Generalizations of Sylvester's determinantal identity

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Due Giorni di Algebra Lineare Numerica

6-7 Marzo 2008

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Hystory

Sylvester was a 19th century mathematician and poet, a collaborator of Cayley, DeMorgan, and Chebyshev. He was primarily an algebraist, and advanced the fields of matrix theory and algebraic equations. He worked by inspiration, and frequently it is difficult to detect a proof in what he confidently asserted. All his work is characterized by powerful imagination and inventiveness.

In 1851 he published, in his words, "a remarkable theorem" showing the relation between determinants and submatrices. However, Sylvester himself stated his identity without proof, and maybe because of that, the equation has historically been regarded as merely a notable curiosity of the Theory of Determinants.

There are at least 7 published proofs of the theorem and later various authors proposed some generalized formulas of this identity with useful applications in several domains.

The classical Sylvester's determinantal identity

$M = (a_{ij})$ a square matrix of order n,
t a fixed integer, $0 \leq t \leq n-1$,
i, j with $t < i, j \leq n$, a given couple of integers.
We define the determinant $a_{i,j}^{(t)}$ of order $(t+1)$ as

$$a_{i,j}^{(t)} = \left| \begin{array}{ccc|c} a_{11} & \cdots & a_{1t} & a_{1j} \\ a_{21} & \cdots & a_{2t} & a_{2j} \\ \vdots & & \vdots & \vdots \\ a_{t1} & \cdots & a_{tt} & a_{tj} \\ \hline a_{i1} & \cdots & a_{it} & a_{ij} \end{array} \right|, \ \text{for } 0 < t \leq n-1 \text{ and } \ a_{i,j}^{(0)} = a_{ij}$$

determinant obtained from the matrix M by extending its leading principal submatrix of order t with the row i and the column j of M.

The classical Sylvester's determinantal identity

Theorem 1 (Sylvester's identity): *Let M be a square matrix of order n and t an integer, $0 \le t \le n-1$. Then, the following identity holds*

$$\det M \cdot \left[a_{t,t}^{(t-1)}\right]^{n-t-1} = \begin{vmatrix} a_{t+1,t+1}^{(t)} & \cdots & a_{t+1,n}^{(t)} \\ \vdots & & \vdots \\ a_{n,t+1}^{(t)} & \cdots & a_{n,n}^{(t)} \end{vmatrix}, \quad \textit{with}\ a_{0,0}^{(-1)} = 1 \qquad (1)$$

J.J. Sylvester stated this theorem in 1851 without proof, later 9 different proofs are presented:

- G. Kowalewski [1948, four proofs]
- F.R. Gantmacher [1959]
- E.H. Bareiss [1968]
- G.I. Malaschonok [1983-1986, two proofs]
- M. Konvalinka [2007]
- ...

Sylvester's identity: some particular cases

- $t = 1$ and $a_{11} \neq 0$, Chió pivotal condensation [Chió,1853]:

$$\det M = \det B / (a_{11})^{n-2}$$

$B = (b_{ij})$, square matrix of order $(n-1)$ with entries

$$b_{ij} = a^{(1)}_{i+1,j+1} = \begin{vmatrix} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{vmatrix}, \text{ for } i, j = 1, \ldots, n-1$$

- $t = n-2$, M matrix partitioned as

$$M = \begin{pmatrix} a & \mathbf{b}^T & e \\ \mathbf{c} & D & \mathbf{f} \\ g & \mathbf{h}^T & l \end{pmatrix} \quad \det M = \begin{vmatrix} D & \mathbf{c} & \mathbf{f} \\ \mathbf{b}^T & a & e \\ \mathbf{h}^T & g & l \end{vmatrix},$$

  - a, e, g, l are scalars,
  - $\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{h}$ are $(n-2)$-length column vectors,
  - D is a $(n-2)$-order square matrix.

Sylvester's identity: some particular cases

applying the Sylvester's identity

$$\det M \cdot \det D = \begin{vmatrix} \begin{vmatrix} D & \mathbf{c} \\ \mathbf{b}^T & a \end{vmatrix} & \begin{vmatrix} D & \mathbf{f} \\ \mathbf{b}^T & e \end{vmatrix} \\ \\ \begin{vmatrix} D & \mathbf{c} \\ \mathbf{h}^T & g \end{vmatrix} & \begin{vmatrix} D & \mathbf{f} \\ \mathbf{h}^T & l \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & \mathbf{b}^T \\ \mathbf{c} & D \end{vmatrix} & \begin{vmatrix} \mathbf{b}^T & e \\ D & \mathbf{f} \end{vmatrix} \\ \\ \begin{vmatrix} \mathbf{c} & D \\ g & \mathbf{h}^T \end{vmatrix} & \begin{vmatrix} D & \mathbf{f} \\ \mathbf{h}^T & l \end{vmatrix} \end{vmatrix}$$

by setting

$$A' = \begin{pmatrix} a & \mathbf{b}^T \\ \mathbf{c} & D \end{pmatrix}, B' = \begin{pmatrix} \mathbf{b}^T & e \\ D & \mathbf{f} \end{pmatrix}, C' = \begin{pmatrix} \mathbf{c} & D \\ g & \mathbf{h}^T \end{pmatrix}, D' = \begin{pmatrix} D & \mathbf{f} \\ \mathbf{h}^T & l \end{pmatrix}$$

we have the simple rule frequently used to obtain recursive algorithms in sequence transformations

$$\boxed{\det M \ \det D = \det A' \ \det D' - \det B' \ \det C'} \qquad (1a)$$

The classical Sylvester's determinantal identity

Sylvester's determinantal identity has been extensively studied, both in the algebraic and in the combinatorial context.
This classical formula is useful for evaluating and transform certain types of determinants, is a very important tools for

