Strategie per l'analisi della sensibilità alle perturbazioni delle matrici tridiagonali di Toeplitz complesse*

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Matrix nearness problems
Normality, Defectiveness, Rank-deficiency
[Henrici 62] [Wilkinson 65, 72, 84a, 84b, 86]
[Ruhe 75, 87] [Demmel 87,90] [Higham 89]
[Alam Bora 04] [Burke Lewis Overton 08]

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Tridiagonal Toeplitz structure

(Structured distance to normality of a *real* tridiagonal Toeplitz matrix in [N., Pasquini, Reichel ETNA 07])

OUTLINE

- Well known formulae, simple computations and remarks

- Matrices in $\mathcal{N}_{\mathcal{T}}$, nearest matrix in $\mathcal{N}_{\mathcal{T}}$, structured distance from normality

- Normalized distance $d_F(T_0, \mathcal{N}_{\mathcal{T}}) / \|T_0\|_F$

- Matrices in $\mathcal{M}_{\mathcal{T}}$, nearest matrix in $\mathcal{M}_{\mathcal{T}}$, structured distance from $\mathcal{M}_{\mathcal{T}}$

- Normalized distance $d_F(T_0, \mathcal{M}_{\mathcal{T}}) / \|T_0\|_F$

- Sensitivity of the eigenvalues to perturbations in the matrix

- Matrices in $\mathcal{S}_{\mathcal{T}}$, nearest matrix in $\mathcal{S}_{\mathcal{T}}$, structured distance from $\mathcal{S}_{\mathcal{T}}$

- Sensitivity of the matrix T_0

WELL KNOWN FORMULAE, SIMPLE COMPUTATIONS AND REMARKS

$$T = T(n; \sigma, \delta, \tau) = \begin{bmatrix} \delta & \tau & & & & \\ \sigma & \delta & \tau & & & \\ & \sigma & . & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & . & . & \tau \\ & & & & & \sigma & \delta \end{bmatrix}$$

$$\boxed{\lambda_h(T) = \delta + 2\sqrt{\sigma\,\tau}\,\cos\tfrac{h\,\pi}{n+1}\,, \qquad h = 1 : n}$$

[Bellman 60]

If $\sigma\tau \neq 0$, T has n simple eigenvalues which lie on the closed segment of the complex plane

$$\delta + te^{i(\arg\sigma + \arg\tau)/2}, \qquad t \in \mathbf{R}$$

$$|t| \leq 2\sqrt{|\sigma|\,|\tau|}\cos\frac{\pi}{n+1}.$$

The eigenvalues are symmetrically arranged with respect to δ. If $\sigma = 0$ and $\tau \neq 0$ [or: if $\sigma \neq 0$ and $\tau = 0$], T has the unique eigenvalue δ which is defective and has geometric multiplicity 1. Right and left eigenvectors are the first and the last column [or: the last and the first column], respectively, of the identity matrix I. The case $\sigma = \tau = 0$ is trivial.

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Right and left eigenvector components:

$$x_{hk} = (\sigma/\tau)^{k/2}\sin\left(\frac{h\,k\,\pi}{n+1}\right), k = 1:n, h = 1:n$$

[Meyer 00]

$$\boxed{y_{hk} = \left(\overline{\tau/\sigma}\right)^{k/2} \sin\left(\frac{h\,k\,\pi}{n+1}\right), k = 1:n, h = 1:n}$$

A structured analysis reduces to the case of $\sigma\,\tau \neq 0$ and to the study of the sensitivity to perturbations on σ and τ.

This is confirmed by the eigenvectors, which do not depend on δ.

Consequently the analysis is lead on the eigenvalues $\lambda_h(T_0)$, $h = 1:n$, of the matrix $T_0 = T(n; \sigma, 0, \tau)$

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Spectral radius $\rho(T) = \max(|\lambda_1|, |\lambda_n|) =$

$$\max\left(\left|\delta + 2\sqrt{\sigma\,\tau}\cos\frac{\pi}{n+1}\right|, \left|\delta + 2\sqrt{\sigma\,\tau}\cos\frac{n\,\pi}{n+1}\right|\right).$$

In particular:

$\rho(T_0) = 2\sqrt{|\sigma|\ |\tau|}\cos\frac{\pi}{n+1}$.

