram07] proves the following statement. Let $P = \cup_{x \in \mathbb{R}^n} P_x$ be a Lipschitz distribution.² Let Ω be an open set and let $X, Y \in \text{Lip}(\Omega, \mathbb{R}^n)$ be vector fields tangent to P at any point of Ω . Then, for almost all $x \in \Omega$ we have $[X, Y](x) \in P_x$. We do not have a proof of such statement for singular distributions.

Example 1.3. Let $F(x_1) := x_1|x_1|$ and $f(x_1) := F'(x_1) = 2|x_1|$. Define $Y_1 := \partial_1 + f(x_1)\partial_2$ and $Y_2 := |x_2 - F(x_1)|\partial_2$. The vector fields Y_1, Y_2 are Lipschitz and satisfy (1.5) because $[Y_1, Y_2] = 0$ a.e. We have $\mathcal{O}^{(0,0)} = \{x : x_2 = F(x_1)\}$ is a one dimensional $C^{1,1}$ graph. For any $\tilde{x} \in \mathbb{R}^2$ with $\tilde{x}_2 > F(\tilde{x}_1)$ we have $\mathcal{O}^{\tilde{x}} = \{x : x_2 > F(x_1)\}$. Finally, if $\tilde{x}_2 < F(\tilde{x}_1)$, we have $\mathcal{O}^{\tilde{x}} = \{x : x_2 < F(x_1)\}$.

The following example shows that, already in the setting of smooth vector fields, the finite type involutivity condition (1.5) is not a property of the distribution, but it concerns the spanning family.

¹The paper [HT07] contains some interesting applications of the Frobenius theorem to foliation properties for rough Levi flat CR manifolds.

²I.e. (compare [Sus08]), P_x is a subspace of $T_x\mathbb{R}^n$ for each x and moreover for any $x \in \mathbb{R}^n$ and for each $v \in P_x$ there is a neighborhood U of x in \mathbb{R}^n and $X \in \text{Lip}(U, \mathbb{R}^n)$ such that $X(x) = v$ and $X(y) \in P_y$ for all $y \in U$.

Example 1.4. ³ Let $x = (x_1, x_2) \in \mathbb{R}^2$. Let $\mathcal{P} = \{Y_1, Y_2\}$ where

$$Y_1 = e^{-1/|x|^2} \partial_1 \quad \text{and} \quad Y_2 = |x|^2 \partial_2.$$

We have $\mathcal{O}^0 = \{0\}$ and $\mathcal{O}^x = \mathbb{R}^2 \setminus \{0\}$ for all $x \in \mathbb{R}^2 \setminus \{0\}$. Note that $T_y(\mathbb{R}^2 \setminus \{0\}) = P_y$ for all $y \in \mathbb{R}^2 \setminus \{0\}$. On the other side, observe that we can write uniquely for all $x \neq 0$,

$$[Y_1, Y_2] = \frac{2x_2}{|x|^2} Y_1 + \frac{2x_1 e^{-1/|x|^2}}{|x|^2} Y_2.$$

Since the function $x \mapsto \frac{2x_2}{|x|^2}$ is unbounded in any neighborhood of the origin, we conclude that (1.5) does not hold. Observe finally that this phenomenon does not occur if we change the family with the (analytic) family $\{|x|^2 \partial_1, |x|^2 \partial_2\}$.

Before closing this introduction we briefly describe the proof. As already appeared in the paper [Her62], the main step is to show that the rank p_y is constant as y belongs to a fixed orbit \mathcal{O}^x . This can be done in the smooth case by rectifying one of the vector fields and by an ODE argument (see [Her62]). This procedure is not available under our low regularity assumptions. However we are able to show constancy of the rank along orbits by an approximation argument which involves Euclidean mollification of the original vector fields and differentiation of suitable wedge products along a given flow. This argument is both a curvilinear and nonsmooth version of the original one in [Her62] and relies on some differential formulas first derived in [NSW85] and improved in [Str11] and [MM12a]. This is achieved in subsection 2.1.

After establishing the constancy of the rank along a given orbit \mathcal{O} , we will need to construct local $C^{1,1}$ coordinates on \mathcal{O} . Here the classical idea is to construct new vector fields which span the same distribution, but whose flows commute (see [Her77]).⁴ This can be done in the constant rank case in a full Euclidean neighborhood of any fixed point. In the singular case, this construction can be done only in a d -neighborhood of a given point and it is not clear whether or not it can be extended in any Euclidean neighborhood. Therefore, since we are working with locally Lipschitz vector fields, the commutativity argument of [RS07, Theorem 5.3] does not work. We need again to work with the smooth approximations of the vector fields. This involves a careful analysis of how the integrability condition (1.5) behaves under mollifications; see subsection 2.2.

³See the unpublished version of [Bal94] available at <http://www2.math.umd.edu/~rvbalan/PAPERS/MyPapers/distrib.pdf>.

⁴It seems that this commutativity argument appears in the original Clebsch's proof, which is prior to Frobenius' one, see the historical paper [Haw05, Theorem 5.3].

