Almost exponential maps and integrability results for a class of horizontally regular vector fields *

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Abstract

We consider a family $\mathcal{H} := \{X_1, \dots, X_m\}$ of C^1 vector fields in $\mathbb{R}^n$ and we discuss the associated $\mathcal{H}$-orbits. Namely, we assume that our vector fields belong to a *horizontal regularity* class and we require that a suitable *s-involutivity* assumption holds. Then we show that any $\mathcal{H}$-orbit $\mathcal{O}$ is a C^1 immersed submanifolds and it is an integral submanifold of the distribution generated by the family of all commutators up to length s. Our main tool is a class of *almost exponential maps* of which we discuss carefully some precise first order expansions.

Contents

1. Introduction and main results

In this paper we discuss the integrability of distributions defined by families of vector fields under a higher order *horizontal regularity* hypothesis and assuming an involutivity condition of order $s \in \mathbb{N}$. The central tool we exploit is given by a class of *almost exponential maps* which we will analyze in details assuming only low regularity on the coefficients of the vector fields.

To start the discussion, fix a family $\mathcal{H} = \{X_1, \dots, X_m\}$ of at least Lipschitz-continuous vector fields. For any $x \in \mathbb{R}^n$ define the Sussmann's *orbit*, or *leaf*

$$\mathcal{O}_{\mathcal{H}}^x := \{e^{t_1 X_{j_1}} \cdots e^{t_p X_{j_p}} x : p \in \mathbb{N}, J := (j_1, \dots, j_p) \in \{1, \dots, m\}^p, t \in \Omega_{J,x}\}, \tag{1.1}$$

where for fixed $x \in \mathbb{R}^n$ we denote by $\Omega_{J,x} \subset \mathbb{R}^p$ the open neighborhood of the origin where the map $t \mapsto e^{t_1 X_{j_1}} \cdots e^{t_p X_{j_p}} x$ is well defined. We equip the leaf $\mathcal{O}_{\mathcal{H}}^x$ with the topology τ_d defined by the Franchi–Lanconelli distance d; see (2.1).

*2010 Mathematics Subject Classification. Primary 53C17; Secondary 53C12. Key words and Phrases: Integrable distribution, Carnot–Carathéodory distance.

Our purpose is to describe a regularity class of order $s \geq 2$ and a *s-involutivity* assumption that ensure that each orbit $\mathcal{O}_{\mathcal{H}}$ is a integral manifold of the distribution generated by the family $\mathcal{P} := \mathcal{P}_s := \{Y_1, \dots, Y_q\}$ of all nested commutators of length at most s constructed from the original family $\mathcal{H}$. To give coordinates on $\mathcal{O}$ we shall use the following *almost exponential maps*. Fix $s \geq 2$ and denote by $\mathcal{P}$ the aforementioned family of commutators. Assign to each Y_j the *length* $\ell_j \leq s$, just its order. Then, let

$$E_{I,x}(h) := \exp_{\mathrm{ap}}(h_1 Y_{i_1}) \cdots \exp_{\mathrm{ap}}(h_p Y_{i_p})x, \tag{1.2}$$

where $I = (i_1, \dots, i_p)$ is a multiindex which fixes p commutators $Y_{i_1}, \dots, Y_{i_p} \in \mathcal{P}$, $h \in \mathbb{R}^p$ belongs to a neighborhood of the origin and $p \in \{1, \dots, n\}$ is suitable. See (2.15) for the definition of the *approximate exponential* $\exp_{\mathrm{ap}}$. We shall use the maps in (1.2) to construct charts, developing a higher order, nonsmooth, quantitative extension of some ideas appearing in a paper by Lobry; see [Lob70]; see Theorem 3.5 and Remarks 3.6 and 3.7 below.

Here is a description of our regularity class. Let $\mathcal{H} = \{X_1, \dots, X_m\}$ and let $s \geq 2$. Assume that $X_j =: f_j \cdot \nabla \in C^1_{\mathrm{Euc}}$ for all j (here and hereafter C^1_{Euc} refers to Euclidean regularity). Assume also that for each $p \leq s$ and $j_1, \dots, j_p \in \{1, \dots, m\}$, all derivatives $X^\sharp_{j_1} \cdots X^\sharp_{j_{p-1}} f_{j_p}$ exist and are locally Lipschitz-continuous functions with respect to distance d associated to the vector fields. Here, following [MM12a], we denote by $X^\sharp f$ the Lie derivative along the vector field X of the scalar function f. Moreover we require that for any commutator $Y_j =: g_j \cdot \nabla \in \mathcal{P}$, all maps of the form $g_j \circ E_{I,x}$ are continuous for all $p \in \{1, \dots, n\}$, $I = (i_1, \dots, i_p)$ and $x \in \mathbb{R}^n$. [1]

Furthermore, we require the following s-involutivity condition. For any $X_j \in \mathcal{H}$ and for any $Y_k \in \mathcal{P}$ with maximal length $\ell_k = s$, at any $x \in \Omega$ where the derivative $X^\sharp_j g_k(x)$ exists one can write for suitable $b^i = b^i(x)$

$$(\mathrm{ad}_{X_j} Y_k)_x := (X^\sharp_j g_k(x) - Y_k f_j(x)) \cdot \nabla = \sum_{i=1}^q b^i Y_{i,x} \quad \text{with } b^i \text{ locally bounded.} \tag{1.3}$$

The class of vector fields satisfying all those assumptions will be denoted by $\mathcal{A}_s$; see Definition 2.5, where a more precise formulation of this assumption is described. Note that in the smooth case we have $\mathrm{ad}_{X_j} Y_k = [X_j, Y_k]$ and ultimately (1.3) is equivalent to the Hermann condition [Her62]

$$[Y_i, Y_j] = \sum_{1 \leq k \leq q} c^k_{ij} Y_k, \quad \text{with } c^k_{ij} \in L^\infty_{\mathrm{loc}}, \tag{1.4}$$

which ensures that any Sussmann's orbit $\mathcal{O}_{\mathcal{P}}$ of the family of commutators $\mathcal{P}$ is a integral manifold of the distribution generated by $\mathcal{P}$. If furthermore $s = 1$, then $\mathcal{P} = \mathcal{H}$ and (1.4) and (1.3) are the same. Note that the appearance of operators of the form $\mathrm{ad}_{X_j} Y_k$ is very natural in the framework of our almost exponential maps; see the non-commutative calculus formulas discussed in [MM12a, Section 3].

Here is the statement of our result.

[1] This condition is widely ensured for instance as soon as we assume that g_j is continuous in the Euclidean topology, or at least in the Sussmann's orbit topology defined on $\mathcal{O}$ by the family $\mathcal{H}$; see [Sus73].

Theorem 1.1. *Let $\mathcal{H} = \{X_1, \dots, X_m\}$ be a family of vector fields of class $\mathcal{A}_s$. Then, for any $x_0 \in \mathbb{R}^n$, the orbit $\mathcal{O} := \mathcal{O}_{\mathcal{H}}^{x_0}$ with the topology τ_d is a C^1 immersed submanifold of $\mathbb{R}^n$ with tangent space $T_y\mathcal{O} = P_y$ for all $y \in \mathcal{O}$.*

Note that this result does not follow from standard ones, because the commutators Y_j are not assumed to be C^1 in the Euclidean sense. In Example 3.14 we exhibit a family of vector fields where our theorem apply, but classical results do not. See also Remark 3.15 for some further comments. Furthermore, let us mention that if $s = 1$, i.e. $\mathcal{H} = \mathcal{P}$, then Theorem 1.1 is a consequence of the Frobenius Theorem for singular C^1 distributions (it is well known to experts that in such case one can prove that orbits are even C^2 smooth). Note that if $s = 1$, in [MM11a] we proved a singular Frobenius-type theorem assuming only Lipschitz-continuity of the involved vector fields, generalizing part of Rampazzo's results [Ram07] to singular distributions; in fact, in [MM11a], orbits are $C^{1,1}$.

On a technical level, the main tool we discuss is the approximate exponential $E_{I,x}$ in (1.2). Introduce the notation $p_x := \dim P_x := \dim \operatorname{span}\{Y_1(x), \dots, Y_q(x)\}$ for all $x \in \mathbb{R}^n$. Fix x, take $p := p_x$ commutators $Y_{i_1}, \dots, Y_{i_p}$, which are linearly independent at x and construct the map E, defined in (1.2). Then, under the hypotheses of Theorem 1.1, we shall show that if the family $\mathcal{H}$ satisfies condition $\mathcal{A}_s$, then E is a C^1_{Euc}, full rank map in a neighborhood of the origin $0 \in \mathbb{R}^p$, whose derivative enjoys the following remarkable expansion

$$E_*(\partial_{h_k}) = Y_{i_k}(E(h)) + \sum_{\ell_j = \ell_{i_k}+1}^{s} a_k^j(h) Y_j(E(h)) + \sum_{i=1}^{q} \omega_k^i(x,h) Y_i(E(h)). \tag{1.5}$$

The functions a_k^j and ω_k^i have a very precise rate of convergence to 0, as $h \to 0$ which will be specified in (3.22) and (3.23). Note that an expansion of $E_*(\partial_{h_k})$ can be obtained either with the Campbell–Hausdorff formula in the smooth case (see [Mor00] or [VSCC92]), or in nonsmooth situations with the techniques of [MM12b]. However, the expansions in the mentioned papers contain some remainders appearing either as formal series, or in integral form. Here we are able to express such reminders via the pointwise terms ω_k^j, improving all previous results. Note also that we are improving the mentioned papers both from a regularity standpoint and because here we do not assume the Hörmander condition. At the authors' knowledge, expansion (1.5) with precise estimates on a_k^j and ω_k^i is new even in the smooth case. As a final remark, observe that Theorem 3.11 contains an explicit detailed proof of the fact that the map E is C^1 smooth, avoiding any use of the Campbell–Hausdorff formula. Note that, even if the vector fields are smooth, such maps are not much more than C^1; see Remark 3.12-(ii).

The useful information one can extract from (1.5) is that $E_*(\partial_{h_k}) \in P_{E(h)}$ (note that we are interested to situations where the inclusion $P_{E(h)} \subset \mathbb{R}^n$ is strict); see Theorem 3.11 for a precise statement. Observe that, if $O \subset \mathbb{R}^p$ is a small open set containing the origin, then $E(O)$ is a C^1 submanifold of $\mathbb{R}^n$ and (1.5) shows that $T_{E(h)}E(O) \subseteq P_{E(h)}$ for all h. This is the starting point to prove that $\mathcal{O}_{\mathcal{H}}^x$ is a integral manifold of the distribution generated by $\mathcal{P}$. Another fact we need to prove is that the dimension of $P_y := \operatorname{span}\{Y_j(y) : 1 \le j \le q\}$ is constant if y belongs to a fixed orbit $\mathcal{O}_{\mathcal{H}}^x$. This is obtained by means of a nonsmooth quantitative curvilinear version of the original Hermann's argument inspired to the work of Nagel, Stein and Wainger [NSW85] and Street [Str11].

To conclude this introduction, we give some references and motivations to study our almost exponential maps E. Such maps appear in [NSW85], and were used by the authors to show equivalence between different control distances; see also [VSCC92]. More recently they have revealed to be a useful tool to study Poincaré inequalities (see [LM00]), subelliptic Sobolev spaces (see [Dan91, Mor00, CRTN01, MM12b]), and geometric theory of Carnot–Carathéodory spaces (see [MM02, FF03, Vit12]). Finally, note that the precise expansion (1.5) will be a fundamental tool in the companion paper [MM11b], where we shall prove a Poincaré inequality on orbits for a family of vector fields satisfying an integrability condition.

2. Preliminaries

Vector fields and the control distance. Consider a family of vector fields $\mathcal{H} = \{X_1, \ldots, X_m\}$ and assume that $X_j \in C^1_{\text{Euc}}(\mathbb{R}^n)$ for all j. Here and later C^1_{Euc} means C^1 in the Euclidean sense. Write $X_j =: f_j \cdot \nabla$, where $f_j \colon \mathbb{R}^n \to \mathbb{R}^n$. The vector field X_j, evaluated at a point $x \in \mathbb{R}^n$, will be denoted by $X_{j,x}$ or $X_j(x)$. All the vector fields in this paper are always defined on the whole space $\mathbb{R}^n$.