- interpolation problems
- extrapolation algorithms
- matrix triangularization

Generalizations of Sylvester's identity

Several authors have deepened the main property of classical Sylvester's identity, some of these have generalized the classical identity obtaining interesting new formulas:

- Yakovlev [1974,1978]
  - control theory
- Gasca, Lopez-Carmona, and Ramirez [1982]
  - interpolation problems
- Mühlbach-Gasca and Beckermann-Gasca [1985,1994]
  - extrapolation formulas and sequence transformations
- Mulders [2001]
  - fraction free algorithms, linear programming.

Notations and Definitions

We introduce the following notations related to a $(n \times m)$ matrix M over a commutative field $\mathbb{K}$.

- An ***index list*** of length $k < n$, $I = (i_1, \ldots, i_k)$, with $1 \leq i_l \leq n$, for $l = 1, \ldots k$ is an integer k-tuple with possible repetition from $\{1, 2, \ldots, n\}$.
- An ***index list without repetition***, with card$(I) = k$.
- An ***ordered index list*** is an index list with the condition $\alpha < \beta \Rightarrow i_\alpha < i_\beta$.

For any $n \in \mathbb{N}^+$ we can define the ordered index list

$$N_n = (1, 2, \ldots, n)$$

Notations and Definitions

Let $I = (i_1, \ldots, i_\alpha) \subseteq N_n$ and $J = (j_1, \ldots, j_\beta) \subseteq N_m$ two ordered index lists. We denote the $(\alpha \times \beta)$ submatrix, extracted from M,

$$M \binom{I}{J} = M \binom{i_1, \ldots, i_\alpha}{j_1, \ldots, j_\beta} = \begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_\beta} \\ \vdots & & \vdots \\ a_{i_\alpha j_1} & \cdots & a_{i_\alpha j_\beta} \end{pmatrix},$$

if $\alpha = \beta$, we denote the corresponding determinant as

$$M \begin{bmatrix} I \\ J \end{bmatrix} = M \begin{bmatrix} i_1, \ldots, i_\alpha \\ j_1, \ldots, j_\alpha \end{bmatrix} = \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_\alpha} \\ \vdots & & \vdots \\ a_{i_\alpha j_1} & \cdots & a_{i_\alpha j_\alpha} \end{vmatrix}.$$

So, $a_{i,j}^{(t)} = M \begin{bmatrix} 1, \ldots, t, i \\ 1, \ldots, t, j \end{bmatrix}$, for $t < i, j \leq n$ and $0 < t \leq n-1$.

Notations and Definitions

With all this notations Sylvester's formula

$$\det M \cdot \left[a_{t,t}^{(t-1)}\right]^{n-t-1} = \begin{vmatrix} a_{t+1,t+1}^{(t)} & \cdots & a_{t+1,n}^{(t)} \\ \vdots & & \vdots \\ a_{n,t+1}^{(t)} & \cdots & a_{n,n}^{(t)} \end{vmatrix}, \quad 0 \leq t \leq n-1$$

can be written as

$$\det M \cdot \left(M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{n-t-1} = \det B_S, \quad 0 \leq t \leq n-1 \tag{2}$$

where B_S is the matrix of order $(n-t)$ with elements

$$b_{ij} = a_{i,j}^{(t)} \quad \text{for} \quad t+1 \leq i, j \leq n$$

Generalization of Yakovlev

Let

$M = (a_{ij})$ *square matrix of order n,*

t *fixed integer, $0 \le t \le n-1$,*

$I = (i_1, \ldots, i_t) \subset N_n$, $J = (j_1, \ldots, j_t) \subset N_n \Rightarrow$ *ordered index lists,*

$I' = (i'_{t+1}, \ldots, i'_n) \subset N_n$, $J' = (j'_{t+1}, \ldots, j'_n) \subset N_n \Rightarrow$ *complementary*
$(I \cup I' = J \cup J' = N_n)$ *ordered index lists.*

From Laplace expansion formula Yakovlev obtained the following identity

$$\det M \left(M \begin{bmatrix} i_1, \ldots, i_t \\ j_1, \ldots, j_t \end{bmatrix} \right)^{n-t-1} = \sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^{\mu} \prod_{\beta=t+1}^{n} M \begin{bmatrix} i_1, \ldots, i_t, i'_{\alpha_\beta} \\ j_1, \ldots, j_t, j'_\beta \end{bmatrix}$$

where $P(\alpha_{t+1}, \ldots, \alpha_n)$ represent the set of all permutations of $(t+1, \ldots, n)$ and μ is the number of inversions needed to pass from $(t+1, \ldots, n)$ to a certain permutation $(\alpha_{t+1}, \ldots, \alpha_n)$.