And, if T_0^{-1} exists ($\sigma\tau \neq 0$, n even),

$$\rho(T_0^{-1}) = \frac{1}{2\sqrt{|\sigma|\ |\tau|}\cos\frac{n\pi}{2(n+1)}}.$$

MATRICES IN $\mathcal{N}_{\mathcal{T}}$, NEAREST MATRIX IN $\mathcal{N}_{\mathcal{T}}$, STRUCTURED DISTANCE FROM NORMALITY

$\mathcal{T}$:= the subspace of $\mathbf{C}^{n\times n}$ formed by the tridiagonal Toeplitz matrices

$\mathcal{N}$:= the algebraic variety of the normal matrices of $\mathbf{C}^{n\times n}$

$\mathcal{N}_{\mathcal{T}}$:= the algebraic variety of the normal tridiagonal Toeplitz matrices of $\mathbf{C}^{n\times n}$

$$\mathcal{N}_{\mathcal{T}} = \mathcal{N} \bigcap \mathcal{T}$$

$$d_F(T, \mathcal{N}_{\mathcal{T}}) := \min_{T^* \in \mathcal{N}_{\mathcal{T}}} \|T - T^*\|_F$$

$$\boxed{T \text{ is normal iff } |\sigma| = |\tau| \,.}$$

(Real case in [N., Pasquini, Reichel 07])

Theorem *Let $T = T(n; \sigma, \delta, \tau)$ be any matrix in $\mathcal{T}$. There is a unique matrix $T^* = T^*(n; \sigma^*, \delta^*, \tau^*) \in \mathcal{N}_{\mathcal{T}}$ that minimizes $\|T^* - T\|_F$ in $\mathcal{N}_{\mathcal{T}}$*

$$\sigma^* = \frac{|\sigma| + |\tau|}{2} e^{i \arg \sigma},$$

$$\delta^* = \delta,$$

$$\tau^* = \frac{|\sigma| + |\tau|}{2} e^{i \arg \tau}.$$

Proof.

$$\sigma^* = \rho^* \, e^{i \,\text{arg}\, \sigma}, \tau^* = \rho^* \; e^{i \,\text{arg}\, \tau},$$

$$u(\rho) = (\rho - |\sigma|)^2 + (\rho - |\tau|)^2,$$

$$\rho^* = \frac{|\sigma| + |\tau|}{2}.$$

$\diamondsuit$

Remark *The eigenvalues of the normal tridiagonal Toeplitz matrix* $T^* = T^*(n; \sigma^*, \delta^*, \tau^*)$ *closest to* $T = T(n; \sigma, \delta, \tau)$ *are*

$$\lambda_h(T^*) = \delta + (|\sigma| + |\tau|) \; e^{i(\text{arg}\,\sigma + \text{arg}\,\tau)/2} \; \cos \frac{h\pi}{n+1},$$

for $h = 1 : n$, *and lie on the closed segment*

$$\delta + t \, e^{i(\text{arg}\,\sigma + \text{arg}\,\tau)/2}, \qquad t \in \mathbf{R}$$

$$|t| \leq (|\sigma| + |\tau|) \; \cos \frac{\pi}{n+1}.$$

Note that, since

$$|\sigma| + |\tau| - 2\sqrt{|\sigma| \; |\tau|} = \left(\sqrt{|\sigma|} - \sqrt{|\tau|}\right)^2,$$

this segment properly contains the preceding one.

$$\boxed{d_F(T, \mathcal{N}_{\mathcal{T}}) = \sqrt{\tfrac{n-1}{2}}\, \left| |\sigma| - |\tau| \right| .}$$

Remark $d_F(T, \mathcal{N}_{\mathcal{T}})$ *is independent of* δ *but it depends on* δ *the closest normal matrix* T^**. Moreover,* $T_1 = T(n, \sigma, \delta_1, \tau)$*,* $T_2 = T(n, \sigma, \delta_2, \tau)$*, implies* $\left\| T_1^* - T_2^* \right\|_F = \|T_1 - T_2\|_F = \sqrt{n}\, |\delta_1 - \delta_2|$ *.*

Theorem $T \notin \mathcal{N}_{\mathcal{T}}$*.*

$$\|\lambda - \lambda^*\|_\infty < \left(\sqrt{|\sigma|} - \sqrt{|\tau|} \right)^2$$

Proof.

$$|\lambda_h(T) - \lambda_h(T^*)| = \left| 2\sqrt{|\sigma|\,|\tau|} - |\sigma| - |\tau| \right| \left| \cos \frac{h\pi}{n+1} \right|$$

$$= \left(\sqrt{|\sigma|} - \sqrt{|\tau|}\right)^2 \left|\cos\frac{h\pi}{n+1}\right| < \left(\sqrt{|\sigma|} - \sqrt{|\tau|}\right)^2.$$