2 Proof of the main result

Notation. Our notation here are similar to [MM12a,MM11]. Let $\mathcal{P} := \{Y_1, \dots, Y_q\}$ be a family of locally Lipschitz continuous vector fields. Write $Y_j =: g_j \cdot \nabla$ and define for any $p, \mu \in \mathbb{N}$, with $1 \leq p \leq \mu$,

$$\mathcal{I}(p, \mu) := \{I = (i_1, \dots, i_p) : 1 \leq i_1 < i_2 < \dots < i_p \leq \mu\}.$$

Define $p_x := \dim \text{span}\{Y_{j,x} : 1 \leq j \leq q\}$. Obviously, $p_x \leq \min\{n, q\}$. Then for any $p \in \{1, \dots, \min\{n, q\}\}$, let

$$Y_I(x) := Y_{I,x} := Y_{i_1,x} \wedge \dots \wedge Y_{i_p,x} \in \bigwedge_p T_x \mathbb{R}^n \sim \bigwedge_p \mathbb{R}^n \quad \text{for all } I \in \mathcal{I}(p, q)$$

and, for all $K \in \mathcal{I}(p, n)$ and $I \in \mathcal{I}(p, q)$

$$Y_I^K(x) := dx^K(Y_{i_1}, \dots, Y_{i_p})(x) := \det \begin{bmatrix} g_{i_1}^{k_1}(x) & \dots & g_{i_p}^{k_1}(x) \\ \vdots & & \vdots \\ g_{i_1}^{k_p}(x) & \dots & g_{i_p}^{k_p}(x) \end{bmatrix}.$$

Here we let $dx^K := dx^{k_1} \wedge \dots \wedge dx^{k_p}$ for any $K = (k_1, \dots, k_p) \in \mathcal{I}(p, n)$.

Let e_1, \dots, e_n be the canonical basis of \mathbb{R}^n . The family $e_K := e_{k_1} \wedge \dots \wedge e_{k_p}$, where $K \in \mathcal{I}(p, n)$, gives an orthonormal basis of $\bigwedge_p \mathbb{R}^n$. Then we have the orthogonal decomposition $Y_I(x) = \sum_K Y_I^K(x) e_K \in \bigwedge_p \mathbb{R}^n$.

Consider the linear system $\sum_{k=1}^p \xi^k Y_{i_k} = W$ for some $W \in \text{span}\{Y_{i_1}, \dots, Y_{i_p}\}$. If $|Y_I| \neq 0$, then the Cramer's rule gives the unique solution

$$\xi^k = \frac{\langle Y_I, \iota^k(W) Y_I \rangle}{|Y_I|^2} \quad \text{for each } k = 1, \dots, p, \quad (2.1)$$

where we let $\iota^k(W) Y_I := Y_{(i_1, \dots, i_{k-1})} \wedge W \wedge Y_{(i_{k+1}, \dots, i_p)}$.

Finally, for $p \in \{1, \dots, \min\{n, q\}\}$, introduce the vector-valued function

$$\Lambda_p(x) := (Y_J^K(x))_{J \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)}.$$

Note that $|\Lambda_p(x)|^2 = \sum_{I \in \mathcal{I}(p, q)} |Y_I(x)|^2 = \sum_{I \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)} Y_I^K(x)^2$.

2.1 Invariance of the dimension p_x on the orbit

Write $Y_j =: g_j \cdot \nabla$. For each $x \in \mathbb{R}^n$, let $P_x := \text{span}\{Y_{1,x}, \dots, Y_{q,x}\}$ and for all $r > 0$ let $P_x^r := \{\sum_{1 \leq j \leq q} c_j Y_{j,x} : |c| \leq r\}$. Finally define $p_x := \dim \text{span}\{Y_{j,x}\}$. Let \mathcal{O}^x be the orbit of the family \mathcal{P} containing x , see (1.3). Equip \mathcal{O} with the topology τ_d .

Theorem 2.1. *Let $Y_j \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$ and assume that for any bounded open set $\Omega \subset \mathbb{R}^n$ there is $C = C_\Omega > 0$ such that*

$$[Y_j, Y_k]_x := (Y_j g_k(x) - Y_k g_j(x)) \cdot \nabla \in P_x^{C_\Omega} \quad \text{for a.e. } x \in \Omega. \quad (2.2)$$

Then, for all $x_0 \in \mathbb{R}^n$, we have $p_x = p_{x_0}$ for all $x \in \mathcal{O}_P^{x_0}$.

Concerning formula (2.2), note that $Y_j g_k(x) = g_j(x) \cdot \nabla g_k(x)$ exists for almost all $x \in \mathbb{R}^n$ by the Rademacher theorem.

In order to prove Theorem 2.1 we start with a measurability lemma.

Lemma 2.2. *Assume that (1.5) holds. Then there are measurable functions $c_{jk}^i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$[Y_j, Y_k]_x = \sum_{1 \leq i \leq q} c_{jk}^i(x) Y_{i,x} \quad \text{for almost all } x \in \mathbb{R}^n \text{ and all } j, k = 1, \dots, q, \quad (2.3)$$

where $\|c_{jk}\|_{L^\infty(\Omega)} \leq C_\Omega$ for all $j, k \in \{1, \dots, q\}$.

Proof. For all $x \in \mathbb{R}^n$ let $Y_x := [Y_{1,x}, \dots, Y_{q,x}] \in \mathbb{R}^{n \times q}$. Let Y_x^\dagger be its Moore–Penrose inverse. Therefore, at any differentiability point x of both g_j and g_k , the vector $c_{jk}(x) := Y_x^\dagger (Y_j g_k(x) - Y_k g_j(x)) \in \mathbb{R}^q$ is the least-norm solution of the system $Y_x \xi = Y_j g_k(x) - Y_k g_j(x)$, with $\xi \in \mathbb{R}^q$. Therefore $|c_{jk}(x)| \leq C_\Omega$ for a.e. $x \in \Omega$. Measurability follows from the approximation formula

$$Y_x^\dagger (Y_j g_k - Y_k g_j)(x) = \lim_{\delta \rightarrow 0} (\delta I + Y_x^T Y_x)^{-1} Y_x^T (Y_j g_k - Y_k g_j)(x).$$

(see the appendix of [MM12b]). The proof is concluded. \square

Recall now some properties of mollifiers from [RS07]. For any $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, let $f^{(\sigma)}(x) := \int_{\mathbb{R}^n} f(x - \sigma y) \chi(y) dy$, where $\chi \geq 0$ is a smooth averaging kernel supported in the unit ball. Let also $Y_j^{(\sigma)} := g_j^{(\sigma)} \cdot \nabla$ be the smooth approximation of the vector field Y_j . Denote for all $p \in \{1, \dots, q\}$ and for all $J \in \mathcal{I}(p, q)$, $Y_J^\sigma := Y_{j_1}^\sigma \wedge \dots \wedge Y_{j_p}^\sigma$. Note that in general $Y_J^\sigma \neq Y_J^{(\sigma)}$, unless $J \in \mathcal{I}(1, q)$.