Define the Franchi–Lanconelli distance [FL83]

$$
\begin{aligned}
d(x, y) := \inf \Big\{ r > 0 : y = e^{t_1 Z_1} \cdots e^{t_\mu Z_\mu} x \text{ for some } \mu \in \mathbb{N} \\
\text{where } \sum |t_j| \le 1 \text{ with } Z_j \in r\mathcal{H} \Big\}.
\end{aligned}
\tag{2.1}
$$

Here and hereafter we let $r\mathcal{H} := \{rX_1, \ldots, rX_m\}$ and $\pm r\mathcal{H} := \{\pm rX_1, \ldots, \pm rX_m\}$. The topology associated with d will be denoted with τ_d. We denote instead by d_{cc} the standard *Carnot–Carathéodory* or *control* distance (see Feffermann–Phong [FP83] and Nagel–Stein–Wainger [NSW85]). In the present paper we shall make a prevalent use of the distance d. It is well known that τ_d is (possibly strictly) stronger than the topology $\tau_{\text{Euc}}|_{\mathcal{O}}$ received by $\mathcal{O}$ from $\mathbb{R}^n$. See [BCH08, Chapter 3] and [AS04, Example 5.5].

In view of the mentioned examples, we need to use the broad definition of submanifold; see [Che46, KN96]. Below, if $\Sigma \subset \mathbb{R}^n$, we denote by $\tau_{\text{Euc}}|_\Sigma$ the induced topology.

Definition 2.1 (Immersed submanifold)**.** *Let $\Sigma \subset \mathbb{R}^n$ and let $\tau \supseteq \tau_{\text{Euc}}|_\Sigma$ be a topology on Σ. We say that Σ is a C^k submanifold if Σ is connected and for all $x \in \Sigma$ there is $\Omega \in \tau$, open neighborhood of x such that Ω is a C^k graph. If moreover $\tau = \tau_{\text{Euc}}|_\Sigma$ then we say that Σ is an* embedded *submanifold.*

Horizontal regularity classes. Here we define our notion of horizontal regularity in terms of the distance d. Note that we *do not* use the control distance d_{cc}.

Definition 2.2. *Let $\mathcal{H} := \{X_1, , \ldots, X_m\}$ be a family of vector fields, $X_j \in C^1_{\text{Euc}}$. Let d be their distance* (2.1) *Let $g : \mathbb{R}^n \to \mathbb{R}$. We say that g is d-*continuous*, and we write $g \in C^0_{\mathcal{H}}(\mathbb{R}^n)$, if for all $x \in \mathbb{R}^n$, we have $g(y) \to g(x)$, as $d(y, x) \to 0$. We say that $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{H}$-*Lipschitz *or d-*Lipschitz *in $A \subset \mathbb{R}^n$ if*

$$
\operatorname{Lip}_{\mathcal{H}}(g; A) := \sup_{x,y \in A,\, x \neq y} \frac{|g(x) - g(y)|}{d(x, y)} < \infty.
$$

We say that $g \in C^1_{\mathcal{H}}(\mathbb{R}^n)$ if the derivative $X_j^\sharp g(x) := \lim_{t\to 0}(f(e^{tX_j}x) - f(x))/t$ is a d-continuous function for any $j = 1, \dots, m$. We say that $g \in C^k_{\mathcal{H}}(\mathbb{R}^n)$ if all the derivatives $X^\sharp_{j_1} \cdots X^\sharp_{j_p} g$ are d-continuous for $p \le k$ and $j_1, \dots, j_p \in \{1, \dots, m\}$. If all the derivatives $X^\sharp_{j_1} \cdots X^\sharp_{j_k} g$ are d-Lipschitz on each Ω bounded set in the Euclidean metric, then we say that $g \in C^{k,1}_{\mathcal{H},\mathrm{loc}}(\mathbb{R}^n)$. Finally, denote the usual Euclidean Lipschitz constant of g on $A \subset \mathbb{R}^n$ by $\mathrm{Lip}_{\mathrm{Euc}}(g; A)$.

We will usually deal with vector fields which are of class at least $C^1_{\mathrm{Euc}} \cap C^{s-1,1}_{\mathcal{H},\mathrm{loc}}$, where $s \ge 1$ is a suitable integer. In this case it turns out that commutators up to the order s can be defined; see Definition 2.3. In the companion paper [MM12a] we study several issues related with this definition.

Definitions of commutator. Our purpose now is to show that, given a family $\mathcal{H}$ of vector fields with $X_j \in C^{s-1,1}_{\mathcal{H},\mathrm{loc}} \cap C^1_{\mathrm{Euc}}$, then commutators can be defined up to length s.

For any $\ell \in \mathbb{N}$, denote by $\mathcal{W}_\ell := \{w_1 \cdots w_\ell : w_j \in \{1, \dots, m\}\}$ the words of length $|w| := \ell$ in the alphabet $1, 2, \dots, m$. Let also $\mathfrak{S}_\ell$ be the group of permutations of ℓ letters. Then for all $\ell \ge 1$, there are functions $\pi_\ell : \mathfrak{S}_\ell \to \{-1, 0, 1\}$ such that

$$[A_{w_1}, [A_{w_2}, \dots [A_{w_{\ell-1}}, A_{w_\ell}]] \dots] = \sum_{\sigma \in \mathfrak{S}_\ell} \pi_\ell(\sigma) A_{\sigma_1(w)} A_{\sigma_2(w)} \cdots A_{\sigma_\ell(w)}, \tag{2.2}$$

for all $A_1, \dots, A_m : V \to V$ linear operators on a vector space V. See [MM12a] for a more formal definition and an in-depth discussion.

We are now ready to define commutators for vector fields in our regularity classes.

Definition 2.3 (Definitions of commutator)**.** *Given a family $\mathcal{H} = \{X_1, \dots, X_m\}$ of vector fields of class $C^{s-1,1}_{\mathcal{H},\mathrm{loc}} \cap C^1_{\mathrm{Euc}}$, define for any function $\psi \in C^1_{\mathcal{H}}$ the operator $X_j^\sharp \psi(x) := \mathcal{L}_{X_j}\psi(x)$, the Lie derivative. Let also $X_j\psi(x) := f_j(x) \cdot \nabla\psi(x)$ where $\psi \in C^1_{\mathrm{Euc}}$. Moreover, let*

$$\begin{aligned} f_w &:= \sum_{\sigma \in \mathfrak{S}_\ell} \pi_\ell(\sigma)\big(X_{\sigma_1(w)} \cdots X_{\sigma_{\ell-1}(w)} f_{\sigma_\ell(w)}\big) \quad \text{for all } w \text{ with } |w| \le s, \\ X_w\psi &:= [X_{w_1},, \dots, [X_{w_{\ell-1}}, X_{w_\ell}]]\psi := f_w \cdot \nabla\psi \quad \text{for all } \psi \in C^1_{\mathrm{Euc}} \quad |w| \le s, \\ X_w^\sharp\psi &:= \sum_{\sigma \in \mathfrak{S}_\ell} \pi_\ell(\sigma) X^\sharp_{\sigma_1(w)} \cdots X^\sharp_{\sigma_{\ell-1}(w)} X^\sharp_{\sigma_\ell(w)}\psi \quad \text{for all } \psi \in C^\ell_{\mathcal{H}} \quad |w| \le s-1. \end{aligned}$$

Finally, for any $j \in \{1, \dots, m\}$ and w with $1 \le |w| \le s$, let

$$\mathrm{ad}_{X_j} X_w\psi := (X_j^\sharp f_w - f_w \cdot \nabla f_j) \cdot \nabla\psi = (X_j^\sharp f_w - X_w f_j) \cdot \nabla\psi \quad \text{for all } \psi \in C^1_{\mathrm{Euc}}. \tag{2.3}$$

Non-nested commutators are precisely defined in [MM12a].

Remark 2.4.

- *Let $Z \in \pm\mathcal{H}$. If $|w| \le s-1$, then there are no problems in defining $\mathrm{ad}_Z X_w$. More precisely, in [MM12a] we show that $\mathrm{ad}_Z X_w = [Z, X_w]$. If instead $|w| = s$, then the function $t \mapsto f_w(e^{tZ}x)$ is Euclidean Lipschitz. In particular it is differentiable for a.e. t. In other words, for any fixed $x \in \mathbb{R}^n$, the limit $\frac{d}{dt} f_w(e^{tZ}x) =: Z^\sharp f_w(e^{tZ}x)$ exists for a.e. t close to 0. Therefore the pointwise derivative $Z^\sharp f_w(y)$ exists for almost all $y \in \mathbb{R}^n$ and ultimately $\mathrm{ad}_Z X_w$ is defined almost everywhere.*

- *Both our definitions of commutator,* X_w *and* $X_w^\sharp$ *are well posed from an algebraic point of view, i.e. they satisfy antisymmetry and the Jacobi identity; see [MM12a].*
- *In [MM12a] we will also recognize that the first order operator* X_w *agrees with* $X_w^\sharp$ *against functions* $\psi \in C^{s-1,1}_{\mathcal{H},\mathrm{loc}} \cap C^1_{\mathrm{Euc}}$ *as soon as* $|w| \le s-1$.

The integrability class $\mathcal{A}_s$.

Definition 2.5 (Vector fields of class $\mathcal{A}_s$)**.** *Let* $\mathcal{H} = \{X_1, \dots, X_m\}$ *be a family in the regularity class* $C^1_{\mathrm{Euc}} \cap C^{s-1,1}_{\mathcal{H},\mathrm{loc}}$. *We say that the family* $\mathcal{H}$ *belongs to the class* $\mathcal{A}_s$ *if, fixed an open bounded set* $\Omega \subset \mathbb{R}^n$, *there is* $C_0 > 1$ *such that the following holds: for any* $Z \in \pm\mathcal{H}$, *for any word* w *with* $|w| = s$, *for each* $x \in \Omega$ *and for a.e.* $t \in [-C_0^{-1}, C_0^{-1}]$, *there are coefficients* $b^v \in \mathbb{R}$ *such that*

$$\mathrm{ad}_Z X_w(e^{tZ}x) = \sum_{1\le|u|\le s} b^u X_u(e^{tZ}x) \quad \text{with} \tag{2.4}$$

$$|b^u| \le C_0 \qquad \text{for all } u \text{ with } 1 \le |u| \le s; \tag{2.5}$$

finally assume that if $1 \le |w| \le s$, *for all* $p \in \{1, \dots, n\}$, *for any* $I \in \mathcal{I}(p,q)$, $x \in \mathbb{R}^n$, *we have at any* h^* *where* $E_{I,x}$ *is defined*

$$f_w(E_{I,x}(h)) \longrightarrow f_w(E_{I,x}(h^*)) \quad \text{as } h \to h^*. \tag{2.6}$$

Remark 2.6.

- *Assumption* (2.6) *will be used only once, in* (3.25)*, but it is essential in order to ensure that the almost exponential maps we define later are actually* C^1_{Euc} *smooth. It is easy to check that assumption* (2.6) *is satisfied as soon as* $f_w : (\mathcal{O}_{\mathcal{H}}, \tau_{\mathcal{H}}) \to \mathbb{R}$ *is continuous, where* $\tau_{\mathcal{H}}$ *denotes the Sussmann's orbit topology defined by the family* $\mathcal{H}$*, see [Sus73]. Note that at this stage assumption* (2.6) *is not ensured by the d-Lipschitz continuity of* f_w.
- *Conditions* (2.4) *and* (2.5) *scale nicely. Namely, letting for all* $r \le 1$, $\widetilde{Z} = rZ$, $\widetilde{X}_w = r^{|w|}X_w$ *with* $|w| = s$, *we have*

$$\mathrm{ad}_{\widetilde{Z}} \widetilde{X}_w(x) = \sum_{1\le|u|\le s} \widetilde{b}^u \widetilde{X}_u(x) \qquad \text{where } |\widetilde{b}^u| \le C_0 r \le C_0 \text{ for all } u. \tag{2.7}$$

- *Let* $\mathcal{H}$ *be a family of vector fields in the class* $C^1_{\mathrm{Euc}} \cap C^{s-1,1}_{\mathcal{H},\mathrm{loc}}$ *satisfying the Hörmander bracket-generating condition of step* s *and assume that each* f_w *with* $|w| \le s$ *is continuous in the Euclidean sense. Then* $\mathcal{H}$ *satisfies* $\mathcal{A}_s$. *The constant* C_0 *in* (2.5) *depends also on a positive lower bound on* $\inf_\Omega |\Lambda_n(x,1)|$, *see* (2.13). *This case is discussed in [MM12a, Section 4].*
- *The pathological vector fields* $X_1 = \partial_{x_1}$ *and* $X_2 = e^{-1/x_1{}^2}\partial_{x_2}$, *in spite of their* C^∞ *smoothness, do not satisfy* (2.5) *for any* $s \in \mathbb{N}$.