$\diamondsuit$

Theorem

$$\frac{\|\lambda - \lambda^*\|_\infty}{d_F(T, \mathcal{N}_\mathcal{T})} < \sqrt{\frac{2}{n-1}}\,\frac{\left(\sqrt{|\sigma|} - \sqrt{|\tau|}\right)}{\left(\sqrt{|\sigma|} + \sqrt{|\tau|}\right)}.$$

Remark

$$\lim_{T \to T^*} \frac{\|\lambda - \lambda^*\|_\infty}{d_F(T, \mathcal{N}_\mathcal{T})} = 0$$

DISTANCE $d_F(T_0, \mathcal{N}_\mathcal{T}) / \|T_0\|_F$

$$\frac{d_F(T_0, \mathcal{N}_\mathcal{T})}{\|T_0\|_F} = \frac{||\tau/\sigma| - 1|}{\sqrt{2}\sqrt{|\tau/\sigma|^2 + 1}}$$

i) in the case of $\sigma\tau \neq 0$, one has

$$0 \leq \frac{d_F(T_0, \mathcal{N}_\mathcal{T})}{\|T_0\|_F} < \frac{\sqrt{2}}{2},$$

and the normalized structured distance decreases from $\sqrt{2}/2$ to 0 when when T grows balanced;

ii) it results $\frac{d_F(T_0, \mathcal{N}_\mathcal{T})}{\|T_0\|_F} = 0$, iff $|\sigma| = |\tau|$;

iii) it results $\frac{d_F(T_0, \mathcal{N}_\mathcal{T})}{\|T_0\|_F} = \frac{\sqrt{2}}{2}$, in the remaining cases $\sigma = 0$, $\tau \neq 0$, and $\sigma \neq 0$, $\tau = 0$.

MATRICES IN $\mathcal{M}_\mathcal{T}$, NEAREST MATRIX IN $\mathcal{M}_\mathcal{T}$, STRUCTURED DISTANCE FROM NORMALITY

$\mathcal{M}$:= the algebraic variety of the matrices of $\mathbf{C}^{n\times n}$ with multiple eigenvalues

$\mathcal{M}_\mathcal{T}$:= the algebraic variety of the tridiagonal Toeplitz matrices of $\mathbf{C}^{n\times n}$ with multiple eigenvalues

$$\mathcal{M}_{\mathcal{T}} = \mathcal{M} \bigcap \mathcal{T}$$

$$d_F(T, \mathcal{M}_{\mathcal{T}}) := \min_{T^+ \in \mathcal{M}_{\mathcal{T}}} \left\| T - T^+ \right\|_F$$

$$\boxed{d_F(T, \mathcal{M}_{\mathcal{T}}) = \sqrt{n-1}\, \min\left\{|\sigma|, |\tau|\right\}.}$$

$$d_F(T^*, \mathcal{M}_{\mathcal{T}}) = \sqrt{n-1}\, \frac{|\sigma| + |\tau|}{2};$$

$$d_F(T^*, \mathcal{M}_{\mathcal{T}}) - d_F(T, \mathcal{M}_{\mathcal{T}}) =$$

$$= \sqrt{n-1}\, \frac{\max\left\{|\sigma|, |\tau|\right\} - \min\left\{|\sigma|, |\tau|\right\}}{2}.$$

NORMALIZED DISTANCE
$d_F(T_0, \mathcal{M}_{\mathcal{T}}) / \|T_0\|_F$

$$\frac{d_F(T_0, \mathcal{M}_{\mathcal{T}})}{\|T_0\|_F} = \frac{1}{\sqrt{\left|\frac{\max\{|\sigma|,|\tau|\}}{\min\{|\sigma|,|\tau|\}}\right|^2 + 1}}.$$

i) It results $0 < \frac{d_F(T_0, \mathcal{M}_{\mathcal{T}})}{\|T_0\|_F} \leq \frac{\sqrt{2}}{2}$;

ii) it results $\frac{d_F(T_0, \mathcal{M}_{\mathcal{T}})}{\|T_0\|_F} = \frac{\sqrt{2}}{2}$ iff $|\sigma| = |\tau|$;

iii) $\lim_{\min\{|\sigma|,|\tau|\} \to 0} \frac{d_F(T_0, \mathcal{M}_{\mathcal{T}})}{\|T_0\|_F} = 0$.