Classical Sylvester identity as particular case

- $I = J = (1, \ldots, t)$, $I' = J' = (t+1, \ldots, n)$.

$$\det M \cdot \left(M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{n-t-1} = \sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^{\mu} \prod_{\beta=t+1}^{n} \underbrace{M \begin{bmatrix} 1, \ldots, t, \alpha_\beta \\ 1, \ldots, t, \beta \end{bmatrix}}_{\Downarrow \; a^{(t)}_{\alpha_\beta,\beta}}$$

by Leibniz formula

$$\det M \cdot \left(M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{n-t-1} = \sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^{\mu} \prod_{\beta=t+1}^{n} a^{(t)}_{\alpha_\beta,\beta} = \det B_S$$

which is exactly the classical Sylvester identity (2).

Generalization of Yakovlev: examples

Example 1:

$n = 6$, $t = 4$,
$I = (i_1, \ldots, i_4) = (1, 3, 5, 6)$, $J = (j_1, \ldots, j_4) = (1, 2, 4, 6)$,
$I' = (i'_5, i'_6) = (2, 4)$, $J' = (j'_5, j'_6) = (3, 5)$.

$$\det M \cdot M \begin{bmatrix} I \\ J \end{bmatrix} = \sum_{P(\alpha_5,\alpha_6)} (-1)^{\boldsymbol{\mu}} \prod_{\beta=5}^{6} M \begin{bmatrix} I, i'_{\alpha_\beta} \\ J, j'_\beta \end{bmatrix}.$$

since $P(\alpha_5, \alpha_6) = \{(5, 6), (6, 5)\}$, and $\boldsymbol{\mu} = (0, 1)$, we obtain

$$\det M \cdot M \begin{bmatrix} I \\ J \end{bmatrix} = M \begin{bmatrix} I, 2 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 5 \end{bmatrix} - M \begin{bmatrix} I, 4 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 2 \\ J, 5 \end{bmatrix}.$$

Example 2:

$n = 6$, $t = 4$,
$I = J = (1, \dots, 4)$,
$I' = J' = (5, 6)$,
$P(\alpha_5, \alpha_6) = \{(5,6), (6,5)\}$, and $\boldsymbol{\mu} = (0, 1)$.

$$\det M \cdot M \begin{bmatrix} I \\ J \end{bmatrix} = M \begin{bmatrix} I, 5 \\ J, 5 \end{bmatrix} \cdot M \begin{bmatrix} I, 6 \\ J, 6 \end{bmatrix} - M \begin{bmatrix} I, 6 \\ J, 5 \end{bmatrix} \cdot M \begin{bmatrix} I, 5 \\ J, 6 \end{bmatrix},$$

that is

$$\det M \cdot M \begin{bmatrix} 1, \dots, 4 \\ 1, \dots, 4 \end{bmatrix} = \begin{vmatrix} a_{5,5}^{(4)} & a_{5,6}^{(4)} \\ a_{6,5}^{(4)} & a_{6,6}^{(4)} \end{vmatrix} = \det B_S$$

which is the classical Sylvester's identity (2).

Gasca, Lopez-Carmona, Ramirez

Gasca, Lopez-Carmona, and Ramirez [1982] proved a first useful generalization of Sylvester's determinantal identity.

This generalized identity has been applied to the derivation of a recurrence interpolation formula for the solution of a multivariate interpolation problem.

The interpolation formula, following Brezinski's method, take the form of a quotient of two determinants and in particular cases leads to Aitken-like and Neville-like algorithms.

Generalization of Gasca, Lopez, Ramirez

Let

$M = (a_{ij})$ *square matrix of order n,*

t, q *fixed positive integers s.t. $n = t + q$,*

$J_k = (j_1^k, \ldots, j_{t+1}^k) \subset N_n,\ k = 1, \ldots, q \Rightarrow$ *set of q ordered index lists,*

$S_k = J_k \cap J_{k+1} = (s_1^k, \ldots, s_t^k),\ k = 1, \ldots, q-1.$

Let B the matrix with elements

$$b_{ij} = M \begin{bmatrix} 1, \ldots, t, & t+i \\ j_1^j, \ldots, j_t^j, & j_{t+1}^j \end{bmatrix} \qquad 1 \le i, j \le q$$

the authors give the generalization of Sylvester's identity

$$\det B = c \cdot \det M \cdot \prod_{k=1}^{q-1} M \begin{bmatrix} 1, \ldots, t \\ s_1^k, \ldots, s_t^k \end{bmatrix}$$

where c is a sign factor ($c = 0$ or $c = \pm 1$) which does not depend on the element a_{ij} of M, but only on the set of J_k.

Classical Sylvester identity as particular case

- $J_k = (1, \ldots, t, t+k), \quad$ for all $k = 1, 2, \ldots, q$
- $S_k = (1, \ldots, t)$ for all k.

The previous generalized identity becomes

$$\det B = \det M \cdot \prod_{k=1}^{q-1} M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix}$$

with $B \equiv B_S$, so

$$\det B_S = \det M \cdot \left(M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{q-1}$$

which is exactly the classical Sylvester identity (2).