Let $\Omega_0 \subset \mathbb{R}^n$ be a fixed open set, bounded in the Euclidean metric. Given a family $\mathcal{H}$ of vector fields of class $C^1_{\mathrm{Euc}} \cap C^{s-1,1}_{\mathcal{H},\mathrm{loc}}$, introduce the constant

$$\begin{aligned} L_0 := \sum_{j_1,\dots,j_s=1}^{m} \Big\{ \sup_{\Omega_0} \Big(|f_{j_1}| + |\nabla f_{j_1}| + \sum_{p\le s} |X^\sharp_{j_1} \cdots X^\sharp_{j_{p-1}} f_{j_p}| \Big) \\ + \mathrm{Lip}_{\mathcal{H}}(X^\sharp_{j_1} \cdots X^\sharp_{j_{s-1}} f_{j_s}; \Omega_0) \Big\}. \end{aligned} \tag{2.8}$$

We shall always choose points $x \in \Omega \Subset \Omega_0$ and we fix a constant $t_0 > 0$ small enough to ensure that

$$e^{\tau_1 Z_1} \cdots e^{\tau_N Z_N} x \in \Omega_0 \quad \text{if } x \in \Omega,\ Z_j \in \mathcal{H},\ |\tau_j| \le t_0 \text{ and } N \le N_0, \tag{2.9}$$

where N_0 is a suitable constant which depends on the data n, m and s.

Proposition 2.7 (measurability). *Let $\mathcal{H}$ be a family of class $\mathcal{A}_s$. Let $|w| = s$ and let $Z \in \pm\mathcal{H}$, Then for any $x \in \Omega$ we can write*

$$\operatorname{ad}_Z X_w(e^{tZ}x) = \sum_{1 \le |v| \le s} b^v(t) X_v(e^{tZ}x) \qquad \text{for a.e. } t \in (-t_0, t_0), \tag{2.10}$$

where the functions $t \mapsto b^v(t)$ are measurable *and for a.e. t we have $|b^v(t)| \le C_0$, where C_0 denotes the constant in* (2.5).

Proof. The statement can be proved arguing as in [MM12a, Proposition 4.1]. □

Wedge products and η-maximality conditions. Following [Str11], denote by $\mathcal{P} := \{Y_1, \dots, Y_q\} = \{X_w : 1 \le |w| \le s\}$ the family of commutators of length at most s. Let $\ell_j \le s$ be the length of Y_j and write $Y_j =: g_j \cdot \nabla$. Define for any $p, \mu \in \mathbb{N}$, with $1 \le p \le \mu$, $\mathcal{I}(p, \mu) := \{I = (i_1, \dots, i_p) : 1 \le i_1 < i_2 < \cdots < i_p \le \mu\}$. For each $x \in \mathbb{R}^n$ define $p_x := \dim \operatorname{span}\{Y_{j,x} : 1 \le j \le q\}$. Obviousely, $p_x \le \min\{n, q\}$. Then for any $p \in \{1, \dots, \min\{n, q\}\}$, let

$$Y_{I,x} := Y_{i_1,x} \wedge \cdots \wedge Y_{i_p,x} \in \textstyle\bigwedge_p T_x\mathbb{R}^n \sim \bigwedge_p \mathbb{R}^n \quad \text{for all } I \in \mathcal{I}(p, q),$$

and, for all $K \in \mathcal{I}(p, n)$ and $I \in \mathcal{I}(p, q)$

$$Y_I^K(x) := dx^K(Y_{i_1}, \dots, Y_{i_p})(x) := \det(g_{i_\alpha}^{k_\beta})_{\alpha,\beta=1,\dots,p}. \tag{2.11}$$

Here we let $dx^K := dx^{k_1} \wedge \cdots \wedge dx^{k_p}$ for any $K = (k_1, \dots, k_p) \in \mathcal{I}(p, n)$.

The family $e_K := e_{k_1} \wedge \cdots \wedge e_{k_p}$, where $K \in \mathcal{I}(p, n)$, gives an othonormal basis of $\bigwedge_p \mathbb{R}^n$, i.e. $\langle e_K, e_H \rangle = \delta_{K,H}$ for all K, H. Then we have the orthogonal decomposition $Y_I(x) = \sum_K Y_J^K(x) e_K \in \bigwedge_p \mathbb{R}^n$, so that the number

$$|Y_I(x)| := \Big(\sum_{K \in \mathcal{I}(p,n)} Y_I^K(x)^2 \Big)^{1/2} = |Y_{i_1}(x) \wedge \cdots \wedge Y_{i_p}(x)|$$

gives the p-dimensional volume of the parallelepiped generated by $Y_{i_1}(x), \dots, Y_{i_p}(x)$.

Let $I = (i_1, \dots, i_p) \in \mathcal{I}(p, q)$ such that $|Y_I| \ne 0$. Consider the linear system $\sum_{k=1}^p \xi^k Y_{i_k} = W$, for some $W \in \operatorname{span}\{Y_{i_1}, \dots, Y_{i_p}\}$. The Cramer's rule gives the unique solution

$$\xi^k = \frac{\langle Y_I, \iota^k(W) Y_I \rangle}{|Y_I|^2} \quad \text{for each } k = 1, \dots, p, \tag{2.12}$$

where we let $\iota_W^k Y_I := \iota^k(W) Y_I := Y_{(i_1,\dots,i_{k-1})} \wedge W \wedge Y_{(i_{k+1},\dots,i_p)}$.

Let $r > 0$. Given $J \in \mathcal{I}(p, q)$, let $\ell(J) := \ell_{j_1} + \cdots + \ell_{j_p}$. Introduce the vector-valued function

$$\Lambda_p(x, r) := \big(Y_J^K(x) r^{\ell(J)} \big)_{J \in \mathcal{I}(p,q), K \in \mathcal{I}(p,n)} =: \big(\widetilde{Y}_J^K(x) \big)_{J \in \mathcal{I}(p,q),\, K \in \mathcal{I}(p,n)}, \tag{2.13}$$

where we adopt the tilde notation $\widetilde{Y}_k = r^{\ell_k} Y_k$ and its obvious generalization for wedge products. Note that $|\Lambda_p(x, r)|^2 = \sum_{I \in \mathcal{I}(p,q)} r^{2\ell(I)} |Y_I(x)|^2$.

Definition 2.8 (η-maximality)**.** *Let $x \in \mathbb{R}^n$, let $I \in \mathcal{I}(p_x, q)$ and $\eta \in (0,1)$. We say that (I, x, r) is η-maximal if* $|Y_I(x)| r^{\ell(I)} > \eta \max_{J \in \mathcal{I}(p_x,q)} |Y_J(x)| r^{\ell(J)}$.

Note that, if (I, x, r) is a candidate to be η-maximal with $I \in \mathcal{I}(p, q)$, then by definition it *must* be $p = p_x = \dim \operatorname{span}\{Y_j(x) : 1 \le j \le q\}$.

Approximate exponentials of commutators. Let $w_1, \dots, w_\ell \in \{1, \dots, m\}$. Given $\tau > 0$, we define, as in [NSW85, Mor00] and [MM12b],

$$\begin{aligned} C_\tau(X_{w_1}) &:= \exp(\tau X_{w_1}), \\ C_\tau(X_{w_1}, X_{w_2}) &:= \exp(-\tau X_{w_2}) \exp(-\tau X_{w_1}) \exp(\tau X_{w_2}) \exp(\tau X_{w_1}), \\ &\;\;\vdots \\ C_\tau(X_{w_1}, \dots, X_{w_\ell}) &:= C_\tau(X_{w_2}, \dots, X_{w_\ell})^{-1} \exp(-\tau X_{w_1}) C_\tau(X_{w_2}, \dots, X_{w_\ell}) \exp(\tau X_{w_1}). \end{aligned} \tag{2.14}$$

Then let

$$e_{\mathrm{ap}}^{tX_{w_1 w_2 \dots w_\ell}} := \exp_{\mathrm{ap}}(tX_{w_1 w_2 \dots w_\ell}) := \begin{cases} C_{t^{1/\ell}}(X_{w_1}, \dots, X_{w_\ell}), & \text{if } t \ge 0, \\ C_{|t|^{1/\ell}}(X_{w_1}, \dots, X_{w_\ell})^{-1}, & \text{if } t < 0. \end{cases} \tag{2.15}$$

By standard ODE theory, there is t_0 depending on ℓ, Ω, Ω_0, $\sup|f_j|$ and $\sup|\nabla f_j|$ such that $\exp_*(tX_{w_1 w_2 \dots w_\ell})x \in \Omega_0$ for any $x \in \Omega$ and $|t| \le t_0$. Define, given $I = (i_1, \dots, i_p) \in \{1, \dots, q\}^p$, $x \in \Omega$ and $h \in \mathbb{R}^p$, with $|h| \le C^{-1}$

$$\begin{aligned} E_{I,x}(h) &:= \exp_{\mathrm{ap}}(h_1 Y_{i_1}) \cdots \exp_{\mathrm{ap}}(h_p Y_{i_p})(x) \\ \|h\|_I &:= \max_{j=1,\dots,p} |h_j|^{1/\ell_{i_j}} \quad \text{and} \quad Q_I(r) := \{h \in \mathbb{R}^p : \|h\|_I < r\}. \end{aligned} \tag{2.16}$$

Gronwall's inequality. We shall refer several times to the following standard fact: for all $a \ge 0$, $b > 0$, $T > 0$ and f continuous on $[0, T]$,

$$0 \le f(t) \le at + b \int_0^t f(\tau) d\tau \quad \forall\, t \in [0, T] \quad \Rightarrow \quad f(t) \le \frac{a}{b}(e^{bt} - 1) \quad \forall\, t \in [0, T]. \tag{2.17}$$

3. Approximate exponentials and regularity of A_s orbits

Let $\mathcal{H} = \{X_1, \dots, X_m\}$ be a family of $\mathcal{A}_s$ vector fields in $\mathbb{R}^n$. The main purpose of this section is to prove that any $\mathcal{H}$-orbit $\mathcal{O}_\mathcal{H}$ with the topology τ_d generated by the distance d is a C^1 integral manifold of the distribution generated by $\mathcal{P}$. Recall our usual notation $\mathcal{P} := \{Y_j : 1 \le j \le q\}$, $P_x := \operatorname{span}\{Y_{j,x} : 1 \le j \le q\}$ and $p_x := \dim P_x$.

3.1. Geometric properties of orbits

In this subsection we look at the properties of orbits $\mathcal{O}_\mathcal{H}$ for vector fields of class $\mathcal{A}_s$. First we study how the geometric determinants $\widetilde{Y}_J^K$ change along a given orbit $\mathcal{O}_\mathcal{H}$. The argument we use is known, see for instance [TW03, MM12b] and especially [Str11]. However, we need to address some issues which appear due to our low regularity assumptions. Ultimately, we will show that the positive integer p_x is constant as $x \in \mathcal{O}_\mathcal{H}$.

Below we shall use the following notation: given $r > 0$, we let $\widetilde{Y}_j = r^{\ell_j} Y_j =: \widetilde{g}_j \cdot \nabla$ and $\widetilde{Z} = rZ$, if $Z \in \pm\mathcal{H}$. Let also $\widetilde{Y}_J^K := r^{\ell(J)} Y_J^K$, where the notation for Y_J^K has been introduced in (2.11).