SENSITIVITY OF THE EIGENVALUES TO PERTURBATIONS IN THE MATRIX

$$f(\sigma, \tau) = \lambda(T_0) = [\lambda_1(T_0), \lambda_2(T_0), \ldots, \lambda n(T_0)],$$

$$f : D \subset \mathbf{C}^2 \to f(D) \subset \mathbf{C}^n, \; D = \{(\sigma, \tau) : \sigma\, \tau \neq 0\},$$

$$\left\| J_f(\sigma, \tau) \right\|_F = \sqrt{\tfrac{n-1}{2}} \sqrt{\left|\tfrac{\sigma}{\tau}\right| + \left|\tfrac{\tau}{\sigma}\right|}\,;$$

$$\left\| \Gamma_f(\sigma, \tau) \right\|_2 = \left\| \Gamma_f(\sigma, \tau) \right\|_F = \sqrt{\tfrac{n}{2}}\,.$$

Remark *Unlike the norms of Γ_f, the norm of J_f depends on σ and on τ. It reaches its absolute minimum $\sqrt{n-1}$ iff $|\sigma| = |\tau|$, i.e. iff T is normal, and tends to $+\infty$ when T grows unbalanced.*

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Since the eigenvalues of T_0 [or: the eigenvalues of T] do not change if the product $\sigma\tau$ remains unchanged, to compute them as eigenvalues of a matrix, it could seem convenient to replace T_0 with $T(n; s, 0, t)$ [or: to replace T with $T(n; s, \delta, t)$], where $s = \sqrt{|\sigma|\ |\tau|}\, e^{i \arg \sigma}$, $t = \sqrt{|\sigma|\ |\tau|}\, e^{i \arg \tau}$. Note that $T(n; s, 0, t)$ and

$T(n; s, \delta, t)$ are similar through a diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ to T_0 and T, respectively. But

$$\frac{d_{h+1}}{d_h} = \sqrt{\frac{|\sigma|}{|\tau|}} \quad h = 1, 2, \ldots, n.$$

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Individual condition numbers

$$\|x_h\|_2 = \sqrt{\sum_{k=1}^{n} \left(\frac{|\sigma|}{|\tau|}\right)^k \sin^2\left(\frac{k\,h\,\pi}{n+1}\right)}, \quad h = 1 : n;$$

$$\|y_h\|_2 = \sqrt{\sum_{k=1}^{n} \left(\frac{|\tau|}{|\sigma|}\right)^k \sin^2\left(\frac{k\,h\,\pi}{n+1}\right)}, \quad h = 1 : n;$$

$$\left|y_h^H\, x_h\right| = \frac{n+1}{2}, \quad h = 1 : n.$$

$$\Downarrow$$

$$\kappa(\lambda_h(T)) = \frac{\|x_h\|_2 \, \|y_h\|_2}{\left| y_h^H \, x_h \right|} =$$

$$= \frac{1 - a^{n+1}}{(n+1)a^{n/2}} \sqrt{f_1(a,n,h) f_2(a,n,h)}, \quad h = 1:n.$$

where $a = \left|\frac{\min\{|\sigma|,|\tau|\}}{\max\{|\sigma|,|\tau|\}}\right|$,

$$f_1(a,n,h) = \frac{1}{1-a} - \frac{1 - a\cos(\frac{2\pi h}{n+1})}{(1-a\cos(\frac{2\pi h}{n+1}))^2 + a^2 \sin^2(\frac{2\pi h}{n+1})},$$

$$f_2(a,n,h) = \frac{1}{1-a} - \frac{\cos(\frac{2\pi nh}{n+1}) - a}{(cos(\frac{2\pi nh}{n+1}) - a)^2 + sin^2(\frac{2\pi nh}{n+1})}.$$

$$\kappa(\lambda_h(T)) = 1, \; h = 1:n \quad iff \quad |\sigma| = |\tau|.$$

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Another way to measure the sensitivity of the eigenvalues

Let $\sigma = \min\{|\sigma|, |\tau|\}$ and consider the tridiagonal perturbation $E_s = T(n; -s, 0, 0)$ to the matrix $T = T(n; \sigma, \delta, \tau)$. For $s \neq \sigma$, one gets a family $T + E_s$ of matrices with simple eigenvalues having as a limit for $s \to \sigma$ the defective matrix $T(n; 0, \delta, \tau)$. The perturbation E_σ ($\|E_\sigma\|_F = \sqrt{n-1}\,|\sigma|$) brings all the eigenvalues in δ.