Let now $\Omega_0 \subset\subset \Omega_1$ be bounded open sets contained in \mathbb{R}^n . Then, since the vector fields are locally Lipschitz, there are $\tilde{\sigma} = \tilde{\sigma}(\Omega_0, \Omega_1)$ and $C > 0$ such that $\sup_{\Omega_0} |Y_J^\sigma - Y_J^{(\sigma)}| \leq C\sigma$, for all $\sigma < \tilde{\sigma}$ for some C depending on $\|Y_j\|_{C^{0,1}(\Omega_1)}$.

Recall the following Friedrichs-type commutator estimate (see [RS07, Lemma 4.5]). For all $\sigma \leq \tilde{\sigma}$, we can write on Ω_0

$$[Y_j^{(\sigma)}, Y_k^{(\sigma)}] = [Y_j, Y_k]^{(\sigma)} + \sigma b_{jk}^\sigma \cdot \nabla,$$

where the smooth functions b_{jk}^σ satisfy $\sup_{\Omega_0} |b_{jk}^\sigma| \leq C$ for some C depending on $\|Y_j\|_{C^{0,1}(\Omega_1)}$, $\|Y_k\|_{C^{0,1}(\Omega_1)}$ and for all $\sigma \leq \tilde{\sigma}$. Then, using (2.3), we get, for possibly

different functions b_{jk}^σ , also depending on the constant C_{Ω_1} in the assumptions of Theorem 2.1

$$[Y_j^{(\sigma)}, Y_k^{(\sigma)}] = \sum_{1 \leq i \leq q} (c_{jk}^i)^{(\sigma)} Y_i^\sigma + \sigma b_{jk}^\sigma \cdot \nabla. \quad (2.4)$$

Here c_{jk}^i are the measurable functions appearing in Lemma 2.2 and the functions b_{jk}^σ are smooth and uniformly bounded as $\sigma \rightarrow 0$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then there is $C > 0$ such that for all $x \in \Omega$, for all subunit path γ with $\gamma(0) = x$ and for all $p \in \{1, \dots, \min\{q, n\}\}$, we have*

$$|\Lambda_p(\gamma(t)) - \Lambda_p(x)| \leq |\Lambda_p(x)|(e^{Ct} - 1) \quad \text{for all } t \in [0, C^{-1}]. \quad (2.5)$$

In particular, for each $p \in \{1, \dots, q\}$ and $I \in \mathcal{I}(p, q)$, we have

$$|Y_I(\gamma(t)) - Y_I(x)| \leq C |\Lambda_p(x)|(e^{Ct} - 1) \quad \text{for all } t \in [0, C^{-1}]. \quad (2.6)$$

Remark 2.4. Let $x \in \Omega$, $p = p_x$, $I \in \mathcal{I}(p_x, q)$ and $\eta \in (0, 1)$ be such that

$$|Y_I(x)| > \eta \max_{K \in \mathcal{I}(p_x, q)} |Y_K(x)|.$$

Then

$$|Y_I(\gamma(t)) - Y_I(x)| \leq C \frac{t}{\eta} |Y_I(x)| \quad \text{for all } t \in [0, C^{-1}]. \quad (2.7)$$

An immediate consequence of Theorem 2.3 is the following corollary.

Corollary 2.5. *For all $x_0 \in \mathbb{R}^n$, the number $p_x := \dim \text{span}\{Y_{1,x}, \dots, Y_{q,x}\}$ is constant if $x \in \mathcal{O}^{x_0}$.*

To prove Theorem 2.3 and Corollary 2.5 we need the following lemma.

Lemma 2.6. *Let $p \leq n$ and let U_1, \dots, U_p and $X = \sum_{\alpha=1}^n f^\alpha \partial_\alpha$ be smooth vector fields in \mathbb{R}^n . For any $K = (k_1, \dots, k_p) \in \mathcal{I}(p, n)$, we have*

$$\begin{aligned} X(dx^K(U_1, \dots, U_p)) &= \sum_{1 \leq \alpha \leq p} dx^K(U_1, \dots, U_{\alpha-1}, [X, U_\alpha], U_{\alpha+1}, \dots, U_p) \\ &\quad + \sum_{1 \leq \gamma \leq n} \sum_{1 \leq \beta \leq p} \partial_\gamma f^{k_\beta} dx^{(k_1, \dots, k_{\beta-1}, \gamma, k_{\beta+1}, \dots, k_p)}(U_1, \dots, U_p). \end{aligned}$$

Proof of Lemma 2.6. See [Str11] or [MM12a]. □

Proof of Theorem 2.3. Fix two bounded open sets $\Omega_0 \subset\subset \Omega_1 \subset \mathbb{R}^n$ such that $\Omega \subset\subset \Omega_0$. Let $x \in \Omega$ and take a subunit path γ such that $\gamma(0) = x$. Note that γ is the a.e. solution of a problem of the form $\dot{\gamma} = \sum_{j=1}^q u_j(t) Y_j(\gamma)$ where $\gamma(0) = x$ and $u \in L^\infty((0, 1), \mathbb{R}^q)$ satisfies $|u(t)| \leq 1$, a.e. We can approximate γ with a path γ^σ obtained as the solution of the problem

$$\dot{\gamma}^\sigma = \sum_{1 \leq j \leq q} u_j(t) Y_j^\sigma(\gamma^\sigma) \quad \text{a.e., with } \gamma^\sigma(0) = x.$$