Lemma 3.1. *Let $\mathcal{H}$ be a family of vector fields of class $\mathcal{A}_s$. Let $p \in \{1, \dots, q \wedge n\}$. Let $x \in \Omega$ and $r_0 > 0$ so that $B_d(x, r_0) \subset \Omega_0$. Let $J \in \mathcal{I}(p, q)$, $K \in \mathcal{I}(p, n)$, $r \in (0, r_0]$ and $\widetilde{Z} \in \pm r\mathcal{H}$. Then the function $[-1, 1] \ni t \mapsto \widetilde{Y}_J^K(e^{t\widetilde{Z}} x)$ is Lipschitz continuous and there is $C > 1$ depending on C_0 and L_0 in (2.5) and (2.8) such that*

$$\Big|\frac{d}{dt}\widetilde{Y}_J^K(e^{t\widetilde{Z}}x)\Big| \le C|\Lambda_p(e^{t\widetilde{Z}}x, r)| \quad \textit{for a.e. } t \in (-1, 1).$$

Proof. Denote $\gamma_t := e^{t\widetilde{Z}}x$ and let $t, \tau \in (-1, 1)$. Then

$$\begin{aligned}|\widetilde{Y}_J^K(\gamma_\tau) - \widetilde{Y}_J^K(\gamma_t)| &= \Big|\sum_{1\le\alpha\le p} dx^K(\dots, \widetilde{Y}_{j_{\alpha+1}}(\gamma_t), \widetilde{Y}_{j_\alpha}(\gamma_\tau) - \widetilde{Y}_{j_\alpha}(\gamma_t), \widetilde{Y}_{j_{\alpha+1}}(\gamma_t), \dots)\Big| \\ &\le C|\tau - t|,\end{aligned}$$

where C depends on L_0 in (2.8). Then $t \mapsto \widetilde{Y}_J^K(\gamma_t)$ belongs to $\mathrm{Lip}_{\mathrm{Euc}}(-1, 1)$. The estimate for the Lipschitz constant here is quite rough and it can be refined through a computation of the derivative. Indeed, we claim that for a.e. $t \in (-1, 1)$ we have

$$\begin{aligned}\frac{d}{dt}\widetilde{Y}_J^K(\gamma_t) = &\sum_{\substack{1\le\alpha\le p\\ \ell_{j_\alpha}\le s-1}} dx^K(\dots, \widetilde{Y}_{j_{\alpha-1}}, [\widetilde{Z}, \widetilde{Y}_{j_\alpha}], \widetilde{Y}_{j_{\alpha+1}}, \dots, \widetilde{Y}_{j_p})(\gamma_t) \\ &+ \sum_{\substack{1\le\alpha\le p\\ \ell_{j_\alpha}=s}} \sum_{1\le\beta\le q} b_\alpha^\beta(\gamma_t) dx^K(\dots, \widetilde{Y}_{j_{\alpha-1}}, \widetilde{Y}_\beta, \widetilde{Y}_{j_{\alpha+1}}, \dots, \widetilde{Y}_{j_p})(\gamma_t) \\ &+ \sum_{1\le\gamma\le n} \sum_{1\le\beta\le p} \partial_\gamma \widetilde{f}^{k_\beta} dx^{(k_1, \dots, k_{\beta-1}, \gamma, k_{\beta+1}, \dots, k_p)}(\widetilde{Y}_{j_1}, \dots, \widetilde{Y}_{j_p})(\gamma_t) \\ &=: (A) + (B) + (C),\end{aligned} \tag{3.1}$$

where we wrote $\widetilde{Z} = \widetilde{f} \cdot \nabla \in C^1_{\mathrm{Euc}}$ and b_α^β are measurable functions with $|b_\alpha^\beta| \le C_0$. To prove (3.1), observe that, if $\ell(Y_{j_\alpha}) \le s - 1$, then $t \mapsto \widetilde{Y}_{j_\alpha}(\gamma_t)$ is $C^1_{\mathrm{Euc}}(-1, 1)$ and

$$\lim_{\tau\to t} \frac{\widetilde{Y}_{j_\alpha}(\gamma_\tau) - \widetilde{Y}_{j_\alpha}(\gamma_t)}{\tau - t} = \widetilde{Z}^\sharp \widetilde{g}_{j_\alpha}(\gamma_t) \cdot \nabla = [\widetilde{Z}, \widetilde{Y}_{j_\alpha}](\gamma_t) + \widetilde{Y}_{j_\alpha}\widetilde{f}(\gamma_t) \cdot \nabla \quad \text{for all } t \in [-1, 1].$$

Note that here we used [MM12a, Theorem 3.1] to claim that $\mathrm{ad}_{\widetilde{Z}} \widetilde{Y}_{j_\alpha} = [\widetilde{Z}, \widetilde{Y}_{j_\alpha}]$. If instead $\ell(Y_{j_\alpha}) = s$, then for almost any t we have

$$\begin{aligned}\lim_{\tau\to t} \frac{\widetilde{Y}_{j_\alpha}(\gamma_\tau) - \widetilde{Y}_{j_\alpha}(\gamma_t)}{\tau - t} &= \widetilde{Z}^\sharp \widetilde{g}_{j_\alpha}(\gamma_t) \cdot \nabla = \mathrm{ad}_{\widetilde{Z}} \widetilde{Y}_{j_\alpha}(\gamma_t) + \widetilde{Y}_{j_\alpha}\widetilde{f}(\gamma_t) \cdot \nabla \\ &= \sum_{\beta=1}^q b_\alpha^\beta(t) \widetilde{Y}_\beta(\gamma_t) + \widetilde{Y}_{j_\alpha}\widetilde{f}(\gamma_t) \cdot \nabla.\end{aligned} \tag{3.2}$$

In the first equality we used the definition of ad. Here $\widetilde{Y}_{j_\alpha}\widetilde{f} := \widetilde{g}_{j_\alpha} \cdot \nabla \widetilde{f}$, is well defined. In the second line we used Proposition 2.7. The term $\widetilde{Y}_{j_\alpha}\widetilde{f}$, in view of Lemma A.1 gives the third line of (3.1).

Next we estimate each line of (3.1), starting with (A).

$$|(A)| \le \left| dx^K(\dots, \widetilde{Y}_{j_{\alpha-1}}(\gamma_t), [\widetilde{Z}, \widetilde{Y}_{j_\alpha}](\gamma_t), \widetilde{Y}_{j_{\alpha+1}}(\gamma_t), \dots)\right| \le C|\Lambda_p(\gamma_t, r)|,$$

for all $t \in [-1, 1]$. Estimate is correct even if $\Lambda_p(\gamma_t, r) = 0$. To estimate (B), recall that $|b_\alpha^\beta| \le C$. Then, for all $t \in [-1, 1]$,

$$|(B)| \le \sum_{1\le\alpha\le p}\sum_{1\le\beta\le q} \left| dx^K(\dots, \widetilde{Y}_{j_{\alpha-1}}, \widetilde{Y}_\beta, \widetilde{Y}_{j_{\alpha+1}}, \dots)\right| \le C|\Lambda_p(\gamma_t, r)|.$$

Finally the estimate of (C) is easy and takes the form

$$|(C)| \le \sup_{B_d(x,r)} |\nabla \widetilde{f}| \max_{K\in\mathcal{I}(p,n)} |\widetilde{Y}_J^K(\gamma_t)| \le C|\Lambda_p(\gamma_t, r)| \quad \text{if } |t| \le 1.$$ □

The previous lemma immediately implies the following proposition.

Proposition 3.2. *Let $\mathcal{H}$ be a family in the regularity class $\mathcal{A}_s$. Let $x \in \Omega$, let $r \le r_0$, where r_0 is small enough so that $B_d(x, r_0) \subset \Omega_0$. Let $\gamma(t) := \gamma_t$ be a piecewise integral curve of $\pm r\mathcal{H}$ with $\gamma(0) = x$. Let $p \in \{1, \dots, q \wedge n\}$. Then we have*

$$\left|\Lambda_p(\gamma(t), r) - \Lambda_p(x, r)\right| \le |\Lambda_p(x, r)|\,(e^{Ct} - 1) \quad \textit{for all } t \in [0, 1]. \tag{3.3}$$

In particular, if $p = p_x$ and (I, x, r) is η-maximal, then

$$|\widetilde{Y}_J(\gamma(t)) - \widetilde{Y}_J(x)| \le \frac{Ct}{\eta}|\widetilde{Y}_I(x)| \quad \textit{for all } J \in \mathcal{I}(p, q) \quad t \in [0, 1]. \tag{3.4}$$

Finally, if x, y belong to the same orbit, then $p_x = p_y$.

Remark 3.3. *As a consequence of the proposition and of the Cramer's rule (2.12), if (I, x, r) is η-maximal, then (I, y, r) is $C^{-1}\eta$-maximal for all $y \in B_d(x, C^{-1}\eta r)$ and we may write for all such y and for any $j \in \{1, \dots, q\}$*

$$\widetilde{Y}_{j,y} = \sum_{k=1}^{p} \frac{b_j^k}{\eta}\widetilde{Y}_{i_k,y}, \tag{3.5}$$

where $|b_j^k| \le C$.

Remark 3.4. *Proposition 3.2 shows that the oscillation of determinants Λ_p on a ball is controlled in terms of the value of Λ_p at the center of the ball. It is not true that the oscillation of a single vector field on a ball can be controlled by its value at the center of the ball. For instance, we can take the vector fields $X = \partial_x$ and $Y = y\partial_y + x\partial_x$. Look at the ball $B((0, y), r)$, where $0 < y \ll r$. Note that (r, y) belongs to such ball, but the oscillation $|Y(0, y) - Y(r, y)| \sim r$ can not be controlled with the value $|Y(0, y)| = |y|$.*

Proof of Proposition 3.2. (See [TW03,MM12b,Str11]). Let $p \in \{1, \dots, q \wedge n\}$. By Lemma 3.1, the map $t \mapsto \Lambda_p(\gamma_t, r)$ is Lipschitz. Moreover, we have for a.e. $t \in [0,1]$,

$$\Big|\frac{d}{dt}\Lambda_p(\gamma_t, r)\Big| = \Big|\Big(\frac{d}{dt}\widetilde{Y}_J^K(\gamma_t)\Big)_{\substack{J\in\mathcal{I}(p,q)\\ K\in\mathcal{I}(p,n)}}\Big| \le C|\Lambda_p(\gamma_t, r)|,$$

by Lemma 3.1. Then the Gronwall's inequality (2.17) provides immediately the required estimate (3.3). Note that this implies that if $\Lambda_p(x, r) = 0$, then $\Lambda_p(\gamma_t, r) = 0$ for all $t \in [0, 1]$. Estimate (3.4) follows immediately.

Let now x and y be a couple of points on the same leaf $\mathcal{O}_{\mathcal{H}}$. Let $1 \le p \le q \wedge n$ and let $I \subset \mathbb{R}$ be an interval. Let $I = [a, b]$ and take $\gamma : I \to \mathbb{R}$ a piecewise integral curve of the vector fields X_j with $\gamma(a) = x$ and $\gamma(b) = y$. Let $A_p := \{t \in I : |\Lambda_p(\gamma(t))| = 0\}$. Note that A_p is closed, because it is the zero set of the continuous function $I \ni t \mapsto |\Lambda_p(\gamma(t))| \in \mathbb{R}$. The set A_p is also open by estimate (3.3). Therefore, either $A_p = \varnothing$ or $A_p = I$ and the proof is concluded. □

The fact we are going to establish in the following theorem will have a key role in Subsection 3.2, when we shall study our almost exponential maps E. See Remark 3.6 below.

Theorem 3.5. *Let $\mathcal{H}$ be a family of vector fields of class $\mathcal{A}_s$. Let (I, x, r) be η-maximal where $x \in \Omega$, $r \le r_0$, $I \in \mathcal{I}(p_x, q)$ and $\eta \in (0, 1)$. Denote $\widetilde{U}_j := r^{\ell_{i_j}} Y_{i_j}$ for $j = 1, \dots, p := p_x$ and $\widetilde{Z} := rZ \in \pm r\mathcal{H}$. Then there is $C > 0$ depending on L_0 and C_0 in (2.8) and (2.5) so that*

$$e_*^{-t\widetilde{Z}}(\widetilde{U}_{j,e^{t\widetilde{Z}}x}) \in P_x \quad \text{for all } t \text{ with } |t| \le C^{-1}\eta. \tag{3.6}$$

Moreover, if we write, for a given test function $\psi \in C^1_{\mathrm{Euc}}(\mathbb{R}^n)$,

$$\widetilde{U}_j(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) =: \sum_{k=1}^{p}\big(\delta_j^k + \theta_j^k(t)\big)\widetilde{U}_k\psi(x), \tag{3.7}$$

then we have

$$|\theta_j^k(t)| \le \frac{C|t|}{\eta} \quad \text{for all } j, k = 1, \dots, p \qquad |t| \le C^{-1}\eta. \tag{3.8}$$

Finally, for any commutator $\widetilde{Y}_h := \widetilde{g}_h \cdot \nabla$, where $h \in \{1, \dots, q\}$, we have at any $t \in (-C^{-1}\eta, C^{-1}\eta)$

$$\widetilde{Y}_h(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) = \sum_{k=1}^{p}\frac{b_h^k(t)}{\eta}\widetilde{U}_k\psi(x), \tag{3.9}$$

where $|b_h^k(t)| \le C$ if $|t| \le C^{-1}\eta$.