Example:

$n = 5$, $t = 2$, $q = 3$,
$J_1 = (1,3,4), J_2 = (1,4,5), J_3 = (2,4,5),$
$S_1 = (1,4), S_2 = (4,5)$.
Let

$$B = \begin{pmatrix} M \begin{bmatrix} 1,2,3 \\ 1,3,4 \end{bmatrix} & M \begin{bmatrix} 1,2,3 \\ 1,4,5 \end{bmatrix} & M \begin{bmatrix} 1,2,3 \\ 2,4,5 \end{bmatrix} \\ M \begin{bmatrix} 1,2,4 \\ 1,3,4 \end{bmatrix} & M \begin{bmatrix} 1,2,4 \\ 1,4,5 \end{bmatrix} & M \begin{bmatrix} 1,2,4 \\ 2,4,5 \end{bmatrix} \\ M \begin{bmatrix} 1,2,5 \\ 1,3,4 \end{bmatrix} & M \begin{bmatrix} 1,2,5 \\ 1,4,5 \end{bmatrix} & M \begin{bmatrix} 1,2,5 \\ 2,4,5 \end{bmatrix} \end{pmatrix},$$

since for this particular case $c = -1$, we have

$$\det B = -\det M \cdot M \begin{bmatrix} 1,2 \\ 1,4 \end{bmatrix} \cdot M \begin{bmatrix} 1,2 \\ 4,5 \end{bmatrix}$$

Generalization of Beckermann, Gasca, Mühlbach

Sylvester's classical identity can be interpreted as an extension of Leibniz's definition of a matrix determinant, known as the Muir's law of extensible minors.

Mühlbach [1990] using the relation between Muir's Law of Extensible Minors, Sylvester's identity and the Schur complement, presented a new principle for extending determinantal identities which generalizes Muir's Law.

As applications of this technique, he derived a generalization of Sylvester's identity, which is the same as the one proposed by Gasca and Mühlbach [1985] .

Later, Beckermann and Mühlbach [1994] gave an approach, shorter and conceptually simpler than the earlier attempts, for a general determinantal identity of Sylvester's type.

Generalization of Beckermann, Gasca, Mühlbach

Let

$M = (a_{ij})$ *matrix of order* $(n \times m)$*,*

q *fixed positive integer,*

$I = (i_1, \ldots, i_\alpha) \subset N_n$, $J = (j_1, \ldots, j_\beta) \subset N_m \Rightarrow$ *ordered index lists,*

$\mathbf{z}_k = (z_{k,j})_{j \in N_m}$, $k = 1, \ldots, q$ *row vectors.*

$M \binom{I}{J}$ is bordered with the q row vectors $\mathbf{z}'_k = (z_{k,j_\lambda})_{\lambda=1,\ldots,\beta}$, $k = 1, \ldots, q$, extracted from the $\mathbf{z}_k$, so we have

$$M \left(\begin{matrix} i_1, \ldots, i_\alpha | \, 1, \ldots, q \\ j_1, \ldots, j_\beta \end{matrix} \right) = \left(\begin{array}{ccc} a_{i_1,j_1} & \cdots & a_{i_1,j_\beta} \\ \vdots & & \vdots \\ a_{i_\alpha,j_1} & \cdots & a_{i_\alpha,j_\beta} \\ \hline z_{1,j_1} & \cdots & z_{1,j_\beta} \\ \vdots & & \vdots \\ z_{q,j_1} & \cdots & z_{q,j_\beta} \end{array} \right)$$

Generalization of Beckermann, Gasca, Mühlbach

Let

$I_1, \ldots, I_q \subset N_n,\ J_1, \ldots, J_q \subset N_m \Rightarrow$ *set of index lists,*

s.t. $\mathrm{card}(J_k) = \mathrm{card}(I_k) + 1$ for $k = 1, \ldots, q$

$I^{(k)} := \bigcap_{i=1}^{k} I_i$ and $J^{(k)} := \bigcup_{j=1}^{k} J_j$, for $k = 1, \ldots, q$

$I_0 \subseteq I^{(q)}$ s.t. $\mathrm{card}(I_0) = \mathrm{card}(J^{(q)}) - q$.

Beckermann and Mühlbach give the generalized formula

$$\det B = c \cdot M \begin{bmatrix} I_0 | \, 1, \ldots, q \\ J^{(q)} \end{bmatrix},$$

where c depend only on $M \begin{pmatrix} I_0 \\ J^{(q)} \end{pmatrix}$ and can be computed explicitly only in some particular cases, B is the matrix with elements $b_{ij} = M \begin{bmatrix} I_i | \, j \\ J_i \end{bmatrix}$, $1 \le i, j \le q$, that is $M \begin{bmatrix} I_i \\ J_i \end{bmatrix}$ bordered with the row vector $\mathbf{z}'_j = (z_{j, j_\lambda})_{\lambda \in J_i}$

Classical Sylvester identity as particular case

- $t, q \in \mathbb{N}^+$, such that $t + q = n$,
- $J_k = (1, \dots, t, t + k),$ for $\quad k = 1, \dots, q$,
- $I_k = (1, \dots, t),$ for $\quad k = 1, \dots, q$,
- $I_0 = I^{(q)} = (1, \dots, t), \quad J^{(q)} = N_n = (1, \dots, n)$,
- $\mathbf{z}'_k = M \binom{t+k}{J_k}$, for $\quad k = 1, \dots, q$.