Since the vector fields are locally Lipschitz, there is C depending on Ω , Ω_0 and on the Lipschitz constants of the vector fields such that $\gamma^\sigma(t) \in \Omega_0$ for all $t \in [0, C^{-1}]$. Recall by (2.4) that for all $Z \in \pm\mathcal{P}$, we may write $[Z^\sigma, Y_j^\sigma] = \sum (c_j^i)^\sigma Y_i^\sigma + \sigma b_j^\sigma \cdot \nabla$ for suitable functions $(c_j^i)^\sigma$ and b_j^σ smooth for all $\sigma > 0$ and bounded uniformly in σ . Namely, there are C and $\tilde{\sigma}$ depending on the choice of Ω_0 and Ω_1 such that we have

$$\sup_{\Omega_0} (|(c_j^i)^\sigma| + |b_j^\sigma|) \leq C \quad \text{for all } \sigma \leq \tilde{\sigma}.$$

Fix $J \in \mathcal{I}(p, q)$ and $K \in \mathcal{I}(p, n)$. Note that each function $t \mapsto Y_h^\sigma(\gamma^\sigma(t))$ is Lipschitz continuous. At any differentiability point t of γ^σ we have $\frac{d}{dt} dx^K(Y_J(\gamma_t^\sigma)) = \sum_{h=1}^q u_h(t) Z_h^\sigma(dx^K(Y_J^\sigma))(\gamma_t^\sigma)$, where $|u_j(t)| \leq 1$. By Lemma 2.6 we get

$$\begin{aligned} Z_h^\sigma(dx^K(Y_J^\sigma))(\gamma_t^\sigma) &= \sum_{\alpha=1}^p \sum_{i=1}^q (c_{j_\alpha}^i)^\sigma(\gamma_t^\sigma) dx^K(\dots, Y_{j_{\alpha-1}}^\sigma, Y_i^\sigma, Y_{j_{\alpha+1}}^\sigma, \dots, Y_{j_p}^\sigma)(\gamma_t^\sigma) \\ &\quad + \sum_{\alpha=1}^p \sigma dx^K(\dots, Y_{j_{\alpha-1}}^\sigma(\gamma_t^\sigma), b_{j_\alpha}^\sigma(\gamma_t^\sigma) \cdot \nabla, Y_{j_{\alpha+1}}^\sigma(\gamma_t^\sigma), \dots) \\ &\quad + \sum_{\gamma=1}^n \sum_{\beta=1}^p \partial_\gamma (g^{k_\beta})^\sigma dx^{(k_1, \dots, k_{\beta-1})} \wedge dx^\gamma \wedge dx^{(k_{\beta+1}, \dots, k_p)}(Y_J^\sigma)(\gamma_t^\sigma) \\ &=: A_1^\sigma + A_2^\sigma + A_3^\sigma. \end{aligned}$$

Denote $\Lambda_p^\sigma(y) := (dx^K Y_J^\sigma(y))_{J \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)}$. Since $(c_{j_\alpha}^i)^\sigma$, $\partial_\gamma g^{k_\beta}(\sigma)$ and $b_{j_\alpha}^\sigma$ are uniformly bounded as σ tends to 0, we get $|A_1^\sigma| + |A_2^\sigma| + |A_3^\sigma| \leq C_1 |\Lambda_p^\sigma(\gamma_t^\sigma)| + \sigma C_2$. This gives for a.e. t

$$\left| \frac{d}{dt} dx^K(Y_J^\sigma(\gamma_t^\sigma)) \right| \leq C_1 |\Lambda_p^\sigma(\gamma_t^\sigma)| + \sigma C_2 \quad \text{for all } J \in \mathcal{I}(p, q) \ K \in \mathcal{I}(p, n). \quad (2.8)$$

Therefore,

$$\begin{aligned} \left| \frac{d}{dt} \Lambda_p^\sigma(\gamma_t^\sigma) \right| &= \left| \left(\frac{d}{dt} \Lambda_p^\sigma(\gamma_t^\sigma) \right)_{J \in \mathcal{I}(p, q), K \in \mathcal{I}(p, n)} \right| \\ &\leq C_1 |\Lambda_p^\sigma(\gamma_t^\sigma)| + C_2 \sigma. \end{aligned} \quad (2.9)$$

Integrating (2.9) and using the Gronwall inequality⁵ we obtain

$$|\Lambda_p^\sigma(\gamma_t^\sigma) - \Lambda_p^\sigma(x)| \leq \left(|\Lambda_p^\sigma(x)| + \frac{C_2}{C_1} \sigma \right) (e^{C_1 t} - 1)$$

for all $t \in [0, C^{-1}]$. Therefore, as $\sigma \rightarrow 0$, we get the conclusion

$$|\Lambda_p(\gamma_t) - \Lambda_p(x)| \leq |\Lambda_p(x)| (e^{C_1 t} - 1) \quad \text{for all } t \in [0, C^{-1}].$$

Estimate (2.6) follows trivially. The proof is concluded. \square

Proof of Corollary 2.5. Let $p \in \{1, \dots, \min\{q, n\}\}$. Let $I \subset \mathbb{R}$ be an interval. Let $\gamma : I \rightarrow \mathbb{R}$ be a subunit path. Let $A_p := \{t \in I : |\Lambda_p(\gamma(t))| = 0\}$. We claim that A_p is open and closed. This will imply that either $A_p = \emptyset$ or $A_p = I$. Then the proof is concluded.

To show the claim, note that the set is closed because it is the zero set of the continuous function $I \ni t \mapsto |\Lambda_p(\gamma(t))| \in \mathbb{R}$. It is open as an easy consequence of estimate (2.5). \square

2.2 Manifold structure of orbits

In order to show our main theorem, we start with the following commutativity lemma, which generalizes to the nonconstant rank the results in [RS07, Ram07].