Remark 3.6. *The geometric interpretation of (3.6) tells that $e_*^{-t\widetilde{Z}} P_{e^{t\widetilde{Z}}x} = P_x$, i.e. the tangent map of the C^1 diffeomorphism $e^{-t\widetilde{Z}}$ maps the (candidate) tangent bundle $\cup_x P_x$ to the orbit $\mathcal{O}$ to itself (we say "candidate" because we do not know yet that $\mathcal{O}$ is a manifold). Theorem 3.5 has an important consequence. Namely, in in Theorem 3.8, it will enable us to show that integral remainders have in fact a pointwise form. Ultimately, we will apply such property in Theorem 3.11 to show that $E_*(\partial_{h_k}) \in P_{E(h)}$.*

Remark 3.7. *The proof below is inspired to an argument due to Lobry; see [Lob70, Lemma 1.2.1]. Here we generalize such argument to a higher order, nonsmooth situation and we get more quantitative estimates. See also [Lob76] and the related discussion by Balan [Bal94]; see finally the paper [Pel10], for an up-to-date bibliography on the subject. Note that Lobry's idea is also used in [AS04, Lemma 5.15].*

Proof of Theorem 3.5. Without loss of generality, we can work with positive values of t. First, we differentiate the left-hand side of (3.7). If $\ell_{i_j} \leq s-1$, then we use [MM12a, Theorem 2.6-(a) and Theorem 3.1-(ii)] which give

$$\frac{d}{dt}\widetilde{U}_j(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) = [\widetilde{Z},\widetilde{U}_j](\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) = \sum_{k=1}^{p}\frac{b_j^k(t)}{\eta}\widetilde{U}_k(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x), \tag{3.10}$$

provided that $0 < t \leq C^{-1}\eta$. Here $|b_j^k(t)| \leq C$. In last equality we used (3.5) with $\widetilde{Y}_h = [\widetilde{Z},\widetilde{U}_j]$.

If instead $\ell_{i_j} = s$, then we need first [MM12a, Theorem 2.6-(b)], then (2.6) and Proposition 2.7 in the present paper. This gives for a.e. $t \in [0, C^{-1}\eta]$

$$\begin{aligned}\frac{d}{dt}\widetilde{U}_j(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) &= \sum_{1\leq h\leq q} b_j^h(t)\widetilde{Y}_h(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) \quad \text{by (3.5)}\\ &= \sum_{1\leq h\leq q}\sum_{1\leq k\leq p} b_j^h(t)b_h^k(t)\frac{1}{\eta}\widetilde{U}_k(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x)\\ &=: \sum_{1\leq k\leq p}\frac{b_j^k(t)}{\eta}\widetilde{U}_k(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x)\end{aligned} \tag{3.11}$$

provided that $0 < t \leq C^{-1}\eta$. In this formula b_j^h, b_h^k and b_j^k denote measurable functions, bounded in term of the admissible constants C_0 and L_0.

By elementary ODE theory, for any fixed ψ, the functions $t \mapsto \widetilde{U}_j(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x)$ with $j = 1,\dots,p$ are uniquely determined by their value $\widetilde{U}_j\psi(x)$ at $t = 0$. Moreover, if we denote by $(a_j^k(t)) \in \mathbb{R}^{p\times p}$ the solution of the Cauchy problem

$$\dot{a}(t) = \frac{b(t)}{\eta}a(t) \quad \text{with} \quad a(0) = I_p \in \mathbb{R}^{p\times p}, \tag{3.12}$$

then we can write

$$e_*^{-t\widetilde{Z}}(\widetilde{U}_{j,e^{t\widetilde{Z}}x}) \equiv \widetilde{U}_j(\psi e^{-t\widetilde{Z}})(e^{t\widetilde{Z}}x) = \sum_{k=1}^{p} a_j^k(t)\widetilde{U}_k\psi(x). \tag{3.13}$$

Then we have proved (3.6). The Cramer's rule (2.12) confirms that the coefficients $a_j^k(t)$ are unique for each t.

To estimate the functions $\theta_j^k := a_j^k(t) - \delta_j^k$, where a_j^k satisfy (3.12), it suffices to use estimate $|b_j^k(t)| \leq C$ if $0 \leq t \leq C^{-1}\eta$. The Gronwall inequality (2.17) gives $|a_j^k(t) - \delta_j^k| \leq C|t|/\eta$ for all $j,k = 1,\dots,p$ and $0 < t \leq C^{-1}\eta$. Therefore (3.8) follows.

To obtain the proof of (3.9) it suffices to repeat the computation in (3.10) starting from $\widetilde{Y}_h$ instead of $\widetilde{U}_j$. This ends the proof. □

Under the hypotheses of Theorem 3.5, iterating the argument, we get for all $x \in \Omega$, $\mu \le N_0$ (see (2.9)), $j \in \{1, \dots, p\}$ and $Z_1, \dots, Z_\mu \in \mathcal{H}$,

$$\widetilde{U}_j(\psi e^{-t_1\widetilde{Z}_1} \cdots e^{-t_\mu \widetilde{Z}_\mu})(e^{t_\mu \widetilde{Z}_\mu} \cdots e^{t_1\widetilde{Z}_1}x) = \sum_{1\le k\le p} (\delta_j^k + \theta_j^k(t))\widetilde{U}_k\psi(x) \tag{3.14}$$

where $|\theta(t)| \le C|t|/\eta$, as soon as $\sum_{j=1}^{\mu}|t_j| \le C^{-1}\eta$. Moreover, for each $h \in \{1, \dots, q\}$, we get, if $x \in \Omega$, for the same values of $(t_1, \dots, t_\mu)$ and for almost all $\tau \in (-C^{-1}\eta, C^{-1}\eta)$,

$$\begin{aligned}&\frac{d}{d\tau}\widetilde{Y}_h(\psi e^{-t_1\widetilde{Z}_1} \cdots e^{-t_\mu\widetilde{Z}_\mu} e^{-\tau\widetilde{X}})(e^{\tau\widetilde{X}} e^{t_\mu\widetilde{Z}_\mu} \cdots e^{t_1\widetilde{Z}_1}x)\\ &\quad= \mathrm{ad}_{\widetilde{X}}\,\widetilde{Y}_h(\psi e^{-t_1\widetilde{Z}_1} \cdots e^{-t_\mu\widetilde{Z}_\mu} e^{-\tau\widetilde{X}})(e^{\tau\widetilde{X}} e^{t_\mu\widetilde{Z}_\mu} \cdots e^{t_1\widetilde{Z}_1}x) = \sum_{k=1}^{p}\frac{b_k(x,t,\tau)}{\eta}\widetilde{U}_k\psi(x),\end{aligned}$$

where $|b_k(x,t,\tau)| \le C$ for a.e. τ. Here $X \in \mathcal{H}$. If we do not care about maximality and choose $r = 1$, we get, for any fixed $(t_1, \dots, t_\mu)$ with $\sum_j |t_j| \le C^{-1}$ and for almost all τ with $|\tau| \le C^{-1}$,

$$\begin{aligned}&\frac{d}{d\tau}Y_h(\psi e^{-t_1Z_1} \cdots e^{-t_\mu Z_\mu} e^{-\tau X})(e^{\tau X} e^{t_\mu Z_\mu} \cdots e^{t_1Z_1}x)\\ &\quad= \mathrm{ad}_X\, Y_h(\psi e^{-t_1Z_1} \cdots e^{-t_\mu Z_\mu} e^{-\tau X})(e^{\tau X} e^{t_\mu Z_\mu} \cdots e^{t_1Z_1}x)\\ &\quad= \sum_{1\le j\le q} b_j(x,t,\tau)Y_j\psi(x),\end{aligned} \tag{3.15}$$

where $|b_j(x,t,\tau)| \le C$ for a.e. τ. Here again $x \in \Omega$ and $\psi \in C^1_{\mathrm{Euc}}$ is a test function. Formula (3.15) will be referred to later.

3.2. Derivatives of almost exponential maps and regularity of orbits

In this subsection we get several information on the derivatives of the approximate exponentials $E_{I,x,r}$ associated with a family $\mathcal{H}$ of $\mathcal{A}_s$ vector fields and we show that each orbit $\mathcal{O}$ with topology τ_d is a C^1 immersed submanifold of $\mathbb{R}^n$ with $T_y\mathcal{O} = P_y$ for all $y \in \mathcal{O}$. We will tacitly but heavily rely on the results of [MM12a, Section 3], namely on formulae

$$\mathrm{ad}_{X_{v_1}} \cdots \mathrm{ad}_{X_{v_k}} X_w = X_{vw} \quad \text{for all } v, w \text{ such that } |v| + |w| = k + |w| \le s \tag{3.16}$$

These formulae have a key role. In the proof of Theorem 3.8 below, we shall follow the arguments of [MM12b, Theorems 3.4 and 3.5], modifying everywhere the remainders O_{s+1} in [MM12b] with our remainders defined in [MM12a]. This will give us a formula with integral remainder, see (3.17). Then, using the results of Subsection 3.1, we shall show that such integral remainder can be specified in a pointwise form.

Theorem 3.8. *Let $1 \le |w| =: \ell \le s$, take $x \in \Omega$ and $t \in [0, t_0]$, where t_0 is small enough to ensure that $C_t x \in \Omega_0$ for all $t \in [0, t_0]$. Let $C_t = C_t(X_{w_1}, \dots, X_{w_\ell})$ be the map defined in* (2.14). *Fix a test function $\psi \in C^1_{\mathrm{Euc}}(\mathbb{R}^n)$. Then we have*

$$\frac{d}{dt}\psi(C_tx) = \ell t^{\ell-1}X_w\psi(C_tx) + \sum_{|v|=\ell+1}^{s} a_v t^{|v|-1}X_v\psi(C_tx) + t^s\sum_{|u|=1}^{s} b_u(x,t)X_u\psi(C_tx),$$

and

$$\frac{d}{dt}\psi(C_t^{-1}x) = -\,\ell t^{\ell-1}X_w\psi(C_t^{-1}x) + \sum_{|v|=\ell+1}^{s} \overline{a}_v t^{|v|-1}X_v\psi(C_t^{-1}x) + t^s\sum_{|u|=1}^{s}\overline{b}_u(x,t)X_u\psi(C_t^{-1}x).$$

Both the sums on v are empty if $|w| = s$. Otherwise, we have the cancellations $\sum_{|v|=\ell+1}(a_v + \overline{a}_v)f_v(x) = 0$ for all $x \in \Omega$. The (real) coefficients b_u and $\overline{b}_u$ are bounded in terms of the constants L_0 and C_0 in (2.8) *and* (2.5)*.*

Remark 3.9. *As already observed, the theorem just stated improves [MM12b, Theorem 3.5], both because we relax regularity assumptions and because we devise a pointwise form of the remainders. In particular, choosing as ψ the identity function, we see that the remainder belongs to the subspace $P_{C_tx} = \operatorname{span}\{Y_{j,C_tx} : j = 1,\dots,q\}$ which can be a strict subspace of $\mathbb{R}^n$.*