The more the original eigenvalue $\lambda_h(T)$ is far from δ, the higher is its sensitivity to the perturbation E_σ, and the higher is its "velocity":

$$v(\lambda_h(T)) = \frac{2\sqrt{|\sigma|\ |\tau|}\left|\cos\frac{h\,\pi}{n+1}\right|}{\sqrt{n-1}\,|\sigma|} =$$

$$= \frac{2}{\sqrt{n-1}}\sqrt{\frac{|\tau|}{|\sigma|}}\left|\cos\frac{h\,\pi}{n+1}\right|.$$

The larger is the ratio $\frac{|\tau|}{|\sigma|}$, the higher is the velocity of the eigenvalues, whereas the closer is the ratio to 1, the lower the velocity is.

MATRICES IN $\mathcal{S}_{\mathcal{T}}$, NEAREST MATRIX IN $\mathcal{S}_{\mathcal{T}}$, STRUCTURED DISTANCE FROM SINGULARITY

$\mathcal{S}$:= the algebraic variety of the singular matrices of $\mathbf{C}^{n\times n}$

$\mathcal{S}_{\mathcal{T}}$:= the algebraic variety of the singular tridiagonal Toeplitz matrices of $\mathbf{C}^{n\times n}$

$$\mathcal{S}_{\mathcal{T}} = \mathcal{S} \bigcap \mathcal{T}$$

$$d_F(T_0, \mathcal{S}_{\mathcal{T}}) := \min_{T^- \in \mathcal{S}_{\mathcal{T}}} \left\| T_0 - T^- \right\|_F$$

i) If T_0 is defective, one has rank $(T_0) = n - 1$. Therefore, $d_F(T_0, \mathcal{S}) = 0$; $d_F(T_0, \mathcal{S}_{\mathcal{T}}) = 0$ and $T^- \equiv T_0$;

ii) if $\sigma\tau \neq 0$ and n is odd, one has: T_0 is singular (rank $(T_0) = n-1$) $\Rightarrow d_F(T_0, \mathcal{S}) = 0$; $d_F(T_0, \mathcal{S}_{\mathcal{T}}) = 0$; $T^{-} \equiv T_0$;

iii) if $\sigma\tau \neq 0$ and n is even, one has: $d_F(T_0, \mathcal{S}_{\mathcal{T}}) = \sqrt{n-1}\,\min\{|\sigma|, |\tau|\}$ and one has

$$0 < \frac{d_F(T_0, \mathcal{S}_{\mathcal{T}})}{\|T_0\|_F} \leq \frac{\sqrt{2}}{2}.$$

SENSITIVITY OF THE MATRIX T_0

If $\sigma\tau \neq 0$ and n is even, one has the following upper bound for the distance from singularity of T_0:

$$d(T_0, \mathcal{S}) = \frac{1}{\|T_0^{-1}\|} \leq \min_{k=1:n} |\lambda_k(T_0)| =$$

$$= 2\sqrt{|\sigma|\,|\tau|}\,\cos\frac{n\pi}{2(n+1)}$$

and the following lower bound for the *matrix condition number* of T_0 (i.e. for the reciprocal of the normalized distance from singularity $d(T_0, \mathcal{S}) / \|T_0\|$):

$$\|T_0\| \|T_0^{-1}\| \geq \frac{\max_{k=1:n} |\lambda_k(T_0)|}{\min_{k=1:n} |\lambda_k(T_0)|} = \frac{\cos \frac{\pi}{n+1}}{\cos \frac{n\pi}{2(n+1)}}.$$

The *structured matrix condition number* of T_0 in the Frobenius norm (i.e. the reciprocal of the structured normalized distance from singularity $d_F(T_0, \mathcal{S}_\mathcal{T}) / \|T_0\|_F$) is greater or equal to $\sqrt{2}$. If $|\sigma| = |\tau| \neq 0$ and n is even, it is equal to $\sqrt{2}$.

Remark *Note that, if $|\sigma| = |\tau| \neq 0$ and n is even, the matrix condition number of T_0 in the 2-norm is :*

$$\|T_0\|_2 \|T_0^{-1}\|_2 = \frac{\cos \frac{\pi}{n+1}}{\cos \frac{n\pi}{2(n+1)}}.$$