Let $\mathcal{P} = \{Y_1, \dots, Y_q\}$ be our family of vector fields and let $\mathcal{O} = \mathcal{O}_{\mathcal{P}}^x$ be a given orbit. We denote d -balls by $B_d(x, r)$ and Euclidean balls by $B(x, r)$. Recall that the distance d equips the orbit with a natural topology τ_d which makes \mathcal{O} connected.

Lemma 2.7. *Assume that the hypotheses of Theorem 1.1 hold. Let $x_0 \in \mathbb{R}^n$ and let $p := p_{x_0}$. Take also $I \in \mathcal{I}(p_{x_0}, q)$ such that $|Y_I(x)| \neq 0$. Then there exist $\varepsilon, \delta > 0$, a neighborhood $U_{x_0} := B_d(x_0, \delta) \subset B(x_0, \varepsilon) \cap \mathcal{O}$ of x_0 in the orbit distance d and a map $\beta \in \text{Lip}(B(x_0, \varepsilon), \mathbb{R}^{p \times p})$ such that, letting*

$$V_j := \sum_{1 \leq k \leq p} \beta_j^k Y_{i_k} \quad \text{for all } j = 1, \dots, p, \quad (2.11)$$

we have

$$\begin{aligned} \text{span}\{V_{1,x}, \dots, V_{p,x}\} &= \text{span}\{Y_{i_1,x}, \dots, Y_{i_p,x}\} \\ &= \text{span}\{Y_{1,x}, \dots, Y_{p,x}, \dots, Y_{q,x}\} \end{aligned} \quad (2.12)$$

⁵for all $a \geq 0$, $b > 0$, $T > 0$ and f continuous on $[0, T]$,

$$0 \leq f(t) \leq at + b \int_0^t f(\tau) d\tau \quad \text{on } t \in [0, T] \quad \Rightarrow \quad f(t) \leq \frac{a}{b} (e^{bt} - 1) \quad \text{on } t \in [0, T]. \quad (2.10)$$

for all $x \in U_{x_0}$. Furthermore, for all $j, k \in \{1, \dots, p\}$, for all $x \in U_{x_0}$ and for all small $|t_j|, |t_k|$, we have the commutativity formula

$$e^{-t_j V_j} e^{-t_k V_k} e^{t_j V_j} e^{t_k V_k} x = x. \quad (2.13)$$

Proof. Let $x_0 \in \mathbb{R}^n$. Let $p = p_{x_0}$ and assume without loss of generality that $(i_1, \dots, i_p) = (1, \dots, p)$, i.e. that $Y_1(x_0), \dots, Y_p(x_0)$ are independent. Recall notation $Y_j := \sum_{\alpha=1}^n g_j^\alpha \partial_\alpha$. Up to reordering coordinates, we may as well assume that the matrix $(g_j^k(x_0))_{j,k=1,\dots,p}$ is nonsingular.

Define for $\sigma \geq 0$ the functions $\beta_j^{k,\sigma} \in \text{Lip}(B(x_0, \varepsilon), \mathbb{R})$ such that

$$\sum_{1 \leq k \leq p} \beta_i^{k,\sigma}(x) g_k^{\ell,\sigma}(x) = \delta_i^\ell \quad (2.14)$$

for all $i, \ell = 1, \dots, p$. Here we let $g_k^{\ell,\sigma} := (g_k^\ell)^{(\sigma)}$. The functions $\beta_i^{k,\sigma}$ are uniquely defined and, if ε is sufficiently small, their Lipschitz constants on $B(x_0, \varepsilon)$ are uniform, as $0 \leq \sigma \leq \tilde{\sigma}$ for some $\tilde{\sigma} > 0$.

Define for all $x \in B(x_0, \varepsilon)$, $\ell = 1, \dots, p$ and $0 \leq \sigma \leq \tilde{\sigma}$,

$$V_{\ell,x}^\sigma := \sum_{1 \leq k \leq p} \beta_\ell^{k,\sigma}(x) Y_{k,x}^\sigma =: \partial_\ell + \sum_{p+1 \leq i \leq n} \varphi_\ell^{i,\sigma}(x) \partial_i,$$

where for $\ell \leq p$ and $i \geq p+1$ we defined $\varphi_\ell^{i,\sigma} = \sum_{k=1}^p \beta_\ell^{k,\sigma} g_k^{i,\sigma}$. Note that $\dim \text{span}\{Y_j^\sigma(x_0) : 1 \leq j \leq q\} \geq \dim \text{span}\{Y_j(x_0) : 1 \leq j \leq q\}$ and inequality can be strict. Observe that, at the level $\sigma = 0$ we have, by Corollary 2.5 and Remark 2.4,

$$\text{span}\{V_{1,x}, \dots, V_{p,x}\} = \text{span}\{Y_{1,x}, \dots, Y_{p,x}\} = \text{span}\{Y_{1,x}, \dots, Y_{p,x}, \dots, Y_{q,x}\} \quad (2.15)$$

for all $x \in U_{x_0}$, where $U_{x_0} := B_d(x_0, \delta) \subset \mathcal{O} \cap B(x_0, \varepsilon)$ is a small d -ball centered at x_0 . Then (2.12) is proved. Note that the second equality of (2.15) could be false on any Euclidean neighborhood of x_0 . We can only claim that it holds if x belongs to a suitable d -neighborhood of x_0 . Note that the topology defined by d can be strictly stronger than the Euclidean one. This phenomenon makes inapplicable the arguments of [RS07, Theorem 5.3] to show the commutativity formula (2.13).