Proof of Theorem 3.8. We prove the statement for $t > 0$. By [MM12b, Theorem 3.5], we know that

$$\frac{d}{dt}\psi(C_tx) = \ell t^{\ell-1}X_w\psi(C_tx) + \sum_{|v|=\ell+1}^{s} a_v t^{|v|-1}X_v\psi(C_tx) + O_{s+1}(t^s,\psi,C_tx), \tag{3.17}$$

where the numbers a_v are suitable algebraic coefficients. Note that formula (3.17) in [MM12b] is proved for smooth vectro fields. Using (3.16) and changing everywhere the remainders in [MM12b] with the remainders introduced in [MM12a, Subsection 2.1], one can check that all computations fit to our setting. Therefore, we only need to deal with the integral remainders introduced and discussed in [MM12a]. Concerning such remainders, recall that

$$O_{s+1}(t^s,\psi,C_tx) = (\text{sum of terms like}) \int_0^t \omega(t,\tau)\frac{d}{d\tau}X_v(\psi\varphi^{-1}e^{-\tau Z})(e^{\tau Z}\varphi C_tx)d\tau$$

where $|v| = s$, $\varphi = e^{tZ_1}\cdots e^{tZ_\nu}$ and $Z, Z_j \in \pm\mathcal{H}$. Next, by (3.15), we may write for a.e. τ

$$\frac{d}{d\tau}X_v(\psi\varphi^{-1}e^{-\tau Z})(e^{\tau Z}\varphi C_tx) = \sum_{1\le|u|\le s} b_u(x,t,\tau)X_u\psi(C_tx),$$

where for any t, x the functions $\tau \mapsto b_u(x,t,\tau)$ are measurable and satisfy $|b_u(t,\tau,x)| \le C$ for a.e. τ. Therefore we get

$$\sum_{1\le|u|\le s}\int_0^t \omega(t,\tau)b_u(x,t,\tau)d\tau\; X_u\psi(C_tx) =: t^s\sum_{1\le|u|\le s} b_u(x,t)X_u\psi(C_tx),$$

where $|b_u(x,t)| \le C$ for all $x \in \Omega$ and $|t| \le t_0$. This ends the proof. □

Our purpose now is to study the maps

$$E(h) := E_{I,x,r}(h) := \exp_{\mathrm{ap}}(h_1\widetilde{Y}_{i_1})\cdots\exp_{\mathrm{ap}}(h_p\widetilde{Y}_{i_p}) = \mathrm{e}_{\mathrm{ap}}^{h_1\widetilde{U}_1}\cdots\mathrm{e}_{\mathrm{ap}}^{h_p\widetilde{U}_p}\,x \tag{3.18}$$

where $1 \leq p \leq q$, $I \in \mathcal{I}(p,q)$, $\widetilde{U}_k := \widetilde{Y}_{i_k}$ and $d_k := \ell_{i_k}$. We always take $x \in \Omega$ and h sufficiently close to the origin so that $E(h) \in \Omega_0$, see (2.9).

Some elementary properties of E are contained in the following lemma. Without loss of generality we choose $r = 1$ and $I = (1, \dots, p)$.

Lemma 3.10. *The map* $h \mapsto \mathrm{e}_{\mathrm{ap}}^{h_1 Y_1} \cdots \mathrm{e}_{\mathrm{ap}}^{h_p Y_p} x =: E_{I,x}(h)$ *satisfies for* $x, x^* \in \Omega$ *and* $h, h^* \in B_{\mathrm{Euc}}(C^{-1})$

$$|E_{I,x}(h) - E_{I,x^*}(h^*)| \leq C\big(\|h - h^*\|_I + |x - x^*|\big). \tag{3.19}$$

Moreover, for any w *with* $1 \leq |w| \leq s$, *the function* $F_{X_w} \colon [-C^{-1}, C^{-1}] \times \Omega \to \mathbb{R}^{n\times n}$, *defined as* $F_{X_w}(t,x) := \nabla_x \mathrm{e}_{\mathrm{ap}}^{tX_w}(x)$, *is continuous.*

Proof. Observe first that, since each $Z \in \pm\mathcal{H}$ is C^1_{Euc}, by the Gronwall inequality we have

$$|e^{\tau Z} y - e^{\tau_0 Z} y_0| \leq C\big(|y - y_0| + |\tau - \tau_0|\big) \quad \text{for all } y, y_0 \in \Omega \quad |\tau|, |\tau_0| \leq C^{-1}. \tag{3.20}$$

Next, assume first that $t \geq t^* \geq 0$. Write $\mathrm{e}_{\mathrm{ap}}^{tX_w} x = e^{\tau Z_1} \cdots e^{\tau Z_\nu} x$, where $Z_1, \dots, Z_\nu \in \pm\mathcal{H}$ are suitable, see (2.15), and $\tau = t^{1/\ell}$, with $\ell := |w|$. Then iterating (3.20) we get

$$\big|\mathrm{e}_{\mathrm{ap}}^{tX_w} x - \mathrm{e}_{\mathrm{ap}}^{t^* X_w} x^*\big| = \big|e^{\tau Z_1} \cdots e^{\tau Z_\nu} x - e^{\tau^* Z_1} \cdots e^{\tau^* Z_\nu} x^*\big| \leq C\big(|x - x^*| + |t - t^*|^{1/\ell}\big).$$

If instead $t > 0 > t^*$, then we get

$$\begin{aligned}\big|\mathrm{e}_{\mathrm{ap}}^{tX_w} x - \mathrm{e}_{\mathrm{ap}}^{t^* X_w} x^*\big| &\leq |\mathrm{e}_{\mathrm{ap}}^{tX_w} x - x| + |x^* - \mathrm{e}_{\mathrm{ap}}^{t^* X_w} x^*| + |x - x^*| \\ &\leq C\big(|t|^{1/\ell} + |t^*|^{1/\ell} + |x - x^*|\big) \leq C\big(|t - t^*|^{1/\ell} + |x - x^*|\big).\end{aligned}$$

This shows (3.19) for $p = 1$. Iterating one gets the general case.

Next we prove existence and continuity of the derivative F_{X_w}. Assume first that $t \geq 0$ and decompose $\mathrm{e}_{\mathrm{ap}}^{tX_w} x = e^{t^{1/\ell} Z_1} \cdots e^{t^{1/\ell} Z_\nu} x$, where $\ell = |w|$ and $Z_1, \dots, Z_\nu \in \pm\mathcal{H}$ are suitable. Euclidean regularity of the vector fields Z_j implies that the functions $(\tau, y) \mapsto F_{Z_j}(\tau, y) := \nabla_y e^{\tau Z_j} y$ are continuous if $y \in \Omega$ and $|\tau|$ is small. Therefore, the chain rule gives

$$\begin{aligned}F_{X_w}(t,x) &= \nabla_x \mathrm{e}_{\mathrm{ap}}^{tX_w}(x) \\ &= F_{Z_1}(t^{1/\ell}, e^{t^{1/\ell} Z_2} \cdots e^{t^{1/\ell} Z_\nu} x) F_{Z_2}(t^{1/\ell}, e^{t^{1/\ell} Z_3} \cdots (x)) \cdots F_{Z_\nu}(t^{1/\ell}, x).\end{aligned}$$

Thus $F_{X_w}\big|_{[0,C^{-1}]\times\Omega}$ is continuous. Note that $F_{X_w}(0,x) = I_n$ for all x. An analogous argument shows that $F_{X_w}\big|_{[-C^{-1},0]\times\Omega}$ is continuous and concludes the proof. ◻

At this point we may deduce the following result. See (3.18) for notation on the map E.

Theorem 3.11. *Let* $\mathcal{H}$ *be an* $\mathcal{A}_s$ *family. Let* $x \in \Omega$ *and let* $r \in (0, r_0)$. *Fix* $p \in \{1, \dots, q\}$ *and* $I \in \mathcal{I}(p,q)$. *Then the function* $E_{I,x,r}$ *is* C^1 *smooth on* $B_{\mathrm{Euc}}(C^{-1})$. *Moreover, for all* $h \in B_{\mathrm{Euc}}(C^{-1})$ *and for any* $k \in \{1, \dots, p\}$ *we have* $E_*(\partial_{h_k}) \in P_{E(h)}$ *and we can write*

$$E_*(\partial_{h_k}) = \widetilde{U}_{k,E(h)} + \sum_{\ell_j = d_k + 1}^{s} a_k^j(h) \widetilde{Y}_{j,E(h)} + \sum_{i=1}^{q} \omega_k^i(x,h) \widetilde{Y}_{i,E(h)}, \tag{3.21}$$

where, for some $C > 1$ depending on L_0 and C_0 in (2.8) and (2.5), we have

$$|a_k^j(h)| \leq C\|h\|_I^{\ell_j - d_k} \quad \text{for all } h \in B_{\mathrm{Euc}}(C^{-1}) \tag{3.22}$$

$$|\omega_i(x,h)| \leq C\|h\|_I^{s+1-d_k} \quad \text{for all } h \in B_{\mathrm{Euc}}(C^{-1}) \quad x \in \Omega. \tag{3.23}$$

Proof. For notational simplicity we delete everywhere the tilde. In fact, the statement holds uniformly in $r \in (0, r_0)$, where r_0 depends on the already mentioned constants L_0 and C_0.

Step 1. We first prove the theorem for $p = 1$. Using the definition of $\exp_{\mathrm{ap}}$ and Theorem 3.8, we easily obtain by a change of variable that for any commutator Y of length $\ell \in \{1, \dots, s\}$ and for all $\psi \in C^1_{\mathrm{Euc}}$,

$$\begin{aligned}\frac{d}{dh}\psi(\mathrm{e}_{\mathrm{ap}}^{hY}(x)) = Y\psi(\mathrm{e}_{\mathrm{ap}}^{hY}(x)) + \sum_{\ell_j = \ell+1}^{s} \alpha_j(h) Y_k \psi(\mathrm{e}_{\mathrm{ap}}^{hY}\, x) \\ + |h|^{(s+1-\ell)/\ell} \sum_{i=1}^{q} b_i(x,h) Y_i \psi(\mathrm{e}_{\mathrm{ap}}^{hY}\, x),\end{aligned} \tag{3.24}$$

for all $x \in K$ and $0 < |h| \leq C^{-1}$, where the sum is empty if $\ell = s$. If $\ell < s$, then $\alpha_j(h) = \ell^{-1} a_j h^{(\ell_j - \ell)/\ell}$ if $h > 0$, while $\alpha_j(h) = -\ell^{-1}\overline{a}_j h^{(\ell_j-\ell)/\ell}$ if $h < 0$. The functions a_j come from the statement of Theorem 3.8. The functions $b_i(x,h)$ can be discontinuous, if we pass from $h > 0$ to $h < 0$, but we have estimate $|b_i(x,h)| \leq C$ uniformly in x, h.

To complete Step 1, we need to show that the function $h \mapsto \frac{d}{dh}\mathrm{e}_{\mathrm{ap}}^{hY} z$ is continuous for all fixed $z \in \Omega$. Continuity at any $h \neq 0$ (say $h > 0$) follows immediately from the decomposition $\mathrm{e}_{\mathrm{ap}}^{hY} = e^{h^{1/\ell}Z_1} \cdots e^{h^{1/\ell}Z_\nu}$, where $Z_j \in \pm\mathcal{H}$. We show now continuity at $h = 0$. Formula (3.24) gives $\left|\frac{\partial}{\partial h}\mathrm{e}_{\mathrm{ap}}^{hY} z - g(\mathrm{e}_{\mathrm{ap}}^{hY} z)\right| \leq C|h|^{1/\ell}$ (recall notation $Y =: g \cdot \nabla$). Therefore, using the l'Hôpital's rule, we get

$$\frac{d}{dh}\mathrm{e}_{\mathrm{ap}}^{hY} z\Big|_{h=0} := \lim_{h\to 0}\frac{\mathrm{e}_{\mathrm{ap}}^{hY} z - z}{h} = \lim_{h \to 0} g(\mathrm{e}_{\mathrm{ap}}^{hY} z) + O(|h|^{1/\ell}) = g(z),$$

where we need the d-continuity of g. This shows existence of the derivative at $h = 0$. To see continuity, just let $h \to 0$ in (3.24).

Step 2. By induction on p, we show that E is C^1 smooth. Assume that $(h_1, \dots, h_{p-1}) \mapsto \mathrm{e}_{\mathrm{ap}}^{h_1U_1} \cdots \mathrm{e}_{\mathrm{ap}}^{h_{p-1}U_{p-1}}(x)$ is C^1 for all choice of $U_1, \dots, U_{p-1}$. We need to show that $(h_1, \dots, h_p) \mapsto \mathrm{e}_{\mathrm{ap}}^{h_1U_1} \cdots \mathrm{e}_{\mathrm{ap}}^{h_pU_p}(x)$ is C^1 smooth.