With this choice we have

- $c = \left(M \begin{bmatrix} 1, \dots, t \\ 1, \dots, t \end{bmatrix} \right)^{q-1}$
- $B \equiv B_S$

then we obtain the classical Sylvester's identity (2)

$$\det B_S = \det M \cdot \left(M \begin{bmatrix} 1, \dots, t \\ 1, \dots, t \end{bmatrix} \right)^{q-1}$$

Example:

$n = 6, m = 7, q = 3,$
$I_1 = (2,3,4), I_2 = (2,4), I_3 = (1,2,4) \Rightarrow I^{(3)} = (2,4),$
$J_1 = (2,3,4,5), J_2 = (2,3,4), J_3 = (2,3,4,7) \Rightarrow J^{(3)} = (2,3,4,5,7).$
Choose any $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, and $I_0 = I^{(3)}$. Let

$$B = \begin{pmatrix} M\begin{bmatrix} 2,3,4\,|\,1 \\ 2,3,4,5 \end{bmatrix} & M\begin{bmatrix} 2,3,4\,|\,2 \\ 2,3,4,5 \end{bmatrix} & M\begin{bmatrix} 2,3,4\,|\,3 \\ 2,3,4,5 \end{bmatrix} \\ M\begin{bmatrix} 2,4\,|\,1 \\ 2,3,4 \end{bmatrix} & M\begin{bmatrix} 2,4\,|\,2 \\ 2,3,4 \end{bmatrix} & M\begin{bmatrix} 2,4\,|\,3 \\ 2,3,4 \end{bmatrix} \\ M\begin{bmatrix} 1,2,4\,|\,1 \\ 2,3,4,7 \end{bmatrix} & M\begin{bmatrix} 1,2,4\,|\,2 \\ 2,3,4,7 \end{bmatrix} & M\begin{bmatrix} 1,2,4\,|\,3 \\ 2,3,4,7 \end{bmatrix} \end{pmatrix},$$

with $\det B \neq 0$. Since $M\begin{bmatrix} 2,4\,|\,1,2,3 \\ 2,3,4,5,7 \end{bmatrix} \neq 0$, there exists c, such that the generalized formula holds.

Beckermann, Gasca and Mühlbach: applications

The generalization of Beckermann and Mühlbach is very useful for obtaining other determinantal identities, since c depends on M but not on the rows corresponding to indexes from $N_n \setminus I_0$.

For example is possible to obtain the identities of Schweins and Monge, also a generalized version of Kronecker's theorem can be proved with this generalization of Sylvester 's identity.

As an application of this general determinantal identity old and new recurrence relations for the E-transforms are derived.

Generalization of Mulders

A recent generalization of Sylvester's identity is due to Mulders [2001]. The author follows the idea of some works of Bareiss, where the Sylvester's identity is used to prove that certain Gaussian elimination algorithms, transforming a matrix into an upper-triangular form, are fraction-free.

Fraction-free (or Integer-preserving) algorithms also have applications in the computation of matrix rational approximants, matrix GCDs, and generalized Richardson extrapolation processes [B. Beckermann,G. Labahn, 2000], fast matrix triangularization [D.A. Bini, L. Gemignani,1998].

They are used for controlling, in exact arithmetic, the growth of intermediate results.

Generalization of Mulders

Let

$M = (a_{ij})$ *matrix of order* $(n \times m)$,

t *fixed integer,* $0 \leq t \leq \min(m, n)$,

$I = (i_1, \ldots, i_t) \subseteq N_n \Rightarrow$ *index list* ***without repetition*** *but not necessarily ordered,*

$J = (j_1, \ldots, j_t) \subseteq N_m \Rightarrow$ *index list* ***with possible repetition*** *and not necessarily ordered.*

We can define an equivalence class $[(i_1, j_1), \ldots, (i_t, j_t)]$ of all possible permutations of the t pairs and define the determinant

$$a^{[(i_1,j_1),\ldots,(i_t,j_t)]} = M \begin{bmatrix} i_1, \ldots, i_t \\ j_1, \ldots, j_t \end{bmatrix},$$

with $a^{[\emptyset]} = M \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix} = 1$ when $t = 0$.

Remark: When J has repeated elements, the determinant is equal to zero.

Generalization of Mulders

We define the following operation $[(i_1, j_1), \ldots, (i_t, j_t)] \leftarrow [(i, j)]$ wich produces a new class of pairs, by adding to the first class the pair (i, j) when $i \notin I$, by replacing the pairs (i_k, j_k) of the first class by the pairs (i, j), when exists $1 \leq k \leq t$ such that $i_k = i$.

For i and j, $1 \leq i \leq n, 1 \leq j \leq m$, we consider the determinant $a^{[(i_1,j_1),\ldots,(i_t,j_t)]\leftarrow[(i,j)]}$ which is evaluated as follows:

- When $i \notin I$, then we simply extend the matrix with the ith row and jth column of M, and we compute the determinant, that is

$$a^{[(i_1,j_1),\ldots,(i_t,j_t)]\leftarrow[(i,j)]} = M \begin{bmatrix} i_1, \ldots, i_t, i \\ j_1, \ldots, j_t, j \end{bmatrix}.$$

- When it exists $1 \leq k \leq t$ such that $i_k = i$, we replace in $a^{[(i_1,j_1),\ldots,(i_t,j_t)]}$ the pair (i_k, j_k) with the pair (i, j), that is we take

$$a^{[(i_1,j_1),\ldots,(i_t,j_t)]\leftarrow[(i,j)]} = M \begin{bmatrix} i_1, \ldots, i_k{=}i, \ldots, i_t \\ j_1, \ldots, \quad j \quad, \ldots, j_t \end{bmatrix}.$$

Generalization of Mulders

We consider now the particular case where $i_k = k, j_k = k$, for all $k = 1, \ldots, t$, that is $I = J = N_t$.