In order to show the commutativity formula (2.13), first observe that

$$[V_j^\sigma, V_k^\sigma]_x \in \text{span}\{\partial_{p+1}, \dots, \partial_n\} \quad \text{for all } j, k \in \{1, \dots, p\} \quad x \in B(x_0, \varepsilon). \quad (2.16)$$

Let $x \in U_{x_0} = B_d(x_0, \delta)$. Take numbers t, r with $|t|, |r|$ small and start from the

following formula, see [RS07, Section 4]

$$\begin{aligned}
& e^{-tV_j^\sigma} e^{-rV_i^\sigma} e^{tV_j^\sigma} e^{rV_i^\sigma} x - x \\
&= \int_0^t d\tau \int_0^r d\varrho [V_i^\sigma, V_j^\sigma] (e^{-\tau V_j^\sigma} e^{-\varrho V_i^\sigma}) (e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x) \\
&= \int_0^t d\tau \int_0^r d\varrho \left\langle [V_i^\sigma, V_j^\sigma] (e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x), \nabla (e^{-\tau V_j^\sigma} e^{-\varrho V_i^\sigma}) (e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x) \right\rangle.
\end{aligned} \tag{2.17}$$

Note that the term with Euclidean gradient is uniformly bounded, i.e.

$$\left| \nabla (e^{-\tau V_j^\sigma} e^{-\varrho V_i^\sigma}) (e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x) \right| \leq C$$

as soon as x is close to x_0 and the positive numbers $|\varrho|, |\tau|$ and σ are sufficiently small.

To prove formula (2.13), it suffices to show that, if $d(x, x_0)$, $|t|$ and $|r|$ are sufficiently small, then

$$[V_i^\sigma, V_j^\sigma] (e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x) \rightarrow 0 \quad \text{as } \sigma \rightarrow 0, \tag{2.18}$$

uniformly in the variables ϱ, τ . Write $\gamma^\sigma := e^{(\varrho-r)V_i^\sigma} e^{\tau V_j^\sigma} e^{rV_i^\sigma} x$ and $\gamma := e^{(\varrho-r)V_i} e^{\tau V_j} e^{rV_i} x$. Note that to show (2.18), we shall use the integrability condition (2.4).

Write first, by (2.4), at any point of the Euclidean ball $B(x_0, \varepsilon)$

$$\begin{aligned}
[V_i^\sigma, V_j^\sigma] &= \sum_{h,k=1}^p [\beta_i^{k,\sigma} Y_k^\sigma, \beta_j^{h,\sigma} Y_h^\sigma] \\
&= \sum_{h,k \leq p} (\beta_i^{k,\sigma} Y_k^\sigma \beta_j^{h,\sigma} - \beta_j^{h,\sigma} Y_k^\sigma \beta_i^{k,\sigma}) Y_h^\sigma + \sum_{h,k \leq p} \beta_i^{k,\sigma} \beta_j^{h,\sigma} \left\{ \sum_{1 \leq s \leq q} (c_{kh}^s)^{(\sigma)} Y_s^\sigma + \sigma b_{kh}^\sigma \cdot \nabla \right\} \\
&= \sum_{h,k,\nu \leq p} (\beta_i^{k,\sigma} Y_k^\sigma \beta_j^{h,\sigma} - \beta_j^{h,\sigma} Y_k^\sigma \beta_i^{k,\sigma}) g_h^{\nu,\sigma} V_\nu^\sigma + \sum_{h,k,s,\nu=1}^p \beta_i^{k,\sigma} \beta_j^{h,\sigma} c_{hk}^{s,(\sigma)} g_s^{\nu,\sigma} V_\nu^\sigma \\
&\quad + \sum_{h,k \leq p} \sum_{s \geq p+1} \beta_i^{k,\sigma} \beta_j^{h,\sigma} c_{hk}^{s,(\sigma)} Y_s^\sigma + \sigma \sum_{h,k \leq p} \beta_i^{k,\sigma} \beta_j^{h,\sigma} b_{kh}^\sigma \cdot \nabla \\
&=: \sum_{\nu \leq p} A_{ij}^{\nu,\sigma} V_\nu^\sigma + \sum_{s \geq p+1} B_{ij}^{s,\sigma} Y_s^\sigma + \sigma C_{ij}^\sigma \cdot \nabla,
\end{aligned} \tag{2.19}$$

We must write this equality at the point γ^σ and then let $\sigma \rightarrow 0$. Note that the terms $Y_k^\sigma \beta_i^{h,\sigma}$ and $c_{h,k}^{s,(\sigma)}$ are mollifiers of L_{loc}^∞ functions. Then we can not establish at this stage their behavior as $\sigma \rightarrow 0$. However by (2.16), we know that the projection of the left hand-side along ∂_ℓ vanishes for all $\ell = 1, \dots, p$.

We claim now that (2.19) can be in fact written at the point γ^σ in the form

$$[V_i^\sigma, V_j^\sigma]_{\gamma^\sigma} = \sum_{k \leq p} \sigma M_{ij}^{k,\sigma} V_{k,\gamma} + \sigma N_{ij}^\sigma \cdot \nabla, \quad (2.20)$$

where both $M_{ij}^{k,\sigma}$ and N_{ij}^σ are bounded uniformly, as $\sigma \rightarrow 0$.⁶ This shows (2.18). Then (2.13) follows and the proof of the lemma is concluded.