Let $U_1, \dots, U_p \in \mathcal{P}$. First of all we show that the map $(h_1, \dots, h_p) \mapsto E_*(\partial_{h_1})$ is continuous. If $h_1 \neq 0$, say $h_1 > 0$, then we decompose for suitable $Z_1, \dots, Z_\mu \in \mathcal{H}$,

$$\mathrm{e}_{\mathrm{ap}}^{h_1U_1} \cdots \mathrm{e}_{\mathrm{ap}}^{h_pU_p}\, x = e^{h_1^{1/d_1}Z_1} \cdots e^{{h_1}^{1/d_1}Z_\mu}\, \mathrm{e}_{\mathrm{ap}}^{h_2U_2} \cdots \mathrm{e}_{\mathrm{ap}}^{h_pU_p}\, x.$$

Note that by standard ODE theory, the map $(\tau_1, \dots, \tau_\mu, z) \mapsto e^{\tau_1 Z_1} \cdots e^{\tau_\mu Z_\mu} z$ is C^1. Therefore, by means of Lemma 3.10, we have existence and continuity of $\partial_1 E(h) = E_*(\partial_{h_1})$ at any point of the form $h = (h_1, h_2, \dots, h_p)$ with $h_1 \neq 0$.

To discuss the case $h_1 = 0$, recall that formula (3.24) gives

$$\Big|\frac{\partial}{\partial h_1}\,\mathrm{e}_{\mathrm{ap}}^{h_1U_1}\cdots\mathrm{e}_{\mathrm{ap}}^{h_pU_p}\,x - U_1(\mathrm{e}_{\mathrm{ap}}^{h_1U_1}\cdots\mathrm{e}_{\mathrm{ap}}^{h_pU_p}\,x)\Big| \le C|h_1|^{1/d_1}.$$

Therefore, using de l'Hôpital's rule, for all $h = (0, h_2, \dots, h_p) =: (0, \widehat{h}_1)$, we get

$$\begin{aligned}\partial_1 E(0,\widehat{h}_1) &:= \lim_{h_1\to 0}\frac{\mathrm{e}_{\mathrm{ap}}^{h_1U_1}\,\mathrm{e}_{\mathrm{ap}}^{h_2U_2}\cdots\mathrm{e}_{\mathrm{ap}}^{h_pU_p}\,x - \mathrm{e}_{\mathrm{ap}}^{h_2U_2}\cdots\mathrm{e}_{\mathrm{ap}}^{h_pU_p}\,x}{h_1}\\ &= \lim_{h_1\to 0} U_1(\mathrm{e}_{\mathrm{ap}}^{h_1U_1}\,\mathrm{e}_{\mathrm{ap}}^{h_2U_2}\cdots\mathrm{e}_{\mathrm{ap}}^{h_pU_p}\,x) + O(|h_1|^{1/d_1}) = U_1(E(0,\widehat{h}_1)),\end{aligned}$$

where we need the d-continuity of U_1. This shows existence of $\partial_1 E(0, \widehat{h}_1)$.

To show continuity of $\partial_{h_1} E$ at $h^* = (0, \widehat{h}_1^*) \in B_{\mathrm{Euc}}(C^{-1})$, write by expansion (3.24)

$$\begin{aligned}&\big|\partial_1 E(h_1,\widehat{h}_1) - \partial_1 E(0,\widehat{h}_1^*)\big|\\ &\quad= \Big|U_1(E(h_1,\widehat{h}_1)) + \sum_{d_1+1\le \ell_j\le s}\alpha_j(h_1)Y_j(E(h_1,\widehat{h}_1))\\ &\qquad + |h_1|^{(s+1-d_1)/d_1}\sum_{1\le i\le q} b_iY_i(E(h_1,\widehat{h}_1)) - U_1(E(0,\widehat{h}_1^*))\Big|\\ &\quad\le C|h_1|^{1/d_1} + |U_1(E(h_1,\widehat{h}_1)) - U_1(E(0,\widehat{h}_1^*))| \to 0,\end{aligned}\tag{3.25}$$

as $(h_1, \widehat{h}_1) \to (0, \widehat{h}_1^*)$, here we used assumption (2.6) for U_1.

To conclude *Step 2*, we show the continuity of $\partial_{h_k} E$ for all $2 \le k \le p$. Write by the chain rule

$$\frac{\partial}{\partial h_k}E(h) = F_{U_1}(h_1, \mathrm{e}_{\mathrm{ap}}^{h_2U_2}\cdots(x))\cdots F_{U_{k-1}}(h_{k-1}, \mathrm{e}_{\mathrm{ap}}^{h_kU_k}\cdots(x))\frac{\partial}{\partial h_k}\,\mathrm{e}_{\mathrm{ap}}^{h_kU_k}\cdots(x). \tag{3.26}$$

This ends the proof, because the right-hand side depends continuosly on $h_1, \dots, h_p$, by Lemma 3.10 and the first part of *Step 2*.

Step 3. We show expansion (3.21) and estimates (3.22) and (3.23) for any p and for all $k = 1, \dots, p$.

Let $U_k = Y_{i_k}$, $d_k := \ell_{i_k}$ and $E_{\langle j,k\rangle}(x) := \mathrm{e}_{\mathrm{ap}}^{h_jU_j}\cdots\mathrm{e}_{\mathrm{ap}}^{h_kU_k}(x)$ for all $1 \le j \le k \le p$. We agree that $E_{\langle j,j-1\rangle}$ denotes the identity function. Observe that the function $z \mapsto E_{\langle j,k\rangle}(z)$ is a C^1 diffeomorphism for any fixed $h_j, h_{j+1}, \dots, h_k$. Then, for $k \in \{1, \dots, p\}$, we may use (3.24) and we get

$$\begin{aligned}E_*(\partial_{h_k}) &= U_kE_{\langle 1,k-1\rangle}(E_{\langle k,p\rangle}(x)) + \sum_{\ell_j=d_k+1}^{s}\alpha_j(h_k)Y_jE_{\langle 1,k-1\rangle}(E_{\langle k,p\rangle}(x))\\ &\quad + |h_k|^{(s+1-d_k)/d_k}\sum_{i=1}^{q} b_iY_iE_{\langle 1,k-1\rangle}(E_{\langle k,p\rangle}(x)),\end{aligned}\tag{3.27}$$

where b_i denote bounded functions and $|\alpha_j(h_k)| \le C|h_k|^{(\ell_j-d_k)/d_k}$.

To get formula (3.21), it suffices to use a rough expansion of each term as follows. Write for $\lambda \in \{1, \dots, p\}$ and $h_\lambda > 0$, $\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} = e^{-h_\lambda^{1/d_\lambda} Z_1} \cdots e^{-h_\lambda^{1/d_\lambda} Z_\nu}$, for suitable $Z_i \in \pm\mathcal{H}$. Then for all $j \in \{1, \dots, q\}$ write

$$\begin{aligned}
Y_j(\psi\, \mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda})(z) &= Y_j(\psi e^{-h_\lambda{}^{1/d_\lambda} Z_1} \cdots e^{-h_\lambda{}^{1/d_\lambda} Z_\nu})(z) \\
&= Y_j\psi(\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} z) + \sum_{|\alpha|=1}^{s-\ell_j} \mathrm{ad}_{Z_\nu}^{\alpha_\nu} \cdots \mathrm{ad}_{Z_1}^{\alpha_1} Y_j\psi(\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} z) \frac{h_\lambda^{|\alpha|/d_\lambda}}{\alpha!} \\
&\quad + O_{s+1}(|h_\lambda|^{(s+1-\ell_j)/d_\lambda}, \psi, \mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} z) \\
&= Y_j\psi(\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} z) + \sum_{\ell_i=\ell_j+1}^{s} c_i |h_\lambda|^{(\ell_i-\ell_j)/d_\lambda} Y_i\psi(\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} x) \\
&\quad + |h_\lambda|^{(s+1-\ell_j)/d_\lambda} \sum_{i=1}^{q} b_i Y_i \psi(\mathrm{e}_{\mathrm{ap}}^{h_\lambda U_\lambda} x),
\end{aligned}$$

where we use the pointwise form of the remainder, see the proof of Theorem 3.8. Here c_i are constants, while b_i are bounded functions. The proof of (3.21) follows from (3.27) via a repeated application of this expansion. If $h_\lambda < 0$, then the terms c_i and b_i may change, but the argument gives the same conclusion. The proof of the theorem is concluded ☐

Remark 3.12.

(i) Let X_w be a commutator of length $|w| \le s$. Define the function $H(t,x) := \frac{d}{dt} \mathrm{e}_{\mathrm{ap}}^{tX_w}(x)$. Under our assumptions $\mathcal{A}_s$ we may claim that $H(t,x)$ exists for all (t,x). However, we can not expect that the function $(t,x) \mapsto H(t,x)$ is continuous in $(-t_0,t_0) \times \Omega$. Indeed, in order to show the continuity of H at a point $(0,\widetilde{x})$, because

$$\begin{aligned}
|H(t,x) - H(0,\widetilde{x})| &\le |H(t,x) - H(0,x)| + |H(0,x) - H(0,\widetilde{x})|. \\
&= \Big|\frac{d}{dt} \mathrm{e}_{\mathrm{ap}}^{tX_w} x - f_w(x)\Big| + |f_w(x) - f_w(\widetilde{x})|.
\end{aligned}$$

The first term can be made small uniformly in x, if $|t|$ is small. In order to make the second term small, we can use only assumption (2.6), which does not ensure any continuity if x and $\widetilde{x}$ belong to different orbits.

(ii) Under our assumptions, we cannot expect that maps $h \mapsto E_{I,x}(h)$ are more than C^1. Indeed, the term $F_{U_1}(h_1, \mathrm{e}_{\mathrm{ap}}^{h_2 U_2} \cdots \mathrm{e}_{\mathrm{ap}}^{h_p U_p} x)$ in (3.26) depends continuously on $h_2, \dots, h_p$, if $\mathcal{H}$ is a C^1 family (recall that $F_{U_1}(h,x) := \nabla \mathrm{e}_{\mathrm{ap}}^{hU_1}(\xi)$ is only continuous in ξ). An inspection of the proof above shows that if $\mathcal{H}$ is a C^2 family and $\mathcal{A}_s$ holds, then $E_{I,x} \in C^{1,1/s}_{\mathrm{loc}}$, but this regularity cannot be improved, even if $X_j \in C^\infty$ or C^ω; see [MM12b, Example 5.7].

Now we can easily prove the regularity of orbits, along the lines of the proof in [AS04].