When $t < i, j \leq \min(n, m)$, we recover the usual definition $a_{i,j}^{(t)}$,

in the other cases, denoting $\widetilde{a}_{i,j}^{(t)} = a^{[(1,1),\ldots,(t,t)]\leftarrow[(i,j)]}$, we have

$$\begin{cases}
\text{If } t < i, j & \widetilde{a}_{i,j}^{(t)} \text{ of order } t+1 & \widetilde{a}_{i,j}^{(t)} = a_{i,j}^{(t)} \\
\text{If } j \leq t < i & \widetilde{a}_{i,j}^{(t)} \text{ of order } t+1 & \widetilde{a}_{i,j}^{(t)} = 0 \\
\text{If } i, j \leq t, i \neq j & \widetilde{a}_{i,j}^{(t)} \text{ of order } t & \widetilde{a}_{i,j}^{(t)} = 0 \\
\text{If } i = j \leq t & \widetilde{a}_{i,j}^{(t)} \text{ of order } t & \widetilde{a}_{i,j}^{(t)} = a^{[(1,1),\ldots,(t,t)]} \\
\text{If } i \leq t < j & \widetilde{a}_{i,j}^{(t)} \text{ of order } t & \widetilde{a}_{i,j}^{(t)} \text{ is a new determinant.}
\end{cases}$$

Generalization of Mulders

Let

$I = J = N_t$,

$0 \leq t \leq \min(n, m)$,

$0 \leq p, q \leq t$ and $1 \leq s \leq \min(n - p, m - q)$.

Mulder formulated the following generalized Sylvester's identity

$$a^{[(1,1),\ldots,(p,p),(p+1,q+1),\ldots,(p+s,q+s)]} \left[\widetilde{a}_{t,t}^{(t-1)}\right]^{s-1} = \begin{vmatrix} \widetilde{a}^{(t)}_{p+1,q+1} & \cdots & \widetilde{a}^{(t)}_{p+1,q+s} \\ \vdots & & \vdots \\ \widetilde{a}^{(t)}_{p+s,q+1} & \cdots & \widetilde{a}^{(t)}_{p+s,q+s} \end{vmatrix}$$

where

$$a^{[(1,1),\ldots,(p,p),(p+1,q+1),\ldots,(p+s,q+s)]} = M \begin{bmatrix} 1, \ldots, p, p+1, \ldots, p+s \\ 1, \ldots, p, q+1, \ldots, q+s \end{bmatrix}$$

is a determinant of order $p + s$.

Remark: This identity is only valid if the determinant on the right is different from zero (one or more columns of this determinant can be zero).

Classical Sylvester identity as particular case

- $I = J = N_t$,
- $m = n, p = q = t$,
- $s = n - t$.

We obtain the classical Sylvester's identity (2).

Example :

$n = 6$, $m = 8$,
$I = J = N_5$,
$p = 3, q = 4, s = 3$.

The left hand side of Mulder's formula is

$$a^{[(1,1),(2,2),(3,3),(4,5),(5,6),(6,7)]} \left[\widetilde{a}^{(4)}_{5,5}\right]^2 = M\begin{bmatrix}1,2,3,4,5,6\\1,2,3,5,6,7\end{bmatrix}\left(M\begin{bmatrix}1,\ldots,5\\1,\ldots,5\end{bmatrix}\right)^2.$$

The right hand side is the following determinant of order 3

$$\begin{vmatrix} \widetilde{a}^{(5)}_{4,5} & \widetilde{a}^{(5)}_{4,6} & \widetilde{a}^{(5)}_{4,7} \\ \widetilde{a}^{(5)}_{5,5} & \widetilde{a}^{(5)}_{5,6} & \widetilde{a}^{(5)}_{5,7} \\ \widetilde{a}^{(5)}_{6,5} & \widetilde{a}^{(5)}_{6,6} & \widetilde{a}^{(5)}_{6,7} \end{vmatrix}$$

Generalization of Mulders: example

that is, thanks to the operation defined,

$$\begin{vmatrix} 0 & M\begin{bmatrix}1,2,3,4,5\\1,2,3,6,5\end{bmatrix} & M\begin{bmatrix}1,2,3,4,5\\1,2,3,6,5\end{bmatrix} \\ M\begin{bmatrix}1,\ldots,5\\1,\ldots,5\end{bmatrix} & M\begin{bmatrix}1,2,3,4,5\\1,2,3,4,6\end{bmatrix} & M\begin{bmatrix}1,2,3,4,5\\1,2,3,4,7\end{bmatrix} \\ 0 & M\begin{bmatrix}1,\ldots,6\\1,\ldots,6\end{bmatrix} & M\begin{bmatrix}1,2,3,4,5,6\\1,2,3,4,5,7\end{bmatrix} \end{vmatrix}$$

Generalization of Mulders: applications

This generalization of Sylvester's identity was used to prove that also certain random Gaussian elimination algorithms are fraction-free.

The fraction-free random Gaussian elimination algorithm, obtained is that way, has been used in the simplex method, to solve linear programming problems exactly and for finding a solution of $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$, with A a $(m \times n)$ matrix.

Let
$M = (a_{ij})$ matrix of order $(n \times m)$,
t, s fixed integers, $0 < t \leq \min(n, m)$,$1 \leq s \leq \min(n - t, m - t)$,
such that it exists $q \in \mathbb{N}$ which satisfies $\min(n, m) = t + qs$.
We can build a sequence of submatrices M_k of M,

$$M_k = M \binom{1, \ldots, t, t+1, \ldots, t+ks}{1, \ldots, t, t+1, \ldots, t+ks}, \text{ for } 0 \leq k \leq q$$

by definition

- $\det M_0 = a_{t,t}^{(t-1)}$
- $\det M_q = a_{r,r}^{(r-1)}$, with $r = \min(n, m)$.