To prove the claim (2.20), observe first that since the vector fields $Y_j = g_j \cdot \nabla$ are locally Lipschitz, we know that $|\gamma^\sigma - \gamma| \leq C\sigma$. Therefore, we have

$$\begin{aligned} |g_s^\sigma(\gamma^\sigma) - g_s(\gamma)| &\leq C\sigma \quad \text{for } 1 \leq s \leq q, \\ |\beta_i^{k,\sigma}(\gamma^\sigma) - \beta_i^k(\gamma)| &\leq C\sigma \quad \text{for } i, k \in \{1, \dots, p\}, \end{aligned} \quad (2.21)$$

where we used the fact that the functions β_i^k are Lipschitz in some neighborhood of x_0 . Moreover, since $d(\gamma, x_0) \leq \delta + C(|r| + |t|)$ is small, in view of (2.15) we can write for $\sigma = 0$ by the Cramer's rule (2.1)

$$Y_{s,\gamma} = \sum_{\nu=1}^p \frac{\langle Y_{I,\gamma}, \iota^\nu(Y_{s,\gamma}) Y_{I,\gamma} \rangle}{|Y_{I,\gamma}|^2} Y_{\nu,\gamma} \quad \text{for all } s \in \{1, \dots, q\} \quad (2.22)$$

(formula (2.22) is nontrivial only if $s \geq p+1$). The ratio is bounded by Remark 2.4. Note also that calculating (2.19) at the point γ^σ , we get

$$[V_i^\sigma, V_j^\sigma]_{\gamma^\sigma} = \sum_{k \leq p} A_{ij}^{k,\sigma}(\gamma^\sigma) V_{k,\gamma^\sigma} + \sum_{\ell \geq p+1} B_{ij}^{\ell,\sigma}(\gamma^\sigma) Y_{\ell,\gamma^\sigma} + \sigma C_{ij}^\sigma(\gamma^\sigma) \cdot \nabla, \quad (2.23)$$

where $|A_{ij}^{k,\sigma}(\gamma^\sigma)|, |B_{ij}^{\ell,\sigma}(\gamma^\sigma)|, |C_{ij}^\sigma(\gamma^\sigma)| \leq C$ for some C depending on $I, x_0, x, |t|$ and $|r|$, but uniformly bounded as $\sigma \rightarrow 0$.

Next write for $k \leq p$, $V_{k,\gamma^\sigma} = V_{k,\gamma} + \sigma D_k^\sigma \cdot \nabla$, where D_k^σ are uniformly bounded by (2.21). Moreover, for $\ell \geq p+1$, using (2.21) and (2.22) write

$$\begin{aligned} Y_{\ell,\gamma^\sigma} &=: Y_{\ell,\gamma} + \sigma E_\ell^\sigma \cdot \nabla = \sum_{\nu \leq p} \frac{\langle Y_{I,\gamma}, \iota^\nu(Y_{\ell,\gamma}) Y_{I,\gamma} \rangle}{|Y_{I,\gamma}|^2} Y_{\nu,\gamma} + \sigma E_\ell^\sigma \cdot \nabla \\ &= \sum_{\nu, k \leq p} \frac{\langle Y_{I,\gamma}, \iota^\nu(Y_{\ell,\gamma}) Y_{I,\gamma} \rangle}{|Y_{I,\gamma}|^2} g_\nu^k(\gamma) V_{k,\gamma} + \sigma E_\ell^\sigma \cdot \nabla. \end{aligned}$$

Here E_ℓ^σ is bounded by (2.21). Inserting into (2.23) this gives

$$[V_i^\sigma, V_j^\sigma]_{\gamma^\sigma} = \sum_{k \leq p} A_{ij}^{k,\sigma}(\gamma^\sigma) (V_{k,\gamma} + \sigma D_k^\sigma \cdot \nabla)$$

⁶In (2.20) we identify tangent spaces to \mathbb{R}^n at different points.

$$\begin{aligned}
& + \sum_{\ell \geq p+1} B_{ij}^{\ell, \sigma}(\gamma^\sigma) \left(\sum_{\nu, k \leq p} \frac{\langle Y_{I, \gamma}, \iota^\nu(Y_{\ell, \gamma}) Y_{I, \gamma} \rangle}{|Y_{I, \gamma}|^2} g_\nu^k(\gamma) V_{k, \gamma} + \sigma E_\ell^\sigma \cdot \nabla \right) \\
& + \sigma C_{ij}^\sigma(\gamma^\sigma) \cdot \nabla.
\end{aligned}$$

Projecting along ∂_k with $k \leq p$ we get, in view of (2.16)

$$A_{ij}^{k, \sigma}(\gamma^\sigma) + \sum_{\ell \geq p+1} \sum_{\nu \leq p} B_{ij}^{\ell, \sigma}(\gamma^\sigma) \frac{\langle Y_{I, \gamma}, \iota^\nu(Y_{\ell, \gamma}) Y_{I, \gamma} \rangle}{|Y_{I, \gamma}|^2} g_\nu^k(\gamma) = O(\sigma) =: \sigma M_{ij}^{k, \sigma} \quad \text{for all } k \leq p.$$

The functions $M_{ij}^{k, \sigma}$ are bounded. Therefore the claim (2.20) is proved and the proof of the lemma is concluded. \square

Now we are ready to show that orbits are submanifolds. The argument is known, see [Her77] for the smooth case, [Ram07, Section 6.0.2] for the nonsmooth constant-rank case.

Proof of Theorem 1.1. Let $x_0 \in \mathbb{R}^n$. Let $p := p_{x_0}$, take $I \in \mathcal{I}(p, q)$ such that $|Y_I(x_0)| \neq 0$ and define for some small $\delta > 0$ and $u \in B(0, \delta) \subset \mathbb{R}^p$ the exponential map

$$\Phi(u) := \exp \left(\sum_{j=1}^p u_j V_j \right) x_0, \quad (2.24)$$

where the vector fields V_j are defined in (2.11). Observe first that $\Phi(u) \in \mathcal{O}$, for all $u \in B(0, \delta)$, where δ is sufficiently small. Indeed, $\Phi(u)$ is solution of the ODE

$$\dot{\gamma} = \sum_{j=1}^p u_j V_j(\gamma(t)) = \sum_{j, k=1}^p u_j \beta_j^k(\gamma(t)) Y_k(\gamma(t)) \quad \text{with } \gamma(0) = x_0.$$

Therefore, since the coefficients β_j^k are bounded in a neighborhood of x , γ is subunit (possibly up to a linear reparametrization). Note the obvious inclusion

$$\Phi(B(0, \delta)) \subset B_d(x_0, C\delta) \quad \text{for all small } \delta > 0. \quad (2.25)$$