Theorem 3.13 (Regularity of $\mathcal{A}_s$ orbits)**.** *Let $\mathcal{H}$ be a system of $\mathcal{A}_s$ vector fields. Then each orbit $\mathcal{O}$ with the topology τ_d is a connected C^1 smooth immersed submanifold of $\mathbb{R}^n$ satisfying $T_x\mathcal{O} = P_x := \mathrm{span}\{X_w(x) : 1 \le |w| \le s\}$ for all $x \in \mathcal{O}$.*

Proof. Let $x_0 \in \mathbb{R}^n$ and let $\mathcal{O} := \mathcal{O}_{\mathcal{H}}^{x_0}$ be its $\mathcal{H}$-orbit. We know from Remark 3.3 that $\dim P_x = \dim P_{x_0} =: p$ is constant in $\mathcal{O}$. For each $x \in \mathcal{O}$ choose $I \in \mathcal{I}(p,q)$ such that $|Y_I(x)| \neq 0$. By Theorem 3.11 and by the implicit function theorem, we may claim that for a suitable $O_{I,x} \subset \mathbb{R}^p$, open neighborhood of the origin, the map $E_{I,x} : O_{I,x} \to \mathbb{R}^n$ is a C^1 full-rank map which parametrizes a C^1 smooth, p-dimensional embedded submanifold $E_{I,x}(O_{I,x}) \subset \mathbb{R}^n$. Note also that $E_{I,x}(O_{I,x}) \subset \mathcal{O}$ and, by Theorem 3.11, $T_{E_{I,x}(h)} E_{I,x}(O_{I,x}) = P_{E_{I,x}(h)}$, for all $h \in O_{I,x}$. Let

$$\begin{aligned}\mathcal{U} := \{E_{I,x}(O) : x \in \mathcal{O}, I \in \mathcal{I}(p,q), |Y_I(x)| \neq 0 \\ \text{and } O \subset O_{I,x} \text{ is a open neighborhood of the origin}\}.\end{aligned}$$

We claim that the family $\mathcal{U}$ can be used as a base for a topology $\tau(\mathcal{U})$ on $\mathcal{O}$. To see that, we need to show that if the intersection of the p-dimensional submanifolds $E_{I,x}(O)$ and $E_{I',x'}(O')$ is nonempty, then it contains a small manifold of the form $E_{I'',x''}(O'')$, if O'' is a sufficiently small neighborhood of the origin. Let $\Sigma := E_{I,x}(O)$ and $\Sigma' = E_{I',x'}(O')$ and let $x'' \in \Sigma \cap \Sigma'$. Recall that both Σ and Σ' are embedded C^1 submanifolds of $\mathbb{R}^n$. Let $I'' \in \mathcal{I}(p,q)$ be such that $|Y_{I''}(x'')| \neq 0$. Let $O'' \subset \mathbb{R}^p$ be a small open neighborhood of the origin. For any $h \in O''$, the point $E_{I'',x''}(h)$ can be written as $e^{\tau_1 Z_1} \cdots e^{\tau_\nu Z_\nu} x$ where $Z_j \in \pm\mathcal{H}$ and $\sum_j |\tau_j| \leq C\|h\|_I$. By a repeated application of Bony's theorem [Bon69, Theorem 2.1], it follows that $E(h) \in \Sigma$, provided that h is sufficiently close to the origin. The same argument applies to Σ'. Thus we have proved that $\mathcal{U}$ can be used as a topology base.

A similar argument shows that any submanifold of the form $E_{I,x}(O) \in \mathcal{U}$ contains a small ball $B_d(x,\sigma)$. Therefore τ_d is stronger than $\tau(\mathcal{U})$. The fact that $\tau(\mathcal{U})$ is stronger that τ_d follows easily from estimate $d(E_{I,x}(h), x) \leq C\|h\|_I$. Finally, since all paths of the form $t \mapsto e^{tZ}x \in (\mathcal{O}, \tau(\mathcal{U})) = (\mathcal{O}, \tau_d)$ are continuous, the orbit is connected.

The C^1 differential structure on $\mathcal{O}$ is given by the family maps $E_{I,x}\big|_O$ where $x \in \mathcal{O}$, $I \in \mathcal{I}(p_x, q)$ is such that $|Y_I(x)| \neq 0$ and $O \subset O_{I,x}$ is an open neighborhood of the origin. ☐

Example 3.14. *Let us consider in $\mathbb{R}^3$ the family $\mathcal{H} = \{X_1, X_2, X_3\}$:*

$$X_1 = a(t)\partial_x \quad X_2 = xa(t)\partial_y \quad \text{and} \quad X_3 = t\partial_t,$$

where the function a satisfies $a(t) = 1 + t^3 \sin\left(\frac{1}{t}\right)$, if $0 < |t| < 1$, $a(0) = 0$, $a \in C^\infty(\mathbb{R}\setminus\{0\})$ and $\inf_{\mathbb{R}} a > 0$. Note that $X_j \in C^1_{\mathrm{Euc}}(\mathbb{R}^3)$ and

$$[X_1, X_2] = a(t)^2\partial_y, \quad [X_1, X_3] = -ta'(t)\partial_x \quad \text{and} \quad [X_2, X_3] = -ta'(t)x\partial_y.$$

If $0 < |t| < 1$, then

$$\frac{d}{dt}(ta'(t)) = \frac{d}{dt}\Big(3t^3 \sin\frac{1}{t} - t^2\cos\frac{1}{t}\Big) = 9t^2 \sin\frac{1}{t} - 5t\cos\frac{1}{t} - \sin\frac{1}{t}$$

is discontinuous at $t = 0$. Therefore X_{13} and $X_{23} \notin C^1_{\mathrm{Euc}}$ and the C^1 singular Frobenius theorem does not apply to the family $\mathcal{P} = \{X_1, X_2, X_3, [X_1, X_2], [X_1, X_3], [X_2, X_3]\}$.

However, we claim that the family $\mathcal{H}$ belongs to our class $\mathcal{A}_2$. To show this claim, we first prove that $X_j \in C^{1,1}_{\mathcal{H},\mathrm{loc}}$. To see that, it suffices to show that $X_3^\sharp X_3^\sharp a \in C^0_{\mathcal{H}}$. But, if $0 < |t| < 1$, we have

$$X_3^\sharp X_3^\sharp a(t) = t\partial_t(ta'(t)) = 9t^3 \sin\frac{1}{t} - 5t^2\cos\frac{1}{t} - t\sin\frac{1}{t}, \tag{3.28}$$

which is a continuous function up to $t = 0$ *(note that, since* $X_3^\sharp a(0) = 0$*, we have* $X_3^\sharp X_3^\sharp a(0) = \lim_{t\to 0} t^{-1}(X_3^\sharp a(e^{tX_3}(0)) - X_3^\sharp a(0)) = 0$*). Since* X_{12}, X_{13} *and* $X_{23} \in C^0_{\mathrm{Euc}}$*, condition* (2.6) *is fulfilled.*

Finally, we have to check the 2*-involutivity, i.e. that for all* i, j, k *we can write* $\mathrm{ad}_{X_i} X_{jk} = \sum_{|w|\leq 2} b^w X_w$ *with* b^w *locally bounded. A computation shows that the nonzero terms are the following (we work with* $0 < |t| < 1$*)*

$$\begin{aligned}
-\mathrm{ad}_{X_1} X_{23} = \mathrm{ad}_{X_2} X_{13} = \frac{1}{2}\mathrm{ad}_{X_3} X_{12} = ta(t)a'(t)\partial_y = \frac{ta'(t)}{a(t)}X_{12} \\
\mathrm{ad}_{X_3} X_{13} = -t\partial_t(ta'(t))\partial_x = \frac{-t\partial_t(ta'(t))}{a(t)}X_1 \\
\mathrm{ad}_{X_3} X_{23} = -xt\partial_t(ta'(t))\partial_y = \frac{-t\partial_t(ta'(t))}{a(t)}X_2.
\end{aligned}$$

Since $\inf_{\mathbb{R}} a > 0$*, one can see with the help of* (3.28) *that both the coefficients* $ta'(t)/a(t)$ *and* $-t\partial_t(ta'(t))/a(t)$ *are locally bounded. Thus, hypothesis* $\mathcal{A}_2$ *is fulfilled and our main theorem applies.*

Note finally that it is very easy to see that there are three orbits of the family $\mathcal{H}$*. Namely,* $\mathcal{O}_1 := \{(x, y, t) : t > 0\}$*,* $\mathcal{O}_2 = \{t = 0\}$ *and* $\mathcal{O}_3 = \{t < 0\}$ *and they are integral manifolds of the distribution generated by the family* $\mathcal{P}$*.*

Remark 3.15. *A natural question concerns sharpness of the* C^1 *regularity of* $\mathcal{O}_{\mathcal{H}}$*. It is reasonable to guess that* C^1 *regularity is not sharp. Actually, we do not have any example of vector fields of class* $\mathcal{A}_s$ *where the integral manifolds* $\mathcal{O}_{\mathcal{H}}$ *are less than* C^2*. However, under our assumptions, maps* $E_{I,x}$ *cannot provide more than* C^1 *regularity, see Remark 3.12-(ii).*

A related issue concerns the regularity of the orbit $\mathcal{O}_{\mathcal{H}}$ *of a generic family of* C^1 *(or even Lipschitz-continuous) vector fields which do not satisfy any involutivity assumptions. This would require a careful discussion of a nonsmooth version of Sussmann's orbit theorem.*

We plan to discuss such questions in a future study.

A. Appendix

Here we prove the multilinear algebra lemma which has been used in the proof of Lemma 3.1. The same formula is proved by [Str11, Lemma 3.6], but here we exploit a slightly different argument, which does not rely on the formalism of Lie derivatives.

Lemma A.1 (Linear algebra)**.** *Let* $p \leq n$ *and let* $U_1, \dots, U_p$ *be constant vector fields in* $\mathbb{R}^n$*. Let* $Z = \sum_{\beta=1}^n f^\beta \partial_\beta \in C^1_{\mathrm{Euc}}$*. Then, for any* $(k_1, \dots, k_p) \in \mathcal{I}(p, n)$*,*

$$\begin{aligned}
&\sum_{\alpha=1}^p dx^{k_1} \wedge \cdots dx^{k_p}\Big(U_1, \dots, U_{\alpha-1}, \sum_{\beta=1}^n U_\alpha f^\beta \partial_\beta, U_{\alpha+1}, \dots, U_p\Big) \\
&= \sum_{\gamma=1}^n \sum_{\beta=1}^p \partial_\gamma f^{k_\beta} dx^{(k_1,\dots,k_{\beta-1})} \wedge dx^\gamma \wedge dx^{(k_{\beta+1},\dots,k_p)}(U_1, \dots, U_p).
\end{aligned} \tag{A.1}$$

Note that in the particular case $p = n$, the right-hand side is $\mathrm{div}(f)\det[U_1, \dots, U_n]$.

Proof. Recall first that if we are given $(V_\alpha^\beta)_{\alpha,\beta} \in \mathbb{R}^{p\times p}$, then the matrix $(\operatorname{cof} V)_\alpha^\beta := \det[V_1, \dots, V_{\alpha-1}, \partial_\beta, V_{\alpha+1}, \dots]$ satisfies

$$\sum_{\mu=1}^{p} V_\mu^\sigma (\operatorname{cof} V)_\mu^\rho = (\det V)\delta_{\sigma\rho} \tag{A.2}$$

To prove the lemma, observe first that $dx^{k_\mu}(\partial_\beta) = 0$ if $\mu \in \{1, \dots, p\}$ and $\beta \notin \{k_1, \dots, k_p\}$. Therefore the left-hand side of (A.1) takes the form

$$\begin{aligned}
&\sum_{\alpha=1}^{p} dx^{k_1} \wedge dx^{k_p} \Big(U_1, \dots, U_{\alpha-1}, \sum_{\beta=1}^{p} U_\alpha f^{k_\beta} \partial_{k_\beta}, U_{\alpha+1}, \dots, U_p\Big) \\
&\quad = \sum_{\substack{\alpha,\beta=1,\dots,p\\ \gamma=1,\dots,n}} U_\alpha^\gamma \partial_\gamma f^{k_\beta} dx^{k_1} \wedge dx^{k_p} \big(U_1, \dots, U_{\alpha-1}, \partial_{k_\beta}, U_{\alpha+1}, \dots, U_p\big) \\
&\quad = \sum_{\substack{\beta=1,\dots,p\\ \gamma=1,\dots,n}} \partial_\gamma f^{k_\beta} \sum_{\alpha=1}^{p} U_\alpha^\gamma \operatorname{cof} \begin{bmatrix} U_1^{k_1} & \dots & U_p^{k_1} \\ \vdots & \vdots & \vdots \\ U_1^{k_p} & \dots & U_p^{k_p} \end{bmatrix}_\alpha^\beta \overset{(A.2)}{=} \sum_{\substack{\beta=1,\dots,p\\ \gamma=1,\dots,n}} \partial_\gamma f^{k_\beta} \det \begin{bmatrix} U_1^{k_1} & \dots & U_p^{k_1} \\ \vdots & \vdots & \vdots \\ U_1^{k_{\beta-1}} & \dots & U_p^{k_{\beta-1}} \\ U_1^{\gamma} & \dots & U_p^{\gamma} \\ U_1^{k_{\beta+1}} & \dots & U_p^{k_{\beta+1}} \\ \vdots & \vdots & \vdots \\ U_1^{k_p} & \dots & U_p^{k_p} \end{bmatrix} \\
&\quad = \sum_{\substack{\beta=1,\dots,p\\ \gamma=1,\dots,n}} \partial_\gamma f^{k_\beta} dx^{(k_1,\dots,k_{\beta-1})} \wedge dx^\gamma \wedge dx^{(k_{\beta+1},\dots,k_p)}(U_1, \dots, U_p),
\end{aligned}$$

as desired. □

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