We set $m_k = t + ks$, for $k = 0, \ldots, q-1$, and we consider the matrix B_k of order s, with elements

$$b_{ij}^{(k)} = M \begin{bmatrix} 1, \ldots, m_k, m_k + i \\ 1, \ldots, m_k, m_k + j \end{bmatrix} = a_{m_k+i, m_k+j}^{(m_k)}, \qquad 1 \leq i, j \leq s.$$

Theorem 2 (Generalized Sylvester's identity)
Let M be a matrix of dimension $n \times m$, and M_0 its square leading principal submatrix of fixed order t, $0 < t \leq \min(n, m)$. Then, chosen s, $q \in \mathbb{N}$, $1 \leq s \leq \min(n-t, m-t)$, such that $\min(n, m) = t + q\, s$, the following identities hold for $0 \leq k \leq q$

$$\frac{\det\ M_k}{[\det\ M_0]^{(s-1)^k}} = \frac{\displaystyle\prod_{\substack{i=1 \\ i \text{ odd}}}^{k-1} [\det B_i]^{(s-1)^{k-1-i}}}{\displaystyle\prod_{\substack{j=0 \\ j \text{ even}}}^{k-2} [\det B_j]^{(s-1)^{k-1-j}}}, \qquad \textit{for } k \text{ even}, \tag{3}$$

$$\det\ M_k \cdot [\det\ M_0]^{(s-1)^k} = \frac{\displaystyle\prod_{\substack{i=0 \\ i \text{ even}}}^{k-1} [\det B_i]^{(s-1)^{k-1-i}}}{\displaystyle\prod_{\substack{j=1 \\ j \text{ odd}}}^{k-2} [\det B_j]^{(s-1)^{k-1-j}}}, \qquad \textit{for } k \text{ odd}. \tag{4}$$

A new generalization: proof

The proof is made by mathematical induction over k.

Empty product $\Rightarrow \prod_{i=\alpha}^{\beta} \odot$ =1 when $\alpha < \beta$

- For $k = 0$, (3) is trivially satisfied since, in the right hand side, we have a ratio of empty products.
- For $k = 1$, the product in the denominator of the right hand side is an empty product, and

  $$\det\ M_1 \cdot [\det\ M_0]^{s-1} = \det B_0,$$

  is exactly the Sylvester's identity (2) applied to M_1 with $s = n - t$, then (4) holds.
- In the inductive step we prove that, if (4) holds for k odd, then (3) holds for $k + 1$ even, moreover if (3) holds for k even, then (4) holds for $k + 1$ odd.

Classical Sylvester identity as particular case

- $k = q = 1$, so $s = \min(n, m) - t$,

we have

$$\det\ M_1 \cdot [\det\ M_0]^{s-1} = \det B_0,$$

where $M_1 = M \begin{pmatrix} 1, \ldots, t+s \\ 1, \ldots, t+s \end{pmatrix}$, $\det M_0 = a_{t,t}^{(t-1)}$, $B_0 = (b_{ij}^{(0)})$

with $b_{ij}^{(0)} = M \begin{bmatrix} 1, \ldots, t, t+i \\ 1, \ldots, t, t+j \end{bmatrix} = a_{t+i,t+j}^{(t)}$, for $1 \leq i, j \leq s$.

Being $B_0 \equiv B_s$, we recover the Sylvester's identity (2).

A new generalization: particular case

Let M a square matrix of order n, $s = 2$, chosen t such that $n - t$ is even, $q = (n-t)/2$, even or odd. We set $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21}^T & A_{22} \end{pmatrix}$,

- A_{11} is a square matrices of order t
- A_{22} is a square matrix of order $(n-t) = 2q$
- A_{12}, A_{21} matrices of dimension $t \times (n-t) = t \times 2q$.

The new formulae become

$$\frac{\det\ M}{\det A_{11}} = \prod_{\substack{i=1 \\ i \text{ odd}}}^{q-1} \det B_i \, / \prod_{\substack{j=0 \\ j \text{ even}}}^{q-2} \det B_j \ , \qquad q \text{ even} \, ,$$

$$\det\ M \ \cdot \det A_{11} = \prod_{\substack{i=0 \\ i \text{ even}}}^{q-1} \det B_i \, / \prod_{\substack{j=1 \\ j \text{ odd}}}^{q-2} \det B_j \ , \qquad q \text{ odd},$$

with B_i matrices of order 2.

A new generalization: examples

For $q = 1$, let $D = A_{11}$ be a square matrix of order $n-2$,

$$A_{22} = \begin{pmatrix} a & e \\ g & l \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} \mathbf{c} & \mathbf{f} \end{pmatrix}, \qquad A_{21} = \begin{pmatrix} \mathbf{b} & \mathbf{h} \end{pmatrix},$$

- a, e, g, l scalars,
- $\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{h}$ column vectors of dimension $(n-2)$.

we have $$\det M \cdot \det D = \det B_0$$

$$\det B_0 = \begin{vmatrix} \begin{vmatrix} D & \mathbf{c} \\ \mathbf{b}^T & a \end{vmatrix} & \begin{vmatrix} D & \mathbf{f} \\ \mathbf{b}^T & e \end{vmatrix} \\ & \\ \begin{vmatrix} D & \mathbf{c} \\ \mathbf{h}^T & g \end{vmatrix} & \begin{vmatrix} D & \mathbf{f} \\ \mathbf{h}^T & l \end{vmatrix} \end{vmatrix},$$

we recover exactly the simple rule (1a) frequently used to obtain recursive algorithms in sequences transformations.

A new generalization: examples

Let now M' a square matrix of order n, with $n = t + 4$, partitioned as

$$M' = \begin{pmatrix} A & B^T & E \\ C & D & F \\ G^T & H^T & L \end{pmatrix} \Rightarrow M = \begin{pmatrix} D & C & F \\ B^T & A & E \\ H^T & G^T & L \end{pmatrix},$$

- D is a square matrices of order t
- A, E, G, L are square matrices of order 2
- B, C, F, H are of dimension $t \times 2$

$\det M' = \det M$, so we obtain

$$\frac{\det M}{\det D} = \frac{\det B_1}{\det B_0},$$

with B_0 and B_1 of order 2.

A new generalization: examples

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & , \end{pmatrix},\; C = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix},\; E = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix},\; F = \begin{pmatrix} \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix},$$

$G = \begin{pmatrix} \mathbf{g}_1 & \mathbf{g}_2 \end{pmatrix}$, $H = \begin{pmatrix} \mathbf{h}_1 & \mathbf{h}_2 \end{pmatrix}$ and $A = (a_{ij}), L = (l_{ij})$ for $i, j = 1, 2$

$$\det B_0 = \begin{vmatrix} \begin{vmatrix} D & \mathbf{c}_1 \\ \mathbf{b}_1{}^T & a_{11} \end{vmatrix} & \begin{vmatrix} D & \mathbf{c}_2 \\ \mathbf{b}_1{}^T & a_{12} \end{vmatrix} \\ \begin{vmatrix} D & \mathbf{c}_1 \\ \mathbf{b}_2{}^T & a_{21} \end{vmatrix} & \begin{vmatrix} D & \mathbf{c}_2 \\ \mathbf{b}_2{}^T & a_{22} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & \mathbf{b}_1{}^T \\ \mathbf{c}_1 & D \end{vmatrix} & \begin{vmatrix} a_{12} & \mathbf{b}_1{}^T \\ \mathbf{c}_2 & D \end{vmatrix} \\ \begin{vmatrix} a_{21} & \mathbf{b}_2{}^T \\ \mathbf{c}_1 & D \end{vmatrix} & \begin{vmatrix} a_{22} & \mathbf{b}_2{}^T \\ \mathbf{c}_2 & D \end{vmatrix} \end{vmatrix},$$

$$\det B_1 = \begin{vmatrix} \begin{vmatrix} D & C & \mathbf{f}_1 \\ B^T & A & \mathbf{e}_1 \\ \mathbf{h}_1{}^T & \mathbf{g}_1^T & l_{11} \end{vmatrix} & \begin{vmatrix} D & C & \mathbf{f}_2 \\ B^T & A & \mathbf{e}_2 \\ \mathbf{h}_1{}^T & \mathbf{g}_1^T & l_{12} \end{vmatrix} \\ \begin{vmatrix} D & C & \mathbf{f}_1 \\ B^T & A & \mathbf{e}_1 \\ \mathbf{h}_2{}^T & \mathbf{g}_2^T & l_{21} \end{vmatrix} & \begin{vmatrix} D & C & \mathbf{f}_2 \\ B^T & A & \mathbf{e}_2 \\ \mathbf{h}_2{}^T & \mathbf{g}_2^T & l_{22} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} A & B^T & \mathbf{e}_1 \\ C & D & \mathbf{f}_1 \\ \mathbf{g}_1^T & \mathbf{h}_1{}^T & l_{11} \end{vmatrix} & \begin{vmatrix} A & B^T & \mathbf{e}_2 \\ C & D & \mathbf{f}_2 \\ \mathbf{g}_1^T & \mathbf{h}_1{}^T & l_{12} \end{vmatrix} \\ \begin{vmatrix} A & B^T & \mathbf{e}_1 \\ C & D & \mathbf{f}_1 \\ \mathbf{g}_2^T & \mathbf{h}_2{}^T & l_{21} \end{vmatrix} & \begin{vmatrix} A & B^T & \mathbf{e}_2 \\ C & D & \mathbf{f}_2 \\ \mathbf{g}_2^T & \mathbf{h}_2{}^T & l_{22} \end{vmatrix} \end{vmatrix}.$$

This application is a generalization of a previous particular case.

A new generalization: applications

This new genaralization of the Sylvester's determinantal identity establish a relation between determinants obtained by bordering a leading principal submatrix of order t of a matrix M, with blocks of s rows and s columns.

This formula can be used in building new extrapolation algorithms for vector sequences.

Conclusions

- We presented several generalizations of the Sylvester's determinantal identity, proposed from various authors, and described in an unified way.
- We proposed a new generalizations of the Sylvester's determinantal identity, which expresses the determinant of a matrix in relation with the determinant of the bordered matrices obtained adding more than one row and one column to the original matrix.
- It is our intention, as a future work, to study also other determinantal formulae, to generalize them, and to try to apply the most adapted ones to the construction of new extrapolation processes.

M. Redivo–Zaglia, M.R. Russo, *"Generalizations of Sylvester's determinantal identity*", submitted