Next, by the commutativity property established in (2.13) we may claim that for all $k \in \{1, \dots, p\}$

$$\frac{\partial}{\partial u_k} \Phi(u) = \frac{\partial}{\partial u_k} e^{u_k V_k} \exp \left(\sum_{j \neq k} u_j V_j \right) x_0 = V_{k, \Phi(u)}.$$

Therefore $\Phi \in C^{1,1}(B(0, \delta), \mathbb{R}^n)$, if the positive number δ is sufficiently small. Moreover, possibly shrinking δ , the set $\Phi(B(0, \delta)) \subset \mathcal{O}$ is a p -dimensional $C^{1,1}$ submanifold embedded in \mathbb{R}^n with $T_x \Phi(B(0, \delta)) = P_x$ for all $x \in \Phi(B(0, \delta))$.

Let $\Sigma := \Phi(B(0, \delta))$ with $\delta > 0$ sufficiently small. We claim that for all $y \in \Sigma$ there is $\sigma > 0$ such that

$$B_d(y, \sigma) \subset \Phi(B(0, \delta)) \quad (2.26)$$

To prove the claim, let $z \in B_d(y, \sigma)$. This means that $z = \gamma(1)$, where $\gamma \in \text{Lip}((0, 1), \mathbb{R}^n)$ satisfies a.e. $\dot{\gamma} = \sum_j c_j Y_j(\gamma)$ with $|c(t)| \leq \sigma$ and $\gamma(0) = y$. We may assume that there is a small Euclidean neighborhood U of y , a neighborhood V of the origin in \mathbb{R}^n and a $C^{1,1}$ diffeomorphism $F : U \rightarrow V$ such that $F(y) = 0$ and $F(\Sigma \cap U) = V \cap \{(x', x'') \in \mathbb{R}^p \times \mathbb{R}^{n-p} : x'' = 0\}$. Choose σ small enough to ensure that $\gamma(1) \in U$ for all $c \in L^\infty(0, 1)$ with $|c(t)| \leq \sigma$ a.e. Since $F \in C^{1,1}$, the Lipschitz path $\eta(t) := F(\gamma(t)) \in V$ satisfies for a.e. $t \in [0, 1]$

$$\dot{\eta}(t) = dF(\gamma(t))\dot{\gamma}(t) = \sum_j c_j(t) Y_j F(\gamma(t)) =: \sum_j c_j(t) (F_* Y_j)(\eta(t)), \quad (2.27)$$

with $\eta(0) = 0$. This Cauchy problem has a unique solution, because $F_* Y_j$ is Lipschitz. Moreover since $Y_j(z) \in T_z \Sigma$ for all $z \in \Sigma$, it must be $F_* Y_j = \sum_{\alpha=1}^n h_j^\alpha(\xi) \partial_{\xi_\alpha}$, where $h_j^\alpha(\xi', 0) = 0$ for all $(\xi', 0) \in V \cap (\mathbb{R}^p \times \{0\})$, $\alpha \geq p+1$ and $1 \leq j \leq p$. Since the coefficients h_j^α are Lipschitz continuous in V , we conclude that it must be $\eta''(t) = 0$ for $t \in [0, 1]$. This ends the proof of inclusion (2.26).

Next observe that we can repeat the construction of the map Φ in (2.24) at any point $x_0 \in \mathcal{O}$ and for any possible choice of $I \in \mathcal{I}(p_{x_0}, q)$ such that $|Y_I(x_0)| \neq 0$.

As a first consequence of inclusion (2.26), we show that the family of sets of the form $\Phi(B_\delta)$ constructed in this way can be used as a basis for a topology $\tau_{\mathcal{O}}$ on \mathcal{O} . To show this, let $x, \tilde{x} \in \mathcal{O}$. Let

$$\Phi_x(u) = \exp\left(\sum_j u_j V_j\right)x \quad \text{and} \quad \tilde{\Phi}_{\tilde{x}}(u) = \exp\left(\sum_j u_j \tilde{V}_j\right)\tilde{x}$$

be the maps constructed as above. Let δ and $\tilde{\delta}$ be sufficiently small to ensure that $\Sigma := \Phi_x(B_\delta)$ and $\tilde{\Sigma} := \tilde{\Phi}_{\tilde{x}}(B_{\tilde{\delta}})$ are manifolds. Assume also that $\Sigma \cap \tilde{\Sigma} \neq \emptyset$. Let now $x^* \in \Sigma \cap \tilde{\Sigma}$ and choose a map $\Phi_{x^*}^*(u) = \exp(\sum_j u_j V_j^*)(x^*)$. We need to prove that, for sufficiently small $\delta^* > 0$, we have $\Phi_{x^*}^*(u) \in \Sigma \cap \tilde{\Sigma}$ for each $u \in B(0, \delta^*)$. But we proved before that $\Phi_{x^*}^*(B(0, \delta^*)) \subset B_d(x^*, C\delta^*)$, if δ^* is sufficiently smooth. Therefore the claim follows from the already proved inclusion (2.26).

A further consequence of inclusions (2.25) and (2.26) is that the topologies $\tau_{\mathcal{O}}$ and τ_d are equivalent. They are both stronger than the Euclidean topology τ_{Euc} restricted to \mathcal{O} . Moreover, on a small piece of leaf of the form $\Sigma = \Phi(B(0, \delta))$, $\tau_{\text{Euc}}, \tau_{\mathcal{O}}$ and τ_d induce all the same topology.

Finally, the family of maps in (2.24) as x_0 varies in a fixed orbit \mathcal{O} , $I \in \mathcal{I}(p_{x_0}, q)$ is such that $|Y_I(x_0)| \neq 0$ and δ is sufficiently small, can be used to give a structure of differentiable manifold to (\mathcal{O}, τ_d) . \